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Stochastic 2D Navier-Stokes Equation and Applications to 2D Turbulence

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by

Shahab Karimi

To my wife

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Abstract

Stochastic 2D Navier-Stokes Equation and Applications to 2D Turbulence

Shahab Karimi

We will consider the 2-dimensional Navier-Stokes equation for an incompressible fluid with periodic boundary condition, and with a random perturbation that is in the form of white noise in time and a deterministic perturbation due to the large deviation principle. Our ultimate goal is to find appropriate conditions on the initial data and the forcing terms so that global existence and uniqueness of a mild solution is guaranteed. We will use the Picard's iteration method to prove existence of local mild solution and then prove the existence of a maximal solution which then leads to global existence. The result is applied to the backward Kolmogorov-Obukhov energy cascade and the forward Kraichnan enstrophy cascade in 2D turbulence.

Contents

Chapter 1

Stochastic 2D Navier-Stokes Equation

1.1 Introduction

The first paper on *Stochastic Navier-Stokes equation* was published in 1973, see [2]. Almost every paper on the local existence in the literature of stochastic Navier-Stokes uses the *Galerkin Approximations*. We will use Picard's iteration which makes the proof much easier. Also we will not restrict ourselves to H^1 and will work with a broader range of spaces. Finally we will find conditions that guarantee the existence of a unique global solution. This chapter is greatly influenced by the following: [4], [5], [7], [11], [19].

1.2 Functional Analysis Foundation

Throughout this chapter we assume that dimension is 2.

Let M be a bounded domain in $\mathbb{R}^2$ with smooth boundary ∂M. Define :

$$H = \{u = (u_1, u_2) | u_1, u_2 \in L^2(M), \nabla . u = 0^1,\ u.n = 0 \ on \ \partial M\}$$

Then H is a closed subspace of $(L^2(M))^2$. We equip H with the L^2 - norm. So $< u, v >= \int_M u.v \ dx$ and $|u| =< u, u >^{1/2}$. Also define [2]

$$V = \{u = (u_1, u_2) \mid u_1, u_2 \in H_0^1(M),\ \nabla . u = 0\}$$

Then V is closed in $H^1(M)$ and dense in H. We use the following inner product:

$$\ll u, v \gg = \int_M (\nabla u_1 . \nabla v_1 + \nabla u_2 . \nabla v_2) \ dx$$

So $||u|| = \ll u, u \gg^{1/2}$. Notice that this norm is well-defined because by Poincare inequality given $u_1,\ u_2 \in H_0^1(M)$ we have that $\int_M {u_1}^2 \ dx \le C_1 \int_M |\nabla u_1|^2 \ dx$ and $\int_M {u_2}^2 \ dx \le C_2 \int_M |\nabla u_2|^2 \ dx$ and so there exists a constant c such that for all $u \in V$, $|u| \le c||u||$. We have the continuous and dense embeddings $V \subset H \subset V'$.

The Stokes operator is $A = -P\Delta$ with the domain $D(A) = H^2(M) \cap V$ where P is the Leray projection from $L^2(M)^d$ onto H [3]. It can be extended to an

[1] in the distributional sense
[2] $H_0^1(M) = cl_{H^1(M)}(C_0^\infty(M))^d$
[3] $P(u) = u - \nabla\Delta^{-1}(\nabla . u)$

unbounded operator on H. It is well known that :

$i) < Av, u > \; = \; \ll u, v \gg for\ u, v \in V.\ So\ A\ can\ be\ viewed\ as\ a\ map\ from$

$V\ to\ V'.\ In\ fact,\ A : V \to V'\ \ is\ an\ isomorphism$

$ii)\ A\ is\ a\ strictly\ positive\ definite,\ self-adjoint\ operator\ on\ H\ and\ so$

$there\ exists\ an\ orthonormal\ basis\ \{e_j\}_{j\in\mathbb{N}}\ \ for\ H\ of\ eigenfunctions$

$of\ A\ with\ \ \{\lambda_j\}_{j\in\mathbb{N}}\ \ being\ the\ corresponding\ eigenvalues\ such\ that$

$0 < \lambda_1 \leq \lambda_2 \leq ...\ \ and\ \lambda_j \to +\infty\ as\ j \to +\infty\ (see\ [8]\ page\ 49-51).$

$iii)\ -A\ is\ the\ infinitesimal\ generator\ of\ an\ analytic\ semigroup$

$S(t) = e^{-tA}.\ We\ have\ ||S(t)|| \leq e^{-t\lambda_1}$

Any $u \in H$ can be expressed in the form of $u = \sum_{j=1}^{\infty} u_j e_j$ where $u_j =< u, e_j >$. By Parseval's identity we have $|u|^2 = \sum {u_j}^2$. Also $||u||^2 =< Au, u >= \sum \lambda_j {u_j}^2$. In order to define fractional powers of the Stokes operator, first define $D(A^\alpha)$ $(\alpha \in \mathbb{R})$ as follows:

$$D(A^\alpha) = \{u \in H \mid \sum |\lambda|_j^{2\alpha} {u_j}^2 < \infty\}$$

We equip $D(A^\alpha)$ with the norm $|u|_{D(A^\alpha)} = \left(\sum \lambda_j^{2\alpha} |u_j|^2\right)^{1/2}$. We have $D(A^0) = H$, and $D(A^{1/2}) = V$. It can be shown that $D(A^\alpha)$ is a Banach space (and in fact a

Hilbert space). Now define $A^\alpha u = \sum {\lambda_j}^\alpha u_j e_j,\ for\ u \in D(A^\alpha)$. One can show that $D(A^{-\alpha})$ is isomorphic to $D(A^\alpha)^{'}$ and $D(A^\alpha) \subset H \subset D(A^\alpha)^{'}$. Finally define the nonlinear term $Bu = P((u.\nabla)u)$. It will be useful to define the trilinear operator b on $V \times V \times V$ as follows:

$$b(u, v, w) = \int ((u.\nabla)v).w\ dx = \sum_{i,j=1}^{2} \int u_i\ \frac{\partial v_j}{\partial x_i}\ w_j\ dx$$

Then b is continuous and $b(u, v, v) = 0\ \ \forall u, v \in V$ [4]. Also $< Bu, v >= b(u, u, v)$.

Periodic Case.

Let $\mathbb{T}^2 = [0, 2\pi]^2$. First let us define $L^2_{per}(\mathbb{T}^2)$ and $H^1_{per}(\mathbb{T}^2)$. They are the spaces of $\mathbb{T}^2$-periodic functions defined on $\mathbb{R}^2$ which belong respectively to $L^2(O)$ and $H^1(O)$ for every bounded open set $O \subset \mathbb{R}^2$. We will assume *zero space average* and will work with the following two spaces:

$$\dot{H}_{per} = \{u = (u_1, u_2) \mid u_1, u_2 \in L^2_{per}(\mathbb{T}^2),\ \int_{\mathbb{T}^2} u\ dx = 0,\ \nabla.u = 0\}$$
$$\dot{V}_{per} = \{u = (u_1, u_2) \mid u_1, u_2 \in H^1_{per}(\mathbb{T}^2),\ \int_{\mathbb{T}^2} u\ dx = 0,\ \nabla.u = 0\}$$

We use the same inner products as before here (Poincare inequality holds for periodic functions with vanishing space average). We have $A = -\Delta$ and eigenfunctions of A can be found explicitly ($\lambda_k = |k|^2$). Therefore we can find an

[4] see [19], page 12-13

explicit basis for $\dot{H}_{per}$. Let $\mathbb{Z}_0^2 = \mathbb{Z}^2 \backslash (0,0)$ and $(\mathbb{Z}_0^2)_+ = \{(k_1,k_2) \in \mathbb{Z}^2 | k_1 > 0 \text{ or } (k_1 = 0 \wedge k_2 > 0\}$ and $(\mathbb{Z}_0^2)_- = \mathbb{Z}_0^2 \backslash (\mathbb{Z}_0^2)_+$.

Define e_k as follows:

$$e_k(x) = \begin{cases} \frac{cos(k.x)}{\sqrt{2}\pi|k|} k^{\perp} & k \in (\mathbb{Z}_0^2)_+ \\ \\ \frac{sin(k.x)}{\sqrt{2}\pi|k|} k^{\perp} & k \in (\mathbb{Z}_0^2)_- \end{cases}$$

where $k^{\perp} = (-k_2, k_1)$. Then $\{e_k\}_{k \in \mathbb{Z}_0^2}$ is an orthonormal basis for $\dot{H}_{per}$ [5]. Define

$$V_s = D(A^{s/2}) = \{u = \sum_{k \in \mathbb{Z}_0^2} c_k e_k \mid \sum |k|^{2s} |c_k|^2 < \infty\}$$

The H^s-norm $|.|_s$ on V_s is equivalent to $|.|_{V_s}$ [6], so we use the notation $|.|_s$ for both the H_s-norm and V_s-norm. We have $V_0 = \dot{H}_{per}$ and $V_1 = \dot{V}_{per}$. Also we have :

$$u = \sum_{k \in \mathbb{Z}_0^2} \hat{u}_k e_k \;\mapsto\; A^{\alpha} u = \sum_{k \in \mathbb{Z}_0^2} |k|^{2\alpha} \hat{u}_k e_k$$

So $|u|_{V_s} = |A^{s/2} u|$. One can show that

$$S(t)(\sum_{k \in \mathbb{Z}_0^2} \hat{u}_k e_k) = \sum_{k \in \mathbb{Z}_0^2} e^{-t|k|^2} \hat{u}_k e_k$$

Remark 1. *From this point on, everything will be restricted to the context of periodic case.*

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[5] see [8], page 52

[6] see [19] page 10

1.3 Formulation of the Stochastic Navier-Stokes Equation

The stochastic Navier-Stokes equation for incompressible fluid with zero mean condition on $\mathbb{T}^2$ reads as follows:

$$du = (\nu \Delta u - (u.\nabla)u + \nabla p)dt + Ldt + dW(t) \tag{1.1}$$

$$\nabla.u = 0$$

$$\bar{u} = 0$$

$$u(x, 0, \omega) = u_0(x, \omega)$$

where u is the velocity of an incompressible fluid, ν is the viscosity, p is the pressure field, the deterministic term L explains the large deviation from mean velocity, and $W(t)$ is the noise. Also u_0 is a random variable in $L^2(\mathbb{T}^2 \times \mathbb{T}^2)$. For simplicity we assume $\nu = 1$ (It is possible to restore the general case). One good model is as follows:[7]

$$L = \sum_{k \in \mathbb{Z}_0^2} \eta_k d_k e^{ik.x} \tag{1.2}$$

where d_k's are constant vectors in $\mathbb{C}^2$ and $d_{-k} = \overline{d_k}$ and they represent the bias in a particular direction in Fourier space. The η_k's are the rates in the k^{th} direction[8].

[7] see [1] chapter 1

[8] Usually $\eta_k = |k|^{1/3}$ is chosen.

Also

$$W(t) = \sum_{k\in\mathbb{Z}_0^2} {c_k}^{1/2} {b_t}^k e^{ik.x} \tag{1.3}$$

where the b_t^k's are mutually independent standard Brownian motions on a fixed measure space with a right continuous filteration, $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$. The c_k's are defined in $\mathbb{C}^2$ and $c_{-k} = \overline{c_k}$ and the series is convergent for almost every $\omega \in \Omega$.

The Leray-Hopf projection in dimension 2 acts as follows:

$$P(ae^{ik.x}) = P_k(a)e^{ik.x}$$

where

$$P_k(a) = (a.\frac{k^\perp}{|k|})\frac{k^\perp}{|k|}$$

(see [3] page 199). Then one can show that:

$$P(\sum_{k\in\mathbb{Z}_0^2} c_k e^{ik.x}) = \sum_{k\in(\mathbb{Z}_0^2)_+} \sqrt{2}\pi \; \frac{(c_k + c_{-k}).k^\perp}{|k|} \; e_k \; + \sum_{k\in(\mathbb{Z}_0^2)_-} \sqrt{2}\pi i \; \frac{(c_k - c_{-k}).k^\perp}{|k|} \; e_k \tag{1.4}$$

By applying the *Leray-Hopf projection* to equation (1.1), the pressure term drops out. Let $\Gamma = P(L)$ and $W_H(t) = P(W(t))$. By (1.4) we can assume that:

$$\Gamma = \sum_{k\in\mathbb{Z}_0^2} d_k e_k \tag{1.5}$$

and

$$W_H(t) = \sum_{k\in\mathbb{Z}_0^2} c_k^{1/2} b_t^k e_k \tag{1.6}$$

We are abusing notation here. Now d_k's and c_k's are in $\mathbb{R}$ and are different from what previously were defined in (1.2) and (1.3). Hence, we will obtain the following SDE in V

$$du(t) + [Au(t) + Bu(t)]dt = \Gamma dt + dW_H(t)$$

$$u(x, 0, \omega) = u_0(x, 0, \omega) \tag{1.7}$$

Suppose that $1 \leq p < \infty$. By a local mild solution of equation (1.7) in the space V_α, we mean a pair (u, τ) where τ is a *strictly positive* stopping time (a.s.) and $u(. \wedge \tau) \in L^p(\Omega; C([0, \tau]; V_\alpha))$ is an $\mathcal{F}_t^{u_0}$-adapted process such that:

$$u(t) = \; S(t)u_0 - \int_0^t S(t-s)B(u_s)ds$$

$$+ \;\; f(t) \;\; + W_A(t) \qquad a.s.\ for\ all\ t \in [0, \tau] \tag{1.8}$$

where

$$f(t) = \int_0^t S(t-s)\Gamma ds \;\; = \;\; \sum_{k \in \mathbb{Z}_0^2} \frac{1 - e^{-t|k|^2}}{|k|^2} d_k e_k \tag{1.9}$$

and

$$W_A(t) = \int_0^t S(t-s)dW_H(s) \tag{1.10}$$

We say that a local mild solution (u, τ) in V_α is unique if for any other local mild solution (u', τ') in V_α, we have $u(t \wedge \tau \wedge \tau') = u'(t \wedge \tau \wedge \tau')$ almost surely. Definition of a maximal mild solution is not a trivial one and it is as follows:

Definition 1. We say that a local mild solution (u, τ) in V_α is maximal provided that:

i) if (u', τ') *is a local mild solution in* V_α *then* $\tau' \leq \tau$ *a.s.*

ii) There exists a sequence $\{\tau_n\}_n$ *of stopping times such that* $\tau_n \uparrow \tau$ *and for all* $n \in \mathbb{N}$, (u, τ_n) *is a local mild solution in* V_α

In definition (1), if $\tau = \infty$ almost surely, then we say that the solution is *global*.

1.4 Local Existence and Maximal Solution

The next two lemmas are important tools to make connections between norms in different V_α's and in particular we will use them to prove that the nonlinear operator B is locally Lipschitz. We will use the notation $|.|^{(1)} \lesssim |.|^{(2)}$ between two norms on a space X if there exists a constant C so that $|x|^{(1)} \leq C|x|^{(2)}$ for all $x \in X$. If the two norms are equivalent we will use the notation $\approx$.

Lemma 1. *(Interpolation Inequality) If* $\alpha < \beta$, $0 \leq \theta \leq 1$ *and* $u \in V_\beta$, *then we have :*

$$|u|_{(1-\theta)\alpha+\theta\beta} \;\lesssim\; |u|_\alpha^{1-\theta}|u|_\beta^\theta \tag{1.11}$$

Proof. cf. [10].

□

Theorem 2. *(**Gagliardo-Nirenberg Inequality**) Suppose M is a smooth bounded domain in $\mathbb{R}^n$ and also $1 < p,\ p_0,\ p_1 < \infty,\ s, s_1 \in \mathbb{R},$ and $0 \leq \theta \leq 1$. Then the following holds :*

$$|u|_{H^s_p} \lesssim |u|^{1-\theta}_{L^{p_0}}\ |u|^{\theta}_{H^{s_1}_{p_1}} \tag{1.12}$$

$$if\ and\ only\ if\quad \frac{n}{p} - s =\ (1-\theta)\frac{n}{p_0} + \theta(\frac{n}{p_1} - s_1),\ \ s \leq \theta s_1$$

Proof. cf. [13].

□

Lemma 3. *For $n = 2$ we have the following:*

$$|u|_{H^s_p} \lesssim |u|^{1-\theta}_{H^{s_0}}\ |u|^{\theta}_{H^{s_1}}$$

$$if\quad \frac{2}{p} - s = (1-\theta)(1-s_0) + \theta(1-s_1),\ \ 0 \leq s_0,\ \ s \leq \theta s_1$$

Proof. In Theorem (2), let $s = 0$, $\theta = 1$, and $s_1 = 1 - \frac{2}{p}$. Then we obtain the following:

$$|u|_{L^p} \lesssim |u|_{H^{1-\frac{2}{p}}}\quad iff\ \ p \geq 2 \tag{1.13}$$

Now again in Theorem (2), let $p_1 = 2$ and $p_0 = \frac{2}{1-s_0}$ and use (1.12) to complete the proof.

□

Lemma 4. *i)* $|u|_{L^\infty} \lesssim |u|_\alpha^{1-\theta}|u|_\beta^\theta$ *where* $0 \le \alpha < \frac{n}{2} < \beta$ *and* $(1-\theta)\alpha + \theta\beta = \frac{n}{2}$

ii) $|\nabla u|_{L^4} \lesssim |u|_2^{1/2}.||u||^{1/2}$

Proof. For part (i) see [19] page 11. Part (ii) is *Ladyzhenskaya's inequality.*

□

Proposition 1. *i) If* ($\alpha > 1$ *and* $\delta \ge 0$)*, or* ($\alpha = 1$ *and* $\delta > 0$)*, and* $u \in V_\alpha$ *then* $Bu \in V_{-\delta}$ *and in fact we have the following:*

$$|Bu - Bv|_{-\delta} \;\lesssim\; (|u|_\alpha + |v|_\alpha)|u-v|_\alpha \tag{1.14}$$

In other words, $B : V_\alpha \to V_{-\delta}$ *is locally Lipschitz.*

ii) If $u \in V_2$*, then* $Bu \in V$ *and we have the following:*

$$||Bu - Bv|| \lesssim (|u|_2 + |v|_2)|u-v|_2 \tag{1.15}$$

Proof. *i*) We have that $|b(u,v,w)| \lesssim |u|_\alpha.|v|_\alpha.|w|_\delta$ for $d = 2, (\alpha > 1$, and $\delta \ge 0)$ or $(\alpha = 1$ and $\delta > 0)$. [9]. Now we have $< Bu - Bv, w >= b(u, u-v, w) + b(u-v, v, w)$. So:

$$| < Bu - Bv, w > | \lesssim |u|_\alpha|u-v|_\alpha|w|_\delta + |u-v|_\alpha|v|_\alpha|w|_\delta$$

[9] see [19] page 12

Therefore:

$$|Bu - Bv|_{-\delta} \;=\; sup_{0\neq w\in V_\delta}\frac{|< Bu - Bv, w >|}{|w|_\delta}$$

$$\leq C\ |u - v|_\alpha(|u|_\alpha + |v|_\alpha)$$

$ii)$ We have $Bu - Bv = P[(u.\nabla)u - (v.\nabla)v] = P([(u-v).\nabla]u) - P((v.\nabla)(u-v))$.

By virtue of Lemma (4) we have:

$$||P[(u-v).\nabla)u]|| \leq ||(u-v).\nabla)u|| = \{\int\sum[\partial_m((u_i - v_i)\partial_i u_j)]^2 dx\}^{1/2}$$

$$\leq \sum|\partial_m[(u_i - v_i)\partial_i u_j]|$$

$$\leq \sum|\partial_m(u_i - v_i)\partial_i u_j| + \sum|(u_i - v_i)\partial_m\partial_i u_j|$$

$$\lesssim |\nabla(u-v)|_{L^4}.|\nabla u|_{L^4} + |u-v|_{L^\infty}.|u|_{H^2} \tag{1.16}$$

$$\lesssim |u-v|_2^{1/2}.||u-v||^{1/2}.|u|_2^{1/2}.||u||^{1/2} + |u-v|_2^{1/2}.|u-v|^{1/2}.|u|_2$$

On the other hand, we have the following:

$$\begin{aligned}
||P[(v.\nabla)(u-v)]|| &\leq \sum |\partial_m(v_i\partial_i(u_j-v_j))| \\
&\leq \sum |\partial_m v_i.\partial_i(u_j-v_j)| + \sum |v_i\partial_i\partial_m(u_j-v_j)| \\
&\lesssim |\nabla v|_{L^4}.|\nabla(u-v)|_{L^4} + |v|_{L^\infty}.|u-v|_{H^2} \\
&\lesssim |v|_2^{1/2}.||v||^{1/2}.|u-v|_2^{1/2}.||u-v||^{1/2} + |v|_2^{1/2}.|v|^{1/2}|u-v|_2
\end{aligned}$$

So using the fact that $|z| \leq |Az|$ and $||z|| \leq |Az|$ we obtain the following:

$$||Bu - Bv|| \lesssim (|u|_2 + |v|_2)|u-v|_2$$

□

It is well known that when $1 < p < \infty$ and $s > 0$, the *Triebel-Lizorkin* space $F^s_{p,2}$ coincides with the Sobolev space H^s_p [10]. On the other hand it was proven in [6] that the fractional derivative norm $|D^\alpha f|_p$ is equivalent to $|f|_{F^s_{p,2}}$ and also the following theorem:

Theorem 5. *Suppose that* $\alpha > 0$, $1 < p_1, \ p_2, \ r < \infty$, $1 < q_1, \ q_2 \leq \infty$, *and* $\frac{1}{r} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2}$. *Then :*

$$|D^\alpha(fg)|_{L^r} \lesssim |D^\alpha f|_{L^{p_1}}|g|_{L^{q_1}} + |D^\alpha g|_{L^{p_2}}|f|_{L^{q_2}} \tag{1.17}$$

[10] e.g. see [18] chapter 3

Now we can prove the following proposition:

Proposition 2. *Assume that* $0 < \alpha < 1$*. Then we have the following:*

$$|Bu|_{2\alpha} \ \lesssim \ |u|_{2\alpha+1}(|u|_{\alpha} + |u|_{\alpha+1}) \ + \ |u|_{\alpha}|u|_{\alpha+1} \tag{1.18}$$

Proof. . We have $|Bu|_{2\alpha} \approx \sum_{i,j=1}^{2} |u_i\partial_i u_j|_{H^{2\alpha}} \approx \sum_{i,j=1}^{2} |D^{2\alpha}(u_i\partial_i u_j)|_{L^2}$. In (1.17), set $f = u_i,\ g = \partial_i u_j,\ r = 2,\ p_1 = \frac{2}{\alpha},\ q_1 = \frac{2}{1-\alpha},\ p_2 = 2$, and $q_2 = \infty$. Then using Lemma (3) for each i, j we have the followings:

$$|D^{2\alpha}u_i|_{p_1} \approx |u_i|_{H^{2\alpha}_{p_1}} \lesssim |u_i|_{H^{1+\alpha}} \qquad (\theta = 1, s_1 = 1+\alpha)$$

$$|\partial_i u_j|_{q_1} \lesssim u_{j\,H^1_{q_1}} \lesssim |u_j|_{H^{\alpha}} \qquad (\theta = 1, s_1 = \alpha)$$

$$|u_i|_{q_2} \leq |u|_{q_2} \lesssim |u|_{\alpha}^{\alpha}\ |u|_{\alpha+1}^{1-\alpha} \qquad (\textit{by interpolation inequality})$$

$$|D^{2\alpha}\partial_i u_j|_{p_2} \lesssim |D^{2\alpha+1}u_j|_{p_2} \approx |u_j|_{H^{2\alpha+1}}$$

So far we have proven the following:

$$|Bu|_{2\alpha} \lesssim |u|_{\alpha+1}.|u|_{\alpha} \ + \ |u|_{\alpha}^{\alpha}\ |u|_{\alpha+1}^{1-\alpha}.|u|_{2\alpha+1} \tag{1.19}$$

Recall the Young inequality: for all $a, b, \kappa > 0$, $r > 1$, and $r' = \frac{r}{r-1}$,

$$ab \leq \frac{\kappa}{r}a^r \ + \ \frac{1}{r'\kappa^{\frac{r'}{r}}}b^{r'} \tag{1.20}$$

By applying the Young inequality ($r = \frac{1}{\alpha}$) we get the following:

$$|u|_{\alpha}^{\alpha}\ |u|_{\alpha+1}^{1-\alpha} \leq c_1 |u|_{\alpha} + c_2 |u|_{\alpha+1}$$

So we obtain (1.18).

□

Proposition 3. *Let* $0 \leq \delta < \frac{1}{2} + n(1 - \frac{1}{p})/2$*. Then we have*

$$|A^{-\delta} Bu|_{L^p} \lesssim |A^{\theta} u|_{L^p} |A^{\rho} u|_{L^p} \tag{1.21}$$

provided that $\delta + \theta + \rho \geq \frac{n}{2p} + \frac{1}{2},\ \ \theta > 0,\ \ \rho > 0,\ \ \rho + \delta > \frac{1}{2}$

Proof. cf. [10].

□

Corollary 1. *In Proposition (3), if* $n = p = 2$ *and* $0 < \delta < 1$*, then we have the following:*

$$|Bu|_{-2\delta} \lesssim |u|_{1-\delta}^{2} \tag{1.22}$$

The following theorem will be used frequently throughout this chapter. Notice that by proposition (1), the nonlinear operator B is locally Lipschitz between two different spaces. In essence, B maps V_{α} into $V_{-\delta}$ which is a bigger space. In order to be able to use a *fixed point* argument to prove local existence of a (mild) solution, one has to work with a map from a space X into itself. One important property of the semigroup $S(t)$ is that when it acts on B, it pulls it back into a smaller space and this will be very useful for Picard's iteration.

Theorem 6. *Let X be a Banach space and A be a linear operator on X (not necessarily bounded). Also set $\rho(A) = \{\lambda \in \mathbb{C} : (\lambda I - A)^{-1}\text{exists and is bounded on } X\}$[11]. If $0 \in \rho(A)$ and $-A$ is the infinitesimal generator of an analytic semigroup $S(t)$, then we have the following:*

i) For every $t > 0$ and $\alpha \geq 0$, $S(t) : X \to D(A^\alpha)$.

ii) For every $x \in D(A^\alpha)$ we have $S(t)A^\alpha x = A^\alpha S(t)x$.

iii) For every $t > 0$ and $x \in X$ we have $|A^\alpha S(t)x| \leq M_\alpha t^{-\alpha}|x|$; so $A^\alpha S(t)$ is bounded and $||A^\alpha S(t)|| \leq M_\alpha t^{-\alpha}$

iv) For $0 < \alpha \leq 1$ and $x \in D(A^\alpha)$ we have $|(S(t) - I)x| \leq C_\alpha t^\alpha |A^\alpha x|$.

Proof. cf. [17] page 74-75.

□

Remark 2. *Obviously the Stokes operator satisfies all the conditions of Theorem (6).*

Proposition 4. *Suppose $1 \leq \alpha < 3$, $u \in C([0, T]; V_\alpha)$ and define:*

$$X(t) = \int_0^t S(t - s)B(u_s)ds$$

[11] This is called the resolvent set.

Then $X(.) \in C([0,T]; V_\alpha)$.[12]

Proof. If $1 \le \alpha < 2$, then choose β and γ so that $\alpha \le \beta < \gamma < 2$. If $2 \le \alpha < 3$, then choose β and γ so that $\alpha - 1 \le \beta - 1 < \gamma < 2$. Then by Proposition (1), $u \in C([0,T]; V_\alpha)$ implies $Bu \in C([0,T]; V_{\beta-\gamma})$. Now we have that:

$$A^{\beta/2}(X(t+h) - X(t)) = A^{\beta/2} \int_t^{t+h} S(t+h-s)B(u_s)ds + A^{\beta/2}(S(h) - I)X(t) \tag{1.23}$$

By virtue of Theorem (6), part *(iii)* we have the following:

$$|A^{\beta/2} \int_t^{t+h} S(t+h-s)B(u_s)ds| \le \int_t^{t+h} |A^{\gamma/2} S(t+h-s) A^{(\beta-\gamma)/2} B(u_s)|ds$$

$$\lesssim \frac{h^{1-\frac{\gamma}{2}}}{1-\frac{\gamma}{2}} \sup_{t \le s \le t+h} |A^{(\beta-\gamma)/2} B(u_s)| \tag{1.24}$$

In (1.24) if we set $t = 0$ and $h = t$, we get $X(t) \in V_\beta$. So we can assume that in particular $X(t) \in V_\gamma$. Therefore $A^{\beta/2}X(t) \in V_{\gamma-\beta}$ and by Theorem (6) part *(iv)* for the second term in (1.23) we have:

$$|(S(h) - I)A^{\beta/2}X(t)| \lesssim h^{(\gamma-\beta)/2}|A^{\gamma/2}X(t)|$$

Hence $\lim_{h\to 0} |X(t+h) - X(t)|_\beta = \lim_{h \to 0} |A^{\beta/2}(X(t+h) - X(t))| = 0$ and therefore $X(.)$ is continuous in V_β.

□

[12] see [2] page 254

The following lemma is a Gronwall type lemma which will be used to prove uniqueness of local mild solution.

Lemma 7. *Let* $-1 < r < 0$ *and* f *and* g *be two continuous non-negative functions on* $[0, T]$*. If*

$$f(t) \;\leq\; \int_0^t (t-s)^r f(s)g(s)ds \qquad for\ t\ \in [0, T]$$

then $f \equiv 0$ *on* $[0, T]$*.*

Proof. Choose $p \in (1, -1/r)$ and then define $q = \frac{p}{p-1}$. By Holder inequality we have

$$\int_0^t (t-s)^r f(s)g(s)ds \;\leq\; [\int_0^t (t-s)^{rp} ds]^{1/p} . [\int_0^t (fg)^q ds]^{1/q}$$

$$= (\frac{t^{1+rp}}{1+rp})^{1/p} . [\int_0^t (fg)^q ds]^{1/q}$$

Now let $F = f^q$ and therefore we have :

$$F(t) \;\leq\; (\frac{t^{1+rp}}{1+rp})^{q/p} . \int_0^t F g^q ds$$

So by the Gronwall's Lemma, $F \equiv 0$ and hence $f \equiv 0$ on $[0, T]$.

□

From this point on, we will use f and W_A as defined in (1.9) and (1.10).

Theorem 8. *Suppose* $1 \le \alpha < 3$, $1 \le p < \infty$, $u_0 \in L^p(\Omega; V_\alpha)$, $f \in C([0,T_0]; V_\alpha)$ *and* $W_A(.) \in L^p(\Omega\ ;\ C([0,T_0]; V_\alpha))$ *for some fixed* $T_0 > 0$. *Then there exists a unique mild solution to equation (1.7) in* V_α.

Proof. Define :

$$a(t,\omega) = S(t)u_0(\omega) + f(t) + W_A(t,\omega)$$

$$g(\omega) = 1 + |u_0(\omega)|_\alpha$$

and

$$\beta = \begin{cases} 1 & \text{if} \quad 2 \le \alpha < 3 \\ \\ \alpha - 1 & \text{if} \quad 1 \le \alpha < 2 \end{cases} \tag{1.25}$$

Then by Proposition (1), we have $u \in V_\alpha$ implies $Bu \in V_\beta$ and there exists a constant K such that:

$$|Bu - Bv|_\beta \le K(|u|_\alpha + |v|_\alpha)|u - v|_\alpha \tag{1.26}$$

Also by Theroem (6) , part (iii), there exists a constant C such that:

$$|A^{(\alpha-\beta)/2}S(t)x| \le Ct^{-(\alpha-\beta)/2}|x| \tag{1.27}$$

Define two stopping times on $(\Omega, \mathcal{F}, (\mathcal{F}_t))$ as follows:

$$\tau_1(\omega) = \inf\{t : \sup_{0\le s\le t} |a(s,\omega)|_\alpha > g(\omega)\} \tag{1.28}$$

and

$$\tau_2(\omega) = \inf\{t : CK\frac{t^{1+(\beta-\alpha)/2}}{1+(\beta-\alpha)/2} \ > \ \frac{1}{8\ g(\omega)}\} \tag{1.29}$$

In (1.28) if for a particular ω the set is empty, define $\tau_1(\omega)$ to be T_0. Now let $\tau = \tau_1 \wedge \tau_2 \wedge T_0$. Since $\lim_{t \to 0^+} a(t,\omega) = u_0(\omega)$ (almost surely in V_α), we conclude that $\tau > 0$ (a.s.). Without loss of generality assume $\lim_{t\to 0^+} a(t,\omega) = u_0(\omega)$ for all $\omega \in \Omega$. Let $\mathcal{H} = L^p(\Omega; C([0,\tau]; V_\alpha))$. From the assumptions we conclude that $u_0,\ W_A,\ f \in \mathcal{H}$ and hence, $a \in \mathcal{H}$.

Now define the following sequence:

$$z_0 = \ a$$

$$z_n = \ z_0 \ - \ \int_0^t S(t-s)B(z_{n-1}(s))ds \qquad\qquad 0 \le t \le \tau$$

Notice that by Proposition (4) we conclude that for each $\omega \in \Omega$, $z_n(t,\omega) \in C([0,\tau(\omega)]; V_\alpha)$.

claim. For almost all $\omega \in \Omega$ and $t \in [0, \tau(\omega)]$ we have the following:

$$|z_n(t,\omega) - z_{n-1}(t,\omega)|_\alpha \ \le \ \frac{g(\omega)}{2^n}$$

proof. We proceed by induction. For $n = 1$ we have:

$$
\begin{aligned}
|z_1 - z_0|_\alpha &\le \int_0^t |A^{(\alpha-\beta)/2} S(t-s) A^{\beta/2} B(z_0)| ds \\
&\le \int_0^t CK(t-s)^{(\beta-\alpha)/2} |A^{\alpha/2} z_0(\omega)|^2 ds \\
&\le CK\ g(\omega)^2 \frac{t^{1+(\beta-\alpha)/2}}{1+(\beta-\alpha)/2} \le \frac{g(\omega)}{2} \qquad \text{by (1.29)}
\end{aligned}
$$

Suppose that the claim is true for $0, 1, ..., n$. For any $i \le n$ we have

$$
\begin{aligned}
|z_i|_\alpha &\le |z_0|_\alpha + \sum_{j=1}^{i} |z_j - z_{j-1}|_\alpha \\
&\le g(\omega) + \sum_{j=1}^{i} \frac{g(\omega)}{2^j} < 2g(\omega) \qquad (1.30)
\end{aligned}
$$

Now we have

$$
\begin{aligned}
|(z_{n+1} - z_n)|_\alpha &\le \int_0^t |A^{(\alpha-\beta)/2} S(t-s) A^{\beta/2} [B(z_n) - B(z_{n-1})]| ds \\
&\le \int_0^t (t-s)^{(\beta-\alpha)/2} CK\ |z_n - z_{n-1}|_\alpha (|z_n|_\alpha + |z_{n-1}|_\alpha) ds \\
&\le CK\ \frac{t^{1+(\beta-\alpha)/2}}{1+(\beta-\alpha)/2} \frac{4g(\omega)^2}{2^n} \le \frac{g(\omega)}{2^{n+1}}
\end{aligned}
$$

■

So we conclude that for each ω, $\{z_n(.,\omega)\}$ is a Cauchy sequence in $C([0,\tau(\omega)];V_\alpha])$ and hence it is convergent to some $z(.,\omega) \in C([0,\tau(\omega)];V_\alpha])$. Note that z is an $\mathcal{F}_t^{u_0}$-adapted process because it holds for every z_n. Since the convergence is uniform, z satisfies

$$z = a - \int_0^t S(t-s)B(z(s))ds \qquad\qquad 0 \le t \le \tau$$

In (1.30) we proved that $|z_n(.,\omega)|_\alpha \le 2k(w)$. So $|z(.,\omega)|_\alpha \le 2k(w)$. For $0 \le t \le \tau$ we have the following :

$$|A^{\alpha/2}(z-a)(t,\omega)| \le \int_0^t |A^{(\alpha-\beta)/2}S(t-s)A^{\beta/2}B(z(s))|ds$$

$$\le \int_0^t CK(t-s)^{(\beta-\alpha)/2}|A^{\alpha/2}z(s)|^2 ds$$

$$\le CK\frac{t^{1+(\beta-\alpha)/2}}{1+(\beta-\alpha)/2}\ 4g(\omega)^2 \ \le\ g(\omega)$$

Thus

$$||z-a||_{\mathcal{H}}^p \ = \mathbb{E}[\sup_{0\le t\le\tau}|A^{\alpha/2}(z-a)|^p]$$

$$\le\ \mathbb{E}[g^p] < \infty \qquad\qquad (since\ u_0 \in L^p(\Omega;V_\alpha))$$

Since $a \in \mathcal{H}$, we conclude that $z \in \mathcal{H}$.

To prove uniqueness, suppose (z', τ) is another local solution. So we will have (almost surely on Ω):

$$\begin{aligned} |(z-z')(t)|_\alpha \ \le\ CK \int_0^t (t-s)^{(\beta-\alpha)/2} |(z-z')(s)|_\alpha (|z(s)|_\alpha + |z'(s)|_\alpha) ds \\ for\ all\ 0 \le t \le \tau(\omega) \qquad (1.31) \end{aligned}$$

So by Lemma (7) we conclude that $|(z-w)(t)|_\alpha = 0$ for all $0 \le t \le \tau(\omega)$. Therefore $z = z'$ almost surely.

$\square$

In the next theorem we will prove existence of maximal solution. Note that it is not an automatic result of the Zorn's Lemma because supremum of a collection of stopping times may not be measurable.

Theorem 9. *(Existence of Maximal Mild Solution) Given the assumptions of Theorem (8), there exists a unique (up to null sets) maximal mild solution in* V_α.

Proof. [13] Let Γ_N be the class of all stopping times such that $\tau \in \Gamma_N$ iff $\tau \le N$ and there exists a local mild solution $u \in L^p(\Omega; C([0,\tau]; V_\alpha))$ (by Theorem 8, $\Gamma_N \neq \emptyset$).

claim For every $k \in \mathbb{N}$, there exists a $\tau \in \Gamma_N$ such that

$$|\{\omega : \sigma \ge \tau + \frac{1}{k}\}| < \frac{1}{k} \qquad for\ all\ \sigma \in \Gamma_N \tag{1.32}$$

[13] see [15] pages 71-72

proof. Start with any $\tau_1 \in \Gamma_N$. If (1.32) does not hold, then there exists $\sigma_1 \in \Gamma_N$ s.t. $|\{\omega : \sigma_1 \geq \tau_1 + \frac{1}{k}\}| \geq \frac{1}{k}$. Now let $\tau_2 = \tau_1 \bigvee \sigma_1$. If (1.32) does not hold for τ_2, then there exists an $\sigma_2 \in \Gamma_N$ s.t. $|\{\omega : \sigma_2 \geq \tau_2 + \frac{1}{k}\}| \geq \frac{1}{k}$, and so on. This process has to stop at most at $n = Nk$ number of steps because at this step we have $\{\sigma_n \geq \tau_n + \frac{1}{k}\} \subset \{\sigma_n \geq \sigma_1 + \frac{Nk}{k}\}$, which has measure zero (by definition of Γ_N). Therefore the process will stop after a finite number of steps and that is when (1.32) holds.

■

Now for every $k \in \mathbb{N}$, choose τ_k such that (1.32) holds. Since Γ_N is closed under $\bigvee$ we can replace τ_i by $\bigvee_{j<i} \tau_j$ to get an increasing sequence of stopping times in Γ_N. It is easy to check that the new sequence still satisfies (1.32). Let us denote this sequence by $\tau_{N,1} \leq \tau_{N,2} \leq ....$ For each $N \in \mathbb{N}$ we obtain an increasing sequence $\{\tau_{N,n}\} \subset \Gamma_N$ which satisfies (1.32):

$$\tau_{1,1} \leq \tau_{1,2} \leq \tau_{1,3}, ...$$

$$\tau_{2,1} \leq \tau_{2,2} \leq \tau_{2,3}, ...$$

$$\tau_{3,1} \leq \tau_{3,2} \leq \tau_{3,3}, ...$$

Now let $\overline{\tau}_N = \bigvee_{n\leq N} \tau_{n,N}$. Thus, $\{\overline{\tau}_N\}_N$ is an increasing sequence and for each N, $\overline{\tau}_N$ satisfies the following:

i) $\tau_N \leq N$ *a.s.*

i) Corresponding to each τ_N *there exists a mild solution* $u_N \in L^p(\Omega; C([0,\tau_N]; D(A^\alpha)))$

$$iii)\ \{\omega : \sigma \geq \overline{\tau}_N + \frac{1}{N}\} < \frac{1}{N} \qquad for\ all\ \sigma \in \Gamma_N \tag{1.33}$$

Define $\overline{\tau} = \sup \overline{\tau}_N$. For $t < \overline{\tau}$, define $\overline{u}(t,\omega) = u_N(t,\omega)$ if $t < \overline{\tau}_N$. By uniqueness of solutions, $\overline{u}$ is well-defined (almost surely). If (u,τ) is a mild solution to the NS equation, one can prove that :

$$\{\sigma \geq \overline{\tau}\} = \bigcap_{N\in\mathbb{N}} \bigcup_{n=N}^{\infty} \{\sigma \wedge N \geq \overline{\tau}_N + \frac{1}{N}\}$$

By (1.33) we conclude that $|\{\sigma \geq \overline{\tau}\}| = 0$ and therefore $\sigma \leq \overline{\tau}$, almost surely. So far we have proved that $(\overline{u},\overline{\tau})$ is a maximal solution. Now if $(\overline{u}',\overline{\tau}')$ is another maximal solution with the corresponding sequence $\{\overline{\tau}'_N\}$, then by definition 1, for each N, $\overline{\tau}'_N \leq \overline{\tau}$ and therefore $\overline{\tau}' = \sup \overline{\tau}'_N \leq \overline{\tau}$ a.s.. Similarly, $\overline{\tau} \leq \overline{\tau}'$ a.s..

□

1.5 a Priori Estimates and Global Existence

Unlike deterministic differential equations that we have finite-time blow up criterion for maximal interval of existence, the stochastic equations show a very different kind of behavior. Namely, the maximal time of existence could depend on the sample space and be completely random. The solution could sometimes be global in time and other times have a finite time blow up. In the next few theorems we will develop some tools (exclusively for equation 1.7) similar to *finite-time blow up criterion* in order to prove global existence of mild solution to equation (1.7).

Proposition 5. *Fix $\omega \in \Omega$ and suppose that $1 \leq \alpha < 3$ and that $u(t,\omega) \in C([0,T);V_\alpha)$ satisfies equation (1.8). Also assume that f, $W_A(.,\omega) \in C([0,T];V_\alpha)$. Then $\lim_{t\to T^-} u(t,\omega)$ exists (in V_α) if $\sup_{0\leq t<T} |u(t,\omega)|_{\alpha-\delta} < \infty$ for some $0 \leq \delta < min\{1,\alpha/2\}$.*

Proof. It suffices to show that $u(t,\omega)$ is uniformly continuous on $[0,T)$. Since $S(t)$, f, and W_A are all uniformly continuous on $[0,T]$ we will only need to prove that the following is uniformly continuous in V_α:

$$F(t) = \int_0^t S(t-s)B(u_s)ds$$

Now consider the following cases :

case I. $1 \leq \alpha < 2$: So $min\{1,\alpha/2\} = \alpha/2$. Choose $\delta_1 \in (max\{0, \frac{1}{2}+\delta-\alpha, 1-2(\alpha-\delta)\}, 1-\alpha/2)$ and then choose $\delta_2 \in (\delta_1, 1-\alpha/2)$. Then by Proposition (3) we have $|A^{-\delta_1}Bu| \lesssim |A^{\alpha-\delta}u|^2$. So $\sup_{0\leq s<T} |A^{-\delta_1}Bu(s)| < \infty$. Now invoking lemma (6) we will obtain the following:

$$|A^{\alpha/2}(F(t_1)-F(t_2))| \le \int_{t_1}^{t_2} |A^{\alpha/2+\delta_1}S(t_2-s)A^{-\delta_1}B(u_s)|ds$$

$$+\int_0^{t_1} |A^{\alpha/2+\delta_2}S(t_1-s)[S(t_2-t_1)-I)]A^{-\delta_2}B(u_s)|ds$$

$$\lesssim \int_{t_1}^{t_2} (t_2-s)^{-(\alpha/2+\delta_1)}ds + \int_0^{t_1} (t_1-s)^{-\alpha/2-\delta_2}(t_2-t_1)^{\delta_2-\delta_1}ds$$

$$= \frac{(t_2-t_1)^{1-\alpha/2-\delta_1}}{1-\alpha/2-\delta_1} + \frac{t_1^{1-\alpha/2-\delta_2}}{1-\alpha/2-\delta_2}.(t_2-t_1)^{\delta_2-\delta_1}$$

which can be arbitrarily small provided that $t_1, t_2 \in [0,T]$ and $|t_1-t_2|$ is small enough.

case II. $2 \le \alpha < 3$: Let $\zeta = (\alpha-\delta-1)/2$ and $\zeta' = (\zeta+\frac{\alpha}{2}-1)/2$. One can show that $0<\zeta'<\zeta<1$. If we use inequality (1.18) for ζ, the dominant term on the right hand side of the inequality would be $|A^{2\zeta+1}u_s|$ which is bounded by the assumption. So we conclude that $\sup_{0\le s<T}|A^{2\zeta}B(u_s)|<\infty$. Now using Lemma

(6) parts (iii) and (iv) we have the following :

$$
\begin{aligned}
|A^{\alpha/2}(F(t_1) - F(t_2))| \leq & \int_{t_1}^{t_2} |A^{\alpha/2-2\zeta} S(t_2 - s) A^{2\zeta} B(u_s)| ds \\
& + \int_0^{t_1} |A^{\alpha/2-2\zeta'} S(t_1 - s)[S(t_2 - t_1) - I)] A^{2\zeta'} B(u_s) ds| \\
& \lesssim \int_{t_1}^{t_2} (t_2 - s)^{-(\alpha/2-2\zeta)} ds \\
& + \int_0^{t_1} (t_1 - s)^{-(\alpha/2-2\zeta')} (t_2 - t_1)^{2(\zeta-\zeta')} ds \\
& = \frac{(t_2 - t_1)^{2\zeta-\alpha/2+1}}{2\zeta - \alpha/2 + 1} + \frac{t_1^{\zeta}}{\zeta}.(t_2 - t_1)^{2(\zeta-\zeta')}
\end{aligned}
$$

This could be arbitrarily small provided that $t_1, t_2 \in [0, T]$ and $|t_1 - t_2|$ is small enough. Thus the proof is complete.

□

Remark 3. *Reading through the proof of Proposition (5) we notice that it is also valid for $\alpha \in (0, 1)$.*

The following is another Gronwall type inequality which will be used in Theorem (10).

Proposition 6. *Let f be a continuous nonnegative function on $[0,T]$, $1 \le p < \infty$, and*

$$f(t) \le A + B\ (\int_0^t f^p(s)ds)^{1/p} \qquad 0 \le t \le T$$

Then $f(t) \le \frac{Ag(t)}{g(t)-1}$ where $g(t) = (1 - exp(-B^p t))^{-1/p}$ for $0 \le t \le T$.

Proof. cf. [21].

□

Theorem 10. *Fix $\omega \in \Omega$ and suppose that the assumptions of proposition (5) hold, except that $\alpha \neq 1, 2$. Then $\lim_{t \to T^-} u(t)$ exists if :*

$$\sup_{0 \le t < T} |u(t)|_{\alpha-1} + \int_0^T |u(t)|_\alpha{}^2 dt < \infty$$

Proof. By Proposition (5) it suffices to prove that $\sup_{0 \le t < T} |u(t)|_{\alpha-\delta} < \infty$ for some $0 \le \delta < min\{1, \alpha/2\}$. Let $M = \sup_{0 \le t < T} |u(t)|_{\alpha-1}$. Notice that $|u(t)|_r$ is also bounded for any $r \le \alpha - 1$.

case I. $1 < \alpha < 2$: We will prove that $||u(t)||$ is bounded on $[0,T)$. Define $\delta_1 = \frac{1}{2} - \frac{\alpha}{4}$, and $q = \frac{4}{2-\alpha}$. Choose $p \in (\frac{4}{2+\alpha}, \frac{4}{4-\alpha})$ and then choose r such that $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$. Obviously $\delta < \alpha/2$. By Proposition (3), we have:

$$|A^{-\delta_1} Bu| \lesssim |A^{\alpha/4} u||A^{1/2} u| \tag{1.34}$$

On the other hand, by Lemma (1) we have:

$$|A^{\alpha/4} u| \lesssim |A^{\frac{\alpha-1}{2}} u|^{\alpha/2} |A^{\alpha/2} u|^{1-\frac{\alpha}{2}} \tag{1.35}$$

So by (1.34) and (1.35) we conclude that:

$$|A^{-\delta_1}Bu| \lesssim |A^{\frac{\alpha-1}{2}}u|^{\alpha/2}|A^{\alpha/2}u|^{1-\frac{\alpha}{2}}|A^{1/2}u| \tag{1.36}$$

Since $u(t)$ satisfies equation (1.8), we have the following:

$$||u(t)|| \leq A + \int_0^t |A^{1-\alpha/4}S(t-s)A^{-\delta_1}B(u_s)|ds \tag{1.37}$$

where

$$A = ||u_0|| + \sup_{0\leq t\leq T} ||f(t)|| + \sup_{0\leq t\leq T} ||W_A(t)|| < \infty$$

By inequality (1.36), and Holder inequality for three functions, we have the following:

$$\int_0^t |A^{1-\alpha/4}S(t-s)A^{-\delta_1}B(u_s)|ds \leq \int_0^t (t-s)^{\alpha/4-1}|A^{\frac{\alpha-1}{2}}u|^{\alpha/2}|A^{\alpha/2}u|^{1-\frac{\alpha}{2}}||u||ds$$

$$\leq \{\int_0^t ((t-s)^{\frac{\alpha}{4}-1}|A^{\frac{\alpha-1}{2}}u|^{\alpha/2})^p ds\}^{1/p} . \{\int_0^t |A^{\alpha/2}u|^2 ds\}^{1/q} . \{\int_0^t ||u||^r ds\}^{1/r}$$

$$\leq B\{\int_0^t ||u||^r ds\}^{1/r} \tag{1.38}$$

where

$$B = (\sup_{0\leq t<T} |A^{\frac{\alpha-1}{2}}u|)^{\frac{p\alpha}{2}} \frac{T^{\frac{1}{p}-1+\frac{\alpha}{4}}}{\frac{1}{p}-1+\frac{\alpha}{4}}\{\int_0^T |A^{\alpha/2}u|^2 ds\}^{1/q} < \infty$$

By (1.37), (1.38), and proposition (6), we conclude that $\sup_{0\leq t<T} ||u(t)|| < \infty$.

case II. $2 < \alpha < 3$: Choose any $\delta \in (0,1)$. Define $\delta_1 = \alpha - 2$, $q = \frac{2}{\alpha-2}$, and $p = \frac{2}{\alpha}$. By Proposition (2) we have that :

$$|Bu|_{2\delta_1} \lesssim \{|u|_{2\alpha-3}(|u|_{\alpha-2} + |u|_{\alpha-1}) \ + \ |u|_{\alpha-2}|u|_{\alpha-1}\} \tag{1.39}$$

By the interpolation inequality (Lemma 1) we have that $|u|_{2\alpha-3} \lesssim |u|_{\alpha-1}^{3-\alpha}|u|_{\alpha}^{\alpha-2}$. Every other term in (1.39) is bounded. Therefore we conclude that :

$$|Bu|_{2\delta_1} \leq K_1|u|_{\alpha}^{\alpha-2} + K_2$$

for some K_1 and K_2 independent of t. Therefore:

$$|Bu|_{2\delta_1}^q \leq 2^{q-1}(K_1^q|u|_{\alpha}^{q(\alpha-2)} + K_2^q) = 2^{q-1}(K_1^q|u|_{\alpha}^2 + K_2^q)$$

Hence $\int_0^T |Bu|_{2\delta_1}^q ds < \infty$. Now similar to the first case we have that :

$$|A^{(\alpha-\delta)/2}\int_0^t S(t-s)B(u_s)ds| \leq \int_0^T |A^{(\alpha-\delta)/2-\delta_1}S(t-s)A^{\delta_1}B(u_s)|ds$$

$$\leq \{\int_0^T (t-s)^{-p(\frac{\alpha-\delta}{2}-\delta_1)}ds\}^{1/p} \ \{\int_0^T |A^{\delta_1}B(u_s)|^q ds\}^{1/q}$$

Both terms are finite and thus the proof is complete (since u_0, f, and W_A are already bounded in V_α)

□

Corollary 2. *Under the hypotheses of Theorem (10), we conclude that if* $\sup_{0\leq t<T}|u(t)|_{\alpha-1} + \int_0^T |u(t)|_{\alpha}^2 dt < \infty$ *then* $\sup_{0\leq t<T}|u(t)|_{\alpha} < \infty$.

Theorem 11. *Suppose the assumptions of Theorem (5) hold, except that $\alpha \neq 1, 2$. In addition suppose that $W_A \in \bigcap_{T>0} L^p(\Omega; C([0,T]; V_\alpha))$ and $f \in C([0,\infty); V_\alpha)$. If $(\overline{u}, \overline{\tau})$ is a maximal solution, then:*

$$\sup_{0 \leq t < \overline{\tau}} |u(t)|_{\alpha-1} + \int_0^{\overline{\tau}} |u(t)|_\alpha^2 dt = \infty$$

almost surely on $\{\omega \ : \ \overline{\tau}(\omega) < \infty\}$

Proof. Define:

$$E_{m,k} = \{\omega \ : \ \overline{\tau}(\omega) \leq m \ \ and \ \ \sup_{0 \leq t < \overline{\tau}(\omega)} |u(t,\omega)|_{\alpha-1} + \int_0^{\overline{\tau}(\omega)} |u(t,\omega)|_\alpha^2 dt \leq k\}$$

and

$$\widetilde{u}_0 = \begin{cases} \lim_{t \to \overline{\tau}(\omega)^-} \overline{u}(t,\omega) \ in \ V_\alpha & \text{if } \omega \in E_{m,k} \\ \\ 0 & \text{otherwise} \end{cases}$$

The $E_{m,k}$'s are measurable. Also By Theorem (10), $\widetilde{u}_0$ is well-defined and obviously it belongs to $L^p(\Omega, V_\alpha)$ (because the limit is always less than or equal to k). In Theorem (8), we assumed that initial condition was given at $t = 0$. It is possible to extend the theorem to a more general case where the initial condition is given at a bounded stopping time τ_0. Let $\overline{\tau}_m = \overline{\tau} \wedge m$. Then, there exists a unique local mild solution $(\widetilde{u}, \widetilde{\tau})$ (with $\widetilde{\tau} > \overline{\tau}_m$) to the following :

$$\begin{aligned} \widetilde{u}(t) &= S(t - \ \overline{\tau}_m)\widetilde{u}_0 + \widetilde{f}(t - \ \overline{\tau}_m) \\ &\quad + \widetilde{W}_A(t - \ \overline{\tau}_m) - \int_{\overline{\tau}_m}^t S(t-s)B(u_s)ds \qquad \overline{\tau}_m \leq t \leq \widetilde{\tau} \end{aligned}$$

where

$$\widetilde{f}(t) = f(t + \overline{\tau}_m) - S(t)f(\overline{\tau}_m)$$

and

$$\widetilde{W}_A(t) = W_A(t + \overline{\tau}_m) - S(t)W_A(\overline{\tau}_m)$$

The reason that this local solution exists is that $\widetilde{f}$, $\widetilde{W}_A \in L^p(\Omega; C([0, m+1]; V_\alpha))$.

Define :

$$\overline{\tau}' = \overline{\tau} + \widetilde{\tau}.\mathbb{1}_{E_{m,k} \cap \{\overline{\tau} \leq \widetilde{\tau}\}}$$

and

$$\overline{u}'(t) = \overline{u}(t)\mathbb{1}_{t<\overline{\tau}} + \widetilde{u}(t)\mathbb{1}_{E_{m,k} \cap \{\overline{\tau} \leq t \leq \widetilde{\tau}\}}$$

When $t < \overline{\tau}$, $\overline{u}' = \overline{u}$ and so it satisfies equation 1.8. If $\bar{t} \leq m$ and $t \in [\overline{\tau}, \widetilde{\tau}]$, then we have:

$$\int_0^t S(t-s)B(\overline{u}'_s)ds = \int_0^{\overline{\tau}} S(t-s)B(\overline{u}_s)ds \;+\; \int_{\overline{\tau}}^t S(t-s)B(\widetilde{u}_s)ds$$

$$= \; S(t-\overline{\tau})\int_0^{\tau} S(\overline{\tau}-s)B(\overline{u}_s)ds \;+\; \int_{\overline{\tau}}^t S(t-s)B(\widetilde{u}_s)ds$$

$$= \; S(t-\overline{\tau})[S(\overline{\tau})u_0 + f(\overline{\tau}) + W_A(\overline{\tau}) - \widetilde{u}_0)]$$

$$+ [S(t-\overline{\tau})\widetilde{u}_0 + \widetilde{f}(t-\overline{\tau}) + \widetilde{W_A}(t-\overline{\tau}) - \widetilde{u}(t)]$$

$$= \; S(t)u_0 + f(t) + W_A(t) - \widetilde{u}(t)$$

Also suppose that $\{\overline{\tau}_{(n)}\}_n$ is a corresponding sequence of stopping times for the maximal solution $(\overline{u}, \overline{\tau})$. Define

$$\overline{\tau'}_{(n)} = \overline{\tau}_{(n)} + \mathbb{1}_{E_{m,k}\cap\{\tau_m \leq \widetilde{\tau}\}}.\,\widetilde{\tau}$$

Then $(\overline{u}', \overline{\tau}')$ together with $\{\overline{\tau'}_{(n)}\}_n$ satisfy the conditions of a maximal mild solution (definition 1). Therefore, $\overline{\tau}' = \overline{\tau}$ almost surely. It implies that $|E_{m,k}| = 0$ because $|\{\tau_m < \widetilde{\tau}\}| = 1$. And since $m, k \in \mathbb{N}$ were arbitrary, we conclude that

$\sup_{0\leq t<\overline{\tau}} |u(t)|_{\alpha-1} + \int_0^{\overline{\tau}} |u(t)|_{\alpha}^2 dt$ is almost nowhere finite on $\{\omega \ : \ \overline{\tau}(\omega) < \infty\}$.

□

Lemma 12. *The followings hold:*

i) $A^{\alpha} \int_0^t S(t-s)X_s ds = A^{\alpha-1}X_t - \frac{d}{dt}A^{\alpha-1}\int_0^t S(t-s)X_s ds.$

ii) $A^{\alpha}S(t)x = -\frac{d}{dt}A^{\alpha-1}S(t)x$

iii) $< \frac{d}{dt}(A^{\alpha-1}v(t)) \ , \ A^{\alpha}v(t) > \ = \ \frac{1}{2}\frac{d}{dt}|A^{\alpha-\frac{1}{2}}v(t)|^2$

Proof. All cases can easily be proven.

□

Proposition 7. *Suppose u is a mild solution to equation (1.7) and let $v = u - \varphi$ where $\varphi = W_A$. Then we have the following :*

$$|A^{\alpha}v(t)|^2 \ + \ \frac{1}{2}\frac{d}{dt}|A^{\alpha-\frac{1}{2}}v(t)|^2 = - < A^{\alpha-1}B(v(t)+\varphi(t)) \ , \ A^{\alpha}v(t) >$$

$$+ < A^{\alpha-1}\sum d_k e_k \ , \ A^{\alpha}v(t) > \tag{1.40}$$

Proof. $v(t)$ satisfies the following:

$$v(t) = S(t)u_0 + f(t) - \int_0^t S(t-s)B(v_s+\varphi_s)ds$$

Applying A^α to both sides of this equation and using Lemma (12) parts (i) and (ii) we will obtain the following:

$$A^\alpha v(t) = -\frac{d}{dt}[A^{\alpha-1}(S(t)u_0 - \int_0^t S(t-s)B(v_s+\varphi_s)ds + f(t))]$$

$$-A^{\alpha-1}B(v_t+\varphi_t) + A^{\alpha-1}\sum d_k e_k$$

$$= -\frac{d}{dt}(A^{\alpha-1}v(t)) - A^{\alpha-1}B(v_t+\varphi_t) + A^{\alpha-1}\sum d_k e_k$$

Now dot both sides of the equation (in L^2) by $A^\alpha v(t)$ and use Lemma (12) part (iii) to get (1.40).

$\square$

Proposition 8. *If $\frac{1}{2} \leq r < \frac{3}{2}$ and $r \neq 1$, then we have the following We have the following inequality:*

$$i)\ < B(z_1+z_2), A^{2r-1}z_1 > \ \leq \ \frac{1}{2}|A^{\frac{r}{2}}z_1|^2 + C_r\ (|A^{r-\frac{1}{2}}z_1|^2.F_r + G_r)$$

where F and G are define as follows:

$$F_r = \begin{cases} |A^{\frac{1}{4}}z_2|^4 & \text{if } r = \frac{1}{2} \\ ||z_1||^2 & \text{if } \frac{1}{2} < r < 1 \\ \sum_i |A^{\frac{r}{2}}z_i|^2 + \sum_i |A^{\frac{r-1}{2}}z_i|^2 & \text{if } 1 < r < \frac{3}{2} \end{cases}$$

and

$$G_r = \begin{cases} |A^{\frac{r}{2}} z_2|^4 & \textit{if } \frac{1}{2} \le r < 1 \\ |A^{r-\frac{1}{2}} z_2|^2 (\sum_i |A^{\frac{r}{2}} z_i|^2 + \sum_i |A^{\frac{r-1}{2}} z_i|^2) + \sum_{i,j} |A^{\frac{r}{2}} z_i|^2 \ |A^{\frac{r-1}{2}} z_j|^2 & \textit{if } 1 < \theta < \frac{3}{2} \end{cases}$$

Proof. **case I.** $r = 1/2$: cf. [4] pages 292-294 (They have proved it for $|z|_4$ instead of $|A^{1/4} z|$. Use the fact that $|z|_4 \le |A^{1/4} z|$ when $n = 2$.)

case II. $1/2 < r < 1$: By corollary 1, we have that $|A^{-(1-r)} Bu| \le C |A^{\frac{r}{2}} u|^2$. Also since $r - \frac{1}{2} < \frac{r}{2} < \frac{1}{2}$, by interpolation inequality (Lemma 1), we have $|A^{\frac{r}{2}} u|^2 \le C' \ |A^{r-\frac{1}{2}} u|.||u||$. Hence:

$$\begin{aligned} | < B(z_1 + z_2) \ , \ A^{2r-1} z_1 > | \ &\lesssim \ |B(z_1 + z_2)|_{D(A^{-(1-r)})} |A^{2r-1} z_1|_{D(A^{1-r})} \\ &\lesssim \ |A^{\frac{r}{2}} (z_1 + z_2)|^2 . |A^r z_1| \\ &\lesssim \ |A^{\frac{r}{2}} z_1|^2 . |A^r z_1| + \ |A^{\frac{r}{2}} z_2|^2 . |A^r z_1| \\ &\lesssim |A^{r-\frac{1}{2}} z_1| . ||z_1|| . |A^r z_1| + \ |A^{\frac{r}{2}} z_2|^2 . |A^r z_1| \qquad (1.41) \end{aligned}$$

For each term in (1.41) use the Young inequality with $r = 2$, $a = |A^{\frac{1+\theta}{2}} z_1|$ and κ small enough, to complete the proof.

case III. $1 < r < \frac{3}{2}$: In (1.18), let $\alpha = r - 1$ and we will obtain the following:

$$| < B(z_1 + z_2)\ ,\ A^{2r-1} z_1 > |\ \lesssim |A^r z_1|.\{|A^{r-\frac{1}{2}}(z_1 + z_2)|\ (|A^{\frac{r}{2}}(z_1 + z_2)| + |A^{\frac{r-1}{2}}(z_1 + z_2)|)$$

$$+ |A^{\frac{r}{2}}(z_1 + z_2)|.|A^{\frac{r-1}{2}}(z_1 + z_2)|\}$$

And similar to part (ii) using the Young inequality will complete the proof.

□

Theorem 13. *Let* $\frac{1}{2} < r < \frac{3}{2}$ *and* $r \neq 1$ *and fix* $\omega \in \Omega$ *and suppose that* $u(t)$ *is a mild solution to equation (1.7) on* $[0, T)$*. Then* $\sup_{0 \le t < T} |A^{r-\frac{1}{2}} u(t)| + \int_0^T |A^r u(t)|^2 dt < \infty$ *if the following terms are finite:*

$$|A^{r-\frac{1}{2}} u_0| \tag{1.42}$$

$$|A^{r-1} \sum \eta_k d_k e_k| \tag{1.43}$$

$$\int_0^T |A^r W_A|^2 \tag{1.44}$$

$$\sup_{0 \le t \le T} |A^\beta W_A(t)| \tag{1.45}$$

$$\textit{where } \beta = max\{\frac{r}{2}, r - \frac{1}{2}\}$$

Proof. Let us use φ for W_A in this proof and let $v = u - \varphi$. Since (1.44) and (1.45) are finite it suffices to show that $\sup_{0 \le t < T} |A^{r-\frac{1}{2}} v(t)| + \int_0^T |A^r \varphi|^2 < \infty$. Notice that because we have the *zero mean condition*, If $r \leq \beta$, then $|A^r z| \leq |A^\beta z|$; and even in the absence of the zero mean condition, if $|A^\beta z|$ is finite then $|A^r z|$ would also be finite. By (1.40) we have :

$$|A^r v(t)|^2 \ + \ \frac{1}{2}\frac{d}{dt}|A^{r-\frac{1}{2}} v(t)|^2 = - < B(v(t)+\varphi(t)) \ , \ A^{2r-1}v(t) >$$

$$+ < A^{r-1}\sum d_k e_k \ , \ A^r v(t) > \tag{1.46}$$

By Young inequality we have:

$$\begin{aligned} < A^{r-1}\sum \eta_k d_k e_k \ , \ A^r v > \ &\leq \ |A^{r-1}\sum d_k e_k|.|A^r v| \\ &\leq |A^{r-1}\sum d_k e_k|^2 \ + \ \frac{1}{4}|A^r v|^2 \end{aligned} \tag{1.47}$$

Also by Proposition (8) we have:

$$| < B(v+\varphi), A^{2r-1}v > | \ \leq \ \frac{1}{2}|A^r v|^2 + C_r \ [|A^{r-\frac{1}{2}}v|^2.F_r + G_r] \tag{1.48}$$

By (1.46),(1.47), and (1.48) we conclude that (provided that $|A^r v(t)| < \infty$ on $[0,T)$):

$$\frac{1}{4}|A^r v|^2 + \frac{1}{2}\frac{d}{dt}|A^{r-\frac{1}{2}}v|^2 \ \leq \ C_r \ [|A^{r-\frac{1}{2}}v|^2.F_r + G_r] + |A^{r-1}\sum d_k e_k|^2 \tag{1.49}$$

Omitting the term $\frac{1}{4}|A^r v|^2$ momentarily and using Gronwall's Lemma we obtain :

$$|A^{r-\frac{1}{2}}v|^2 \leq (|A^{r-\frac{1}{2}}v(0)|^2 + 2t|A^{r-1}\sum d_k e_k|^2 + 2C_r\int G_r).exp(2C_r\int F_r) \tag{1.50}$$

Relations (1.49) and (1.50) hold as long as $\frac{1}{2} \leq r < \frac{3}{2}$, $r \neq 1$, and $|A^r v(t)| < \infty$ on $[0,T)$. Denote by M_r the supremum of the right hand side of (1.50) on $[0,T]$. Then taking integral from zero to T of (1.49) we obtain:

$$\int_0^T |A^r v|^2 \leq 4\ M_r + 4\ C_r(M_r \int F_r + \int G_r) + 4T\ |A^{r-1} \sum d_k e_k|^2 \tag{1.51}$$

Therefore $\sup_{0\leq t<T} |A^{r-\frac{1}{2}} v| + \int_0^T |A^r v|^2$ is bounded by a function of $|A^{r-\frac{1}{2}} v_0|$, $|A^{r-1} \sum d_k e_k|$, and F_r and G_r. Now starting with $r = 1/2$ we obtain:

$$\sup_{0\leq t<T} |v(t)|^2 \leq \ (|u_0|^2 + |A^{-\frac{1}{2}} \sum \eta_k d_k e_k|^2 + 2C_{\frac{1}{2}} \int_0^T |A^{\frac{1}{4}} \varphi|^4)$$

$$\times\, exp(2C_{\frac{1}{2}} \int_0^T |A^{\frac{1}{4}} \varphi|^4) = M_{\frac{1}{2}} < \infty \tag{1.52}$$

and

$$\int_0^T ||v||^2 \leq 4M_{\frac{1}{2}} + 4C_{\frac{1}{2}}(1 + M_{\frac{1}{2}}) \int_0^T |A^{\frac{1}{4}} \varphi|^4 + 4T\ |A^{-\frac{1}{2}} \sum \eta_k d_k e_k|^2 < \infty \tag{1.53}$$

If $\frac{1}{2} < r < 1$, then (1.52) and (1.53) still hold true and moreover :

$$\sup_{0\leq t<T} |A^{r-\frac{1}{2}} v(t)|^2 \leq \ (|A^{r-\frac{1}{2}} u_0|^2 + |A^{r-1} \sum \eta_k d_k e_k|^2 + 2C_r \int_0^T |A^{\frac{r}{2}} \varphi|^4)$$

$$\times\ exp(2C_r \int_0^T ||v||^2) = M_r < \infty \tag{1.54}$$

and

$$\int_0^T |A^r v|^2 \leq \; 4M_r + 4C_r(M_r \int_0^T ||v||^2 + \int_0^T |A^{\frac{r}{2}}\varphi|^4)$$

$$+ 4T \left|A^{r-1} \sum \eta_k d_k e_k\right|^2 < \infty \tag{1.55}$$

And finally if $1 < r < \frac{3}{2}$, then (1.54) and (1.55) hold true for $r' = r - \frac{1}{2}$. We will only need to show that $\int F_r < \infty,$ *and* $\int G_r < \infty$. We have that F_r and G_r are combinations of the following terms: $|A^{r-\frac{1}{2}}\varphi|^2$, $|A^{\frac{r}{2}}\varphi|^2$, $|A^{\frac{r-1}{2}}\varphi|^2$, and $|A^{\frac{r}{2}}v|^2$, $|A^{\frac{r-1}{2}}v|^2$. The first three are bounded on $[0,T]$ because $\frac{r-1}{2} < \frac{r}{2} < r - \frac{1}{2}$. So we can pull the supremum of these terms out of the integral while computing $\int F_r$, and $\int G_r$.

Also since $\frac{r}{2} < 1$ by the previous case (1.54), $|A^{\frac{r}{2}}v|$ is bounded (see corollary 2) and we can pull the supremum of it out of the integral. The only problematic term might be $\int_0^T |A^{r-\frac{1}{2}}v|^2$. However, by (1.55) this terms is also bounded.

$\square$

Lemma 14. *Let* $\sum_{k\in\mathbb{Z}_0^2} d_k e_k \in V_{\alpha-2}$ *and* f *be as defined in (1.9). Then* $f \in C([0,\infty);V_\alpha)$.

Proof. One can easily prove that:

$$f(t) = \sum_{k\in\mathbb{Z}_0^2} \frac{1-e^{-t|k|^2}}{|k|^2} d_k e_k \tag{1.56}$$

and

$$\text{if } \ 0 < t_1 < t_2 \ , \text{ then } \frac{|e^{-t_1|k|^2} - e^{-t_2|k|^2}|}{|t_1 - t_2|} \ \le \ |k|^2 e^{-t_1|k|^2} \le \frac{e^{-1}}{t_1}$$

So $0 < t_1 < t_2$ implies:

$$|f(t_1) - f(t_2)|_\alpha = \{\sum |k|^{2\alpha-4} (e^{-t_1|k|^2} - e^{-t_2|k|^2})^2 d_k^2\}^{1/2}$$

$$\le \ \frac{e^{-1}|t_1 - t_2|}{t_1} \ |\sum d_k e_k|_{\alpha-2}$$

Therefore, $f(t)$ is uniformly continuous on any closed interval in $(0, \infty)$. To prove continuity at zero, notice that

$$|f(t) - f(0)|_\alpha^2 = \ |\sum_{0 \ne |k|^2 \le n} \frac{1 - e^{-t|k|^2}}{|k|^2} d_k e_k|_\alpha^2$$

$$+ \ \ |\sum_{|k|^2 > n} \frac{1 - e^{-t|k|^2}}{|k|^2} d_k e_k|_\alpha^2$$

$$\le (1 - e^{-t})^2 \ |\sum d_k e_k|_{\alpha-2}^2 \ + \ \sum_{|k|^2 > n} |k|^{2\alpha-4} d_k^2$$

When $t \to 0$ and $n \to \infty$, the right hand side approaches zero. So f is continuous at 0.

$\square$

Theorem 15. *(Global Existence) Let* $1 < \alpha < 3$, $\alpha \neq 2$, $u_0 \in L^p(\Omega; V_\alpha)$, $\sum d_k e_k \in V_{\alpha-2}$, $W_A \in \bigcap_{T>0} L^p(\Omega; C([0,T]; V_\alpha))$. *Then equation (1.7) has a unique global mild solution in* V_α.

Proof. By Theorem (9), existence of a unique maximal local mild solution (u, τ) is guaranteed. By the assumptions and Theorem (13), $\sup_{0 \le t < \tau} |u(t)|_{\alpha-1} + \int_0^\tau |u|_\alpha^2$ is finite almost everywhere on $\{w : \tau(\omega) < \infty\}$. So by Theorem (11) and by virtue of Lemma (14) we conclude that $\tau = \infty$ almost surely.

□

The condition

$$W_A \in \bigcap_{T>0} L^p(\Omega; C([0,T]; V_\alpha)) \tag{1.57}$$

in Theorem (15) is difficult to verify. In the next paragraph we will provide some sufficient and easier conditions to replace (1.57).

Let $\mathcal{H}$ be a separable and Q be a symmetric non-negative operator with $trQ < \infty$. So there exists a complete orthonormal basis $\{v_j\}_j$ of eigenvectors of Q with bounded sequence of corresponding eigenvalues λ_j. Then there exists a Q-Wiener process which can be represented in the form of $\mathcal{W}_t = \sum_{j=1}^\infty \sqrt{\lambda_j} B_t^j v_j$ where $\{B_t^j\}_j$ is a family of mutually independent 1-dimensional standard Brownian motions. Let $\mathcal{H}_0 = Im(Q^{1/2})$ (which is a Hilbert space). Now suppose that $\mathcal{A}$ generates a contraction semigroup $\mathcal{S}(t)$ [14]. Also assume that Φ is a Hilbert-Schmidt operator

[14] i.e. $||\mathcal{S}(t)|| \le e^{at}$ for some $a \in \mathbb{R}$

from $\mathcal{H}_0$ to K. Then the stochastic convolution $\mathcal{W}_\mathcal{A}$ is defined as follows:

$$\mathcal{W}_\mathcal{A} = \int_0^t \mathcal{S}(t-s)\Phi \, d\mathcal{W}_s \tag{1.58}$$

Theorem 16. *Let $p \geq 2$ and $T > 0$. Then the stochastic convolution $\mathcal{W}_\mathcal{A}(t)$ defined in (1.58) has a continuous version and there exists a constant $C_{p,T}$ such that :*

$$\mathbb{E}(\sup_{0\leq t\leq T} |\mathcal{W}_{\mathcal{A}(t)}|_K^p) \ \leq \ C_{p,T}(\int_0^T ||\Phi||_{HS}^2)^{p/2}$$

where $||\Phi||_{HS}$ denotes the Hilbert-Schmidt norm of Φ.

Proof. cf. [16] page 15.

□

Proposition 9. *Recall the definition of $W(t)$ in (1.6) and W_A in (1.10). If $p \geq 2$ and for some $q, s \in \mathbb{R}$ we have $\sum |k|^q c_k^s < \infty$ and $\sum |k|^{2\alpha-q} c_k^{1-s} < \infty$ then the condition (1.57) holds.*

Proof. Define $Q(e_k) = c_k^s |k|^q e_k$. Then Q is a trace class operator on H and $W(t) = \sum c_k^{s/2} |k|^{q/2} b_t^k e_k$ is a Q - Wiener process. Then define $\Phi : H_0 \to V_\alpha$ by $\Phi(e_k) = |k|^{-q/2} c_k^{\frac{1-s}{2}} e_k$. The operator Φ is a Hilbert-Schmidt operator because $||\Phi||_{HS} = \sum |k|^{2\alpha-q} c_k^{1-s} < \infty$. On the other hand, we have that :

$$\int_0^t S(t-s)\Phi dW_s = \int_0^t S(t-s) c_k^{1/2} db_s^k e_k = W_A(t)$$

and therefore by Theorem (16) we conclude that for every $T > 0$:

$$W_A \in L^p(\Omega; C([0,T]; V_\alpha))$$

$\square$

Chapter 2

Applications in Turbulence

2.1 Introduction

[1] The kinetic energy and the means-squared vorticity are both constants of motion for viscosity zero in two-dimensional turbulence. This implies that there exist two inertial ranges $E(k) \sim \epsilon^{2/3} k^{-5/3}$ and $E(k)\eta^{2/3} \sim k^{-3}$, where the kinetic energy per unit mass is $\int_0^\infty E(k)dk$, ϵ is the rate of the energy cascade per unit mass and η is the rate of the mean-square vorticity cascade per unit mass, see [12]. In experiments and observations in the atmosphere[2], the $-5/3$ range is found to consist of a backward (towards smaller values of k) energy cascade with a zero energy flow, whereas the -3 range consists of a forward (towards larger values

[1]This chapter is heavily based on [1]

[2]see [9]

of k) vorticity cascade with zero energy flow. It is conjectured that if energy is pumped into the systems into a band of wavenumers $\sim k_0$, then two steady states result up to viscous corrections. One a $-5/3$ range for $k << k_0$ and the other a -3 range for $k >> k_0$. The energy increases steadily with time as the $-5/3$ channels energy to ever lower scales until large eddies of the range of the whole fluid are excited. This accounts for the existence of large high and low pressure zones in the atmosphere on scales of the atmosphere itself.

It was pointed out by Onsager that the $-5/3$ cascade corresponded to Hölder continues functions of Hölder index $1/3$, whereas the -3 cascade corresponds to Hölder continuous functions of Hölder index $1/2$. In this dissertation the existence of both of these solutions of the two-dimensional Navier-Stokes equation, in a noisy environment, will be proven.

2.2 Forward and Backward Cascades in 2D

[3] Turbulence phenomenon occurs when the Reynold's number is large ($>$ 4000). It is a dimensionless quantity which is defined by $Re = \frac{LU}{\nu}$, where L is a characteristic length, U is a characteristic velocity of the flow, and ν is the kinematic viscosity of the fluid. A turbulent flow is highly irregular and chaotic but in simple terms it consists of many eddies of different sizes. The larger ed-

[3] some parts borrowed from [14]

dies are independent of viscosity but the smaller ones are controlled by viscosity. In essence, as famously quoted by L. F. Richardson (1922), the large eddies are unstable and eventually break up into smaller eddies, and so on. The energy is passed down from the large scales of the motion to smaller scales (it is called energy cascade) until reaching a sufficiently small length scale such that the viscosity of the fluid can effectively dissipate the kinetic energy. Therefore, there are three length scales: energy-containing scale $\mathcal{I}$, inertia subrange where the energy is transferred from large eddies to smaller ones, and dissipation scale η. Denote by $E(k)$ the energy spectrum for scalar wavenumber k (The larger k, the smaller the eddy). Essentially $E(k)$ represents the density of contributions of to the kinetic energy per unit scalar wave number. Therefore we have :

$$\mathcal{E} = \int_0^\infty E(k)dk$$

where $\mathcal{E}$ represents the kinetic energy.

At equilibrium, the rate of energy transfer from one scale to the next has to remain the same for all scales, so that no group of eddies sharing the same scale sees its total energy level increase or decrease over time. So by conservation of energy we conclude that $\mathcal{I}$ the rate at which energy is supplied at the largest scale is equal to that dissipated at the shortest scale. Let ϵ be the energy dissipation rate when the turbulence is fully developed. According to Kolmogorov theory (1941) the following hypotheses hold true:

i) For very high Re, the turbulent motions with length scales much smaller than l are statistically independent of the components of the motion at the energy-containing scales. The energy-containing scales of the motion may be inhomogeneous and anisotropic, but this information is lost in the cascade so that at much smaller scales the motion is locally homogeneous and isotropic If the energy is transferred over many stages to the large wave numbers where it is dissipated, then the time scales characteristic of the interactions at large wave numbers must be very much smaller than the time scale of the energy-containing eddies. The motion of these large wave numbers is close to a state of statistical equilibrium.

ii) For very high Re, the statistics of components in the equilibrium range, being independent of the larger scales, is universally and uniquely determined by the viscosity ν and the rate of energy dissipation ϵ.

iii) At very high Re the statistics of scales in the inertial subrange ($\mathcal{I}^{-1} \ll k \ll \eta^{-1}$) are universally and uniquely determined by the scale k and the rate of energy dissipation ϵ.

Then, in the inertial subrange the energy spectrum E(k) of the turbulence must be of the following form:

$$E(k) \sim \epsilon^{2/3} k^{-5/3} \qquad (\mathcal{I}^{-1} \ll k \ll \eta^{-1})$$

This is called the *Kolmogorov's 5/3 law*. Some corrections were added to this law by Kolmogorov and Obukhov in 1962 to take care of the influence of intermittency. For example, see [1] pages 48 and 49.

Remark 4. *One conclusion of the Kolmogorov's 5/3 law is that largest eddies contain the bulk of the kinetic energy.*

Remark 5. *There has not been a direct proof of the Kolmogorov's 5/3 law based solely on the Navier-Stokes equation.*

In dimension 2, there is the interesting so-called backward energy cascade phenomenon which was first explained in the paper Inertial Ranges in Two-Dimensional Turbulence, by R. H. Kraichnan [12]. In essence, the inertial range is divided into two subranges: the energy cascade range (backward cascade) and the enstrophy cascade range (forward cascade). The enstrophy is defined as follows:

$$\mathcal{C} = \frac{1}{2}\int_{\mathbb{T}^2} \nabla \times u^2 dx$$

One can show that

$$\mathcal{C} = \int_0^\infty k^2 E(k) dk$$

One very useful tool in the literature of turbulence is *structure functions*. By definition, a structure function of order p is the L^p-norm of velocity differences in equation (1.1). More precisely :

$$S_p(l) = \mathbb{E}[|u(x+l,t) - u(x,t)|^p] \tag{2.1}$$

where u is a solution to equation (1.1). One can show that when the second order structure function is a power law, $S_2(l) \propto l^{\xi_2}$, then the energy spectrum is also a power law of the form $E(k) \propto k^{-(1+\xi_2)}$ [4].

It can be shown that the solution to equation (1.1) is given as follows[5]:

$$u = e^{K(t)} M_t u_0 + \sum c_k^{1/2} \int_0^t e^{K(t-s)} M_{t-s} db_s^k e_k$$

$$+ \sum d_k \int_0^t e^{K(t-s)} M_{t-s} |k|^{1/3} dt e_k$$

where

$$K = \nu\Delta + \nabla\Delta^{-1} tr(\nabla u \nabla)$$

,

$$M_t = exp\{- \int u(B_s, s) dB_s - \frac{1}{2} \int_0^t |u(B_s, s)|^2 ds\}$$

The eigenvalues of the operator K in the inertial range are given by:

$$\lambda_k = 4\pi^2 \nu |k|^2 + C|k|^{2/3} \tag{2.2}$$

In (2.2), C is a constant times a Sobolev space norm of u. Now suppose that u is a solution to equation (1.1) and let us assume that $\lambda_k \approx c|k|^\alpha$. Since there is no intermittency correction in dimension 2, one can use the computations on page

[4]see [20] page 56
[5]see [1] page 42

98 of [1] to show that:

$$S_2 \leq \frac{4}{c^2} \sum_{k \in \mathbb{Z}_0^2} \frac{[d_k^2(1-e^{-\lambda_k t})^2 + c/2\ c_k(1-e^{-2\lambda_k t})]}{|k|^\alpha} \sin^2(\pi k.(x-u)) \tag{2.3}$$

Now if $|x-y| \ll 1$ and $t \to \infty$, we obtain the following:

$$S_2 \lesssim \ \sum \frac{\sin^2(\pi k.(x-y))}{|k|^\alpha} \ \propto \ |x-y|^\alpha \tag{2.4}$$

In the inertial range, if u is Holder continuous of order $\frac{1}{2}$ we will have cascade of enstrophy and if it is Holder continuous of order $\frac{1}{3}$ we will have cascade of energy. On the other hand, by Sobolev embedding theorem we know that $H^{4/3} \subset C^{1/3}$ and $H^{3/2} \subset C^{1/2}$ and for both cases we have the global existence theory from chapter 1.

Now suppose that we are in the inertial range and that $|x-y| \ll 1$. If we are closer to the dissipation scale, in which cascade of enstrophy occurs, then the effect of viscosity is significant and therefore $\nu \gg 1$. Thus, by (2.2) we have $\lambda_k \approx 4\pi^2\nu|k|^2$ and hence by (2.4), $S(2) \propto |k|^2$ and as a result, $E(k) \propto |k|^{-3}$. On the other hand, if we are closer to the larger scales in which energy cascade occurs, i.e. $\nu \ll 1$, then by (2.2) we have $\lambda_k \approx C|k|^{2/3}$ and therefore by (2.4), $S(2) \propto |k|^{2/3}$ and as a result, $E(k) \propto |k|^{-5/3}$. This explains the power laws of cascades of energy and enstrophy in 2 dimensions.

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