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Poincaré Inequalities Under Gauge Transformations

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To my loving wife, Jessica

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Curriculum Vitæ

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Education

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"The Trivariate von Neumann Inequality in Dimension Three." Senior project, California Polytechnic State University, San Luis Obispo, 2005. Microform.

Abstract

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For connections on trivial vector bundles compatible with compact gauge groups, we establish conditions on the vector bundle and gauge group under which translation of a connection by a constant connection matrix is achievable by a gauge transformation. These conditions may be roughly characterized as either restricting the base manifold to be one-dimensional or restricting the gauge group to take values in an abelian Lie group.

These results are then used to prove Poincaré inequalities on the gauge equivalent connection matrices, with some additional refinement of these results when the data considered is compactly supported and Coulomb.

Contents

Chapter 1

Introduction

1.1 Gauge theory: a PDEs perspective

Key within the study of gauge theory is the concept of gauge invariance. Under certain classes of transformations dependent on symmetries, the essential structure of some equations holds. We see this in the study of Maxwell's equations and Yang-Mills' equations [1], and the study of potentials and fields is still pursued by those advancing our understanding of electromagnetism and quantum mechanics.

Though within the study of physics it is common for many objects to be assumed smooth, mathematicians have been able to use partial differential equation results to develop gauge theory under fewer restrictions. Study of Lie groups and Lie algebras, vital to the concept of gauge transformations, is already well developed beyond assumptions of objects being fully smooth. Gauge theory can be developed with controls over Sobolev norms of objects, and controlled singularities can be introduced while maintaining some important results. Furthermore, some mathematicians have relied upon gauge fixing techniques to establish results in partial differential equations.

Tristan Rivieré's works [2] [3] use a gauge fixing technique to establish a sub-criticality results for systems of partial differential equations with an antisymmetric structure.

Karen Uhlenbeck's work [4] in establishing local existence of a Coulomb gauge allows for techniques dependent on divergence-free conditions to be locally applied. To achieve this result, Uhlenbeck relies heavily on solutions to systems of partial differential equations with Dirichlet and Neumann-like boundary conditions.

Simon Donaldson's individual work [5] and joint text with Kronheimer [6] include an alternative proof of Uhlenbeck's local Coulomb gauge existence result. Rather than depending heavily on boundary value problems, they instead are able to recast a key problem to be solved on the boundaryless compact 4-sphere.

Recasting existence results from relying upon boundary value problems to relying upon solving systems of partial differential equations on compact manifolds inspires efforts into other alternative analytic techniques available with the use of changes of gauge.

Furthermore, the usefulness of Poincaré inequalities within regularity analysis inspires investigation into conditions under which appropriate gauge transformations may lend to adapted Poincaré inequalities.

With the incorporation of gauge transformations into these inequalities, our results are distinct from those of Shartser [7], which does address some types Poincaré inequalities on scalar valued forms.

The following includes advances made under the guidance of Professor Denis Labutin towards expanding the body of work on gauge theory within the context of Sobolev spaces.

1.2 On notation and standard assumptions

We typically take a base manifold M to be a (nonempty) connected compact domain $M \subset \mathbb{R}^m$, unless otherwise specified. For purposes of Poincaré inequalities, we may then reference convex subdomains $M' \subset M$ and consider Sobolev norms $L^p_k = L^p_k(M')$ over these subdomains. We also note the classical result that compact convex domains $M' \subset \mathbb{R}^m$ have Lipschitz $\partial M'$, often required in the analysis of partial differential equations.

Though some results may be generalized to nontrivial manifolds, the intention to develop local results for partial differential equations allows these assumptions to serve well, as local results for nontrivial manifolds $\mathbb{M}$ will often begin by assuming the existence of local coordinate maps, or charts $M \to \mathbb{M}$, representing all objects on some domains $M \subset \mathbb{R}^m$.

For $0 \leq k \leq m$, the linear spaces of k-forms on M are denoted Ω^k_M, with $\Omega^0_M = C^\infty(M)$. We often will use a standard basis $\{\mathbf{d}u^\alpha\}_{1\leq\alpha\leq m}$ for Ω^1_M.

We use Einstein summation notation throughout, such as $A_\alpha\ \mathbf{d}u^\alpha := \sum_\alpha A_\alpha\ \mathbf{d}u^\alpha$. Particularly when expressing 2-forms, it should be clear in context whether $\mathbf{d}u^j \wedge \mathbf{d}u^k$ is considered ordered with $j < k$ or unordered.

We equip domain M with an inner product g and a volume-form $\mathbf{vol}_g \in \Omega^m_M$ respective of g. Though Hodge theory and the Hodge-star $\star := \star_{g,\mathbf{vol}_g}$ may be developed with more general inner products, it often suffices to accept a standard Riemannian (M, g).

We take vector bundle $\eta = M \times N$, with fibers for $x \in M$ satisfying $\eta_x \cong \mathbb{R}^n$ or $\eta_x \cong \mathbb{C}^n$. The linear space of the sections of η are then denoted by $\Gamma(\eta)$. We also take TM to be the tangent bundle on M, with tangent spaces T_xM for each $x \in M$.

We also take Lie group G that acts on the fibers of η, and assume it respects given inner products. With G comes its Lie algebra $\mathfrak{G} = \mathfrak{Lie}(G)$, and local identification of the Lie algebra and Lie group through the exponential map $\exp(\cdot) = e^\cdot : \mathfrak{G} \to G$. With many results dependent on e^χ being bounded in L^∞ for every appropriate $\chi : M \to \mathfrak{G}$, we must typically restrict the discussion to *compact* Lie groups G, such as $G \subset SO(n)$ or $G \subset SU(n)$, depending on choice of N.

The dual bundle $\eta^* = (M \times N)^*$ is canonically defined by $\eta^* := M \times N^*$, with fibers

$\eta^*_x = \{x\} \times N^*$ for $x \in M$. Furthermore, there are similarly defined canonical bundles $\eta^{\otimes j} \otimes \eta^{*\otimes k}$ for tensor products. We denote the automorphism bundle, sometimes called the *principle bundle*, by $Aut(\eta)$, and this has fibers $Aut(\eta)_x \cong GL(n, F)$ where $N = F^n$ for a field $F = \mathbb{R}$ or $F = \mathbb{C}$. A gauge group $\mathscr{G} \subset Aut(\eta)$ then is a bundle with fibers $\mathscr{G}_x \cong G$ for a given Lie group G. We identify the sections $\Gamma(\mathscr{G})$ of the gauge group with

$$\mathscr{D} = C^\infty(M, G),$$

and such a map $\sigma \in \mathscr{D}$ is representative of a gauge transformation. The adjoint bundle $Ad(\eta)$ then has fibers $(Ad(\eta))_x \cong \mathfrak{G}$ for the Lie algebra $\mathfrak{G} = \mathfrak{Lie}(G)$.

By extending the concept of sections to include E-valued maps for a general bundle E, we may denote such sections as $\Gamma(E)$. We may then extend the forms Ω^k_M to the tensor bundles, with

$$\Omega^k_M(E) = \Omega^k_M \otimes \Gamma(E).$$

This extension demands a selection of how to calculate wedge products on tensor-valued forms, which we resolve by in context with use of commutators $[\cdot, \cdot]$ for matrix-valued forms. Sections are then analogous to 0-forms taking on tensor-values, and we may write $\Gamma(\eta) = \Omega^0_M(\eta)$, $\Gamma(Aut(\eta)) = \Omega^0_M(Aut(\eta))$, $\Gamma(Ad(\eta)) = \Omega^0_M(Ad(\eta))$, and so on.

The (smooth) connections compatible with a gauge group $\mathscr{G}$ with respect to base connection $\mathbf{d}$ are then denoted

$$\mathfrak{A} := \{\nabla = \mathbf{d} + A \mid A \in C^\infty(M, Ad(\eta) \otimes T^*M)\}.$$

More generally, one may view such $A \in \Omega^1_M(End(\eta))$, where $End(\eta) := M \times End(N)$ is the endomorphism bundle, in the cases where consideration of $\mathscr{G}$ compatibility in unnecessary. The former is far more common in our work.

We often take a standard orthonormal basis $\{e_j\}_{1\le j\le n}$ for N, and its respective dual basis $\{\varepsilon^k\}_{1\le k\le n}$ for N^*. With preference of representation often dependent on computations being performed, we may express a connection matrix A representing a connection $\nabla = \mathbf{d} + A$ as

$$A = A_\alpha\ \mathbf{d}u^\alpha = \omega^j_k \otimes e_j \otimes \varepsilon^k = (A^j_{k\alpha}\ e_j \otimes \varepsilon^k)\ \mathbf{d}u^\alpha,$$

with $A^j_{k\alpha} \in C^\infty(M)$, $A_\alpha = A^j_{k\alpha}\ e_j \otimes \varepsilon^k \in \Omega^0_M(Ad(\eta))$, and $\omega^j_k = A^j_{k\alpha}\mathbf{d}u^\alpha \in \Omega^1_M(Ad(\eta))$.

It should be mentioned that there is no universally accepted convention on indexing components of tensor-valued forms by Greek and Latin, but is is best practice to strive to keep indices related to the base manifold M separated from the tensor indexing associated with the fibers N. Within sections, we adhere to this, but the reader should be aware that sections focusing on k-forms will often revert to taking advantage of multi-indices using capital Latin indexing, whereas in most sections focusing on 1-forms it is more convenient to use Greek indexing on the forms.

We notate the Sobolev spaces $L^p_k = W^{k,p}$, and extend the Sobolev norms from scalar-valued maps on M to tensor and form valued maps by summing the Sobolev norms of the scalar coordinates under a basis representation with respect to

$$\{e_j\}, \{\varepsilon^k\}, \{\mathbf{d}u^\alpha\}, \{\mathbf{d}u^\alpha \wedge \mathbf{d}u^\beta\}_{\alpha<\beta},$$

and so forth. Furthermore, note the canonical identification of forms with vector fields allows a Sobolev norm extension to all tensor form valued maps. This convention for calculating Sobolev norms is equivalent to others used by authors such as Uhlenbeck [4], but we use this convention for convenience in application to our results.

Chapter 2

Background

Much of the following background material may be found in a number of sources addressing differential manifolds, such as Lang's text [8], differential equations on manifolds, such as Arnold's text [9], or on gauge theory, such as Bleecker's text [10]. What is presented here primarily takes the perspective as presented in a Fall 2013 seminar offered by my advisor, Denis Labutin.

2.1 Connections and Curvature

2.1.1 Connections on trivial vector bundles $M \times N$

One may define a vector bundle on any smooth m-dimensional manifold $\mathbb{M}$ by associating at each point $x \in \mathbb{M}$ a common vector space N, commonly taken to be $N = \mathbb{R}^n$ or $N = \mathbb{C}^n$. For the bundle $\eta = \mathbb{M} \times N$, we then may define the fiber of η at $x \in \mathbb{M}$ to be $\eta_x := \{x\} \times N$.

The linear space of all sections of η is given by

$$\Gamma(\eta) := \{\mathfrak{s} \mid \mathfrak{s}(x) = (x, s(x)) \in \eta_x,\ s \in C^\infty(\mathbb{M}, N)\},$$

and it is common to naturally identify a section $\mathfrak{s} \in \Gamma(\eta)$ with the smooth map s taking values in N, and we do so liberally.

What should be emphasized when working on a bundle η is that all maps are inherently dependent on position on the base manifold $x \in \mathbb{M}$, and hence the formal definitions of maps retain this information in their image. In practice, we understand this universal dependence on position in the base manifold $\mathbb{M}$, and in this context focus mostly on the identified maps that take values associated with the structure of N, with little additional emphasis on the distinction that $\eta_x \cong N$ for every $x \in \mathbb{M}$.

For a thorough study of covariant derivatives and connections on vector bundles, one may introduce a local chart identifying a domain $M \subset \mathbb{R}^m$ with a subset of $\mathbb{M}$, allowing all maps on $\mathbb{M}$ to be understood locally in terms of charted behavior on M. However, as many of the results we pursue in the area of partial differential equations are local in nature, we bypass these technicalities and immediately assume all maps are understood in terms of a *trivial* base manifold M, and that the vector bundle on which we work is the trivial vector bundle $\eta = M \times N$. As an aside, Forster's lectures [11] §30 proves the triviality of holomorphic bundles on non-compact $\mathbb{M}$, further supporting the usefulness of consideration of trivial bundles.

The outer differentiation $\mathbf{d} : \Omega^k_M \to \Omega^{k+1}_M$ is already understood for scalar forms in terms of the wedge product. That is, given standard coordinate basic forms $\{\mathbf{d}u^j\}_{j=1}^m$ on $M \subset \mathbb{R}^m$, and an ordered multi-index $J = (j_1, \cdots, j_k)$ with $\mathbf{d}u^J := \mathbf{d}u^{j_1} \wedge \cdots \wedge \mathbf{d}u^{j_k} \in \Omega^k_M$, we have $f = f_J\ \mathbf{d}u^J \in \Omega^k_M$ for $f_J \in C^\infty(M, \mathbb{R})$ implies $\mathbf{d}f := (\partial_j f_J)\ \mathbf{d}u^j \wedge \mathbf{d}u^J \in \Omega^{k+1}_M$, where $\mathbf{d}u^j \wedge \mathbf{d}u^J$ is generally unordered when given by concatenated multi-index (j, J).

With the extension of the forms to tensor-valued bundles E, given by $\Omega_M^k(E) = \Omega_M^k \otimes \Gamma(E)$, we have for the trivial bundle $\eta = M \times N$ that

$$\mathbf{d} : \Omega_M^k(\eta) \to \Omega_M^{k+1}(\eta).$$

With a notational agreement $s = \mathbf{d}u^J \otimes s_J = s_J \; \mathbf{d}u^J \in \Omega_M^k(\eta)$, and perspective of form coordinates being 1-tensors, where given standard basis $\{e_\alpha\}_{\alpha=1}^n$ of N, each $s_J = (s_J)^\alpha \otimes e_\alpha \in \Gamma(\eta)$, and each $(s_J)^\alpha \in \Omega_M^0 = C^\infty(M, \mathbb{R})$, we have that

$$\begin{aligned} \mathbf{d}s &:= \partial_j \left[(s_J)^\alpha\right] \mathbf{d}u^j \wedge \mathbf{d}u^J \otimes e_\alpha \\ &= (\mathbf{d}s_J) \wedge \mathbf{d}u^J = \partial_j(s_J) \; \mathbf{d}u^j \wedge \mathbf{d}u^J. \end{aligned}$$

Equivalently, we may also notate s in terms of tensor coordinates being forms, with $s = \omega^\alpha \otimes e_\alpha \in \Omega_M^k(\eta)$, where each $\omega^\alpha = (\omega^\alpha)_J \; \mathbf{d}u^J \in \Omega_M^k$, and alternatively define

$$\begin{aligned} \mathbf{d}s &:= \partial_j \left[(\omega^\alpha)_J\right] \mathbf{d}u^j \wedge \mathbf{d}u^J \otimes e_\alpha \\ &= (\mathbf{d}\omega^\alpha) \otimes e_\alpha. \end{aligned}$$

These definitions are indeed equivalent, with $(s_J)^\alpha = (\omega^\alpha)_J$, and notational choice is dependent on application. Furthermore, though s was taken to be 1-tensor valued above, the first definition generalizes to sections of other bundles, most importantly to $\Gamma(End(\eta))$, as we shall see later.

A covariant derivative of a section $s \in \Gamma(\eta) = \Omega_M^0(\eta)$ in the direction of some $\mathcal{X} \in T_xM$, where again TM is the tangent bundle, is denoted by $\nabla_{\mathcal{X}} s$, and at each $x \in M$, $(\nabla_{\mathcal{X}} s)_x \in \eta_x = \{x\} \times N$. Given the vector fields $Vect(M)$, we then define covariant derivatives, or "connections," to be the bilinear maps

$$\begin{aligned} \nabla \ : \ Vect(M) \times \Gamma(\eta) &\to \Gamma(\eta) \\ (\mathcal{X}, s) &\mapsto \nabla_{\mathcal{X}} s \end{aligned}$$

that satisfy for all $\phi, \psi \in C^\infty(M, \mathbb{R})$, $\mathcal{X}, \mathcal{Y} \in Vect(M)$, and $s \in \Gamma(\eta)$, it holds that

i.) ∇ is linear and tensorial in $Vect(M)$, so $\nabla_{(\phi\mathcal{X}+\psi\mathcal{Y})} s = \phi \nabla_{\mathcal{X}} s + \psi \nabla_{\mathcal{Y}} s$,

ii.) and the Leibniz rule holds, with $\nabla_{\mathcal{X}}(\phi s) = \mathcal{X}(\phi) s + \phi \nabla_{\mathcal{X}} s$.

Equivalently, we may also interpret the covariant derivatives ∇ as maps

$$\nabla \ : \ \Omega^0_M(\eta) \to \Omega^1_M(\eta),$$

recalling $\Omega^0_M(\eta) = \Gamma(\eta)$ and $\Omega^1_M(\eta) = T^*M \otimes \Gamma(\eta)$, that satisfy for all $s, s_1, s_2 \in \Omega^0_M(\eta)$, $c_1, c_2 \in \mathbb{R}$, and $\phi \in C^\infty(M, \mathbb{R})$,

i.) linearity holds, with $\nabla(C_1 s_1 + C_2 s_2) = C_1 \nabla s_1 + C_2 \nabla s_2$,

ii.) and the Leibniz rule holds, with $\nabla(\phi s) = \mathbf{d}\phi \otimes s + \phi \nabla s$.

It should be noted that, as a function of the sections $s \in \Gamma(\eta) = \Omega^0_M(\eta)$, covariant derivatives are *not* tensorial, but rather obey a Leibniz rule.

The basic example of a covariant derivative is then $\nabla = \mathbf{d}$, the Euclidean coordinate derivative with which we extended outer differentiation onto tensor valued forms. Furthermore, we may note that for all sections $s \in \Omega^0_M(\eta)$ and $\phi \in C^\infty(M, \mathbb{R})$, for any two covariant derivatives $\nabla, \tilde{\nabla}$, it then holds

$$\begin{aligned} (\nabla - \tilde{\nabla})(\phi s) &= \mathbf{d}\phi \otimes s + \phi \nabla s - \mathbf{d}\phi \otimes s - \phi \tilde{\nabla} s \\ &= \phi (\nabla - \tilde{\nabla}) s, \end{aligned}$$

and therefore the difference between two covariant derivatives is tensorial. Because of this, we can then conclude that every covariant derivative can be expressed in terms of some "connection matrix"

$$A = A_j \, \mathbf{d}u^j = \omega^\alpha_\beta \; e_\alpha \otimes \varepsilon^\beta \in \Omega^1_M(End(\eta)),$$

where each $A_j \in \Omega^0_M(End(\eta))$, $\omega^\alpha_\beta = A^\alpha_{\beta j} \; \mathbf{d}u^j \in \Omega^1_M$, and $\{e_\alpha\}^n_{\alpha=1}$ and $\{\varepsilon^\beta\}^n_{\beta=1}$ are respectively dual standard bases for N and N^*. That is, we may write

$$\nabla = \mathbf{d} + A,$$

and for a section $s = s^\alpha \otimes e_\alpha \in \Gamma(\eta) = \Omega^0_M(\eta)$, with each $s^\alpha \in \Omega^0_M$, we can equivalently express the action of ∇ on s as

$$\begin{aligned}
\nabla s &= \mathbf{d}s + As \\
&= \partial_j s \; \mathbf{d}u^j + A_j(s) \; \mathbf{d}u^j \\
&= \mathbf{d}s^\alpha \otimes e_\alpha + \omega^\alpha_\gamma s^\gamma \otimes e_\alpha \\
&= \partial_j(s^\alpha) \; \mathbf{d}u^j \otimes e_\alpha + A^\alpha_{\gamma j} s^\gamma \; \mathbf{d}u^j \otimes e_\alpha,
\end{aligned}$$

with emphasis on the 1-form structure, the tensor structure, or both.

For each connection $\nabla = \mathbf{d} + A$ acting sections of the bundle $\eta = M \times N$, where

$$\begin{aligned}
\nabla \;:\; \Omega^0_M(\eta) &\to \Omega^1_M(\eta) \\
s &\mapsto \mathbf{d}s + As,
\end{aligned}$$

we may extend the covariant derivative to the vector-bundle valued forms as well. We then have for $s = s_J \, \mathbf{d}u^J \in \Omega^k_M(\eta)$ with ordered multi-index J,

$$\begin{aligned} \mathbf{d}^\nabla \ : \ \Omega^k_M(\eta) &\to \Omega^{k+1}_M(\eta) \\ s &\mapsto \mathbf{d}s + A \wedge s, \end{aligned} \tag{2.1}$$

where $A \wedge s = (A_j \, \mathbf{d}u^j) \wedge (s_J \, \mathbf{d}u^J) = A_j s_J \, \mathbf{d}u^j \wedge \mathbf{d}u^J$, each $A_j s_J$ is understood in terms of the standard tensor product with $A_j s_J = \left((A_j)^\alpha_\beta \, e_\alpha \otimes \varepsilon^\beta\right)\left((s_J)^\gamma \, e_\gamma\right) := (A_j)^\alpha_\gamma (s_J)^\gamma \, e_\alpha$, and the multi-index concatenation (j, J) is generally initially unordered.

Additionally it is important to extend the exterior derivative to endomorphism-bundle valued forms. Beyond this, it is also possible to extend the exterior derivative ∇ to act on the tensor-valued bundles $\eta^{\otimes i} \otimes \eta^{*\otimes j}$, for which $i = j = 1$ retrieves $End(\eta) \cong \eta \otimes \eta^*$, and then extend that further to general tensor-bundle valued forms, but we will not address the full generality here.

Initially, we want an extension of $\nabla = \mathbf{d} + A$ to act upon sections $\Gamma(End(\eta)) = \Omega^0_M(End(\eta))$, where fibers $End(\eta)_x \cong GL(n, F)$ for $N = F^n$, where we typically have $F = \mathbb{R}$ or $F = \mathbb{C}$. More generally, we need this covariant derivative extension to act upon tensor-valued forms in $\Omega^k_M(End(\eta))$, and indeed this is possible.

Take $A = A_j \, \mathbf{d}u^j$. We have that an extension of $\nabla = \mathbf{d} + A$ is given by

$$\begin{aligned} \tilde{\nabla} \ : \ \Omega^0_M(End(\eta)) &\to \Omega^1_M(End(\eta)) \\ S &\mapsto \mathbf{d}S + \mathcal{A}S, \end{aligned}$$

where $\mathcal{A} \in \Omega^1_M(End(End(\eta))$, and is given by

$$\mathcal{A}S := [A, S] = [A_j, S] \, \mathbf{d}u^j,$$

and we use the standard matrix commutator $[A_j, S] = A_jS - SA_j$.

The behavior of $\tilde{\nabla}$ on sections $S = S_j\ \mathbf{d}u^j \in \Gamma(End(\eta))$, then extends to $\mathbf{d}^{\tilde{\nabla}}$ acting on endomorphism-valued forms as follows. Given $S = S_J\ \mathbf{d}u^J \in \Omega^k_M(End(\eta))$, where each $S_J = (S_J)^\alpha_\beta\ e_\alpha \otimes \varepsilon^\beta$, and each J is an ordered multi-index, we then take

$$\begin{aligned} \mathbf{d}^{\tilde{\nabla}}\ :\ \Omega^k_M(End(\eta)) &\to \Omega^{k+1}_M(End(\eta)) \\ S &\mapsto \mathbf{d}S + \mathcal{A} \wedge S, \end{aligned} \tag{2.2}$$

where $\mathcal{A} \wedge S = [A \wedge S] := [A_j, S_J]\ \mathbf{d}u^j \wedge \mathbf{d}u^J$. We then have that $\mathbf{d}^{\tilde{\nabla}}$ serves as a exterior derivative. It should be noted that, for vector-bundle valued forms $s \in \Omega^k_M(\eta)$, and endomorphism-bundle valued forms $S \in \Omega^p_M(End(\eta))$, this extension satisfies the Leibniz rule

$$\begin{aligned} \mathbf{d}^{\nabla}(S \wedge s) &= \left(\mathbf{d}^{\tilde{\nabla}} S\right) \wedge s + (-1)^p S \wedge \mathbf{d}^{\nabla} s \\ &= (\mathbf{d}S + [A \wedge S]) \wedge s + (-1)^p S \wedge (\mathbf{d}s + A \wedge s), \end{aligned}$$

where $\mathbf{d}^{\nabla}$ is the extension of ∇ to the vector-bundle valued forms given in (2.1), and the multiplication of tensor-valued forms is understood in terms of the standard tensor multiplication.

2.1.2 Curvature of a connection ∇

As a connection matrix A representing a connection $\nabla = \mathbf{d} + A$ is interpreted by physicists as a potential, so too is the curvature F^{∇} interpreted by physicists as a field, and is sometimes referred to as the "curvature field."

There is also a variety of notations to represent the dependence of the curvature F^{∇} on the connection matrix A, and we see throughout literature the notational equivalences

$F^\nabla = F(\nabla) = F(A) = F^A$, with the latter options more agreeable when a standard base connection $\mathbf{d} + \mathbf{0}$ is understood.

Given $A = A_j \ \mathbf{d}u^j \in C^\infty(M, \mathfrak{G} \otimes T^*M)$ representing a connection $\nabla = \mathbf{d} + A \in \mathfrak{A}$ on a bundle $\eta = M \times N$, and the extensions $\mathbf{d}^\nabla \ : \ \Omega^k_M(\eta) \to \Omega^{k+1}_M(\eta)$, the *curvature* $F^\nabla = F(A)$ is defined as

$$F^\nabla := \mathbf{d}^\nabla \circ \mathbf{d}^\nabla.$$

It is then straightforward to show the action of F^∇ on a section $s = \Omega^0_M(\eta)$ is given by

$$\begin{aligned} F^\nabla s &= (\mathbf{d} + A \wedge \cdot)(\mathbf{d} + A)s \\ &= (\mathbf{d} + A \wedge \cdot)(\mathbf{d}s + As) \\ &= \mathbf{d}^2 s + \mathbf{d}(As) + A \wedge \mathbf{d}s + A \wedge As \\ &= \mathbf{0} + (\mathbf{d}A)s + (-1)^1 A \wedge \mathbf{d}s + A \wedge \mathbf{d}s + A \wedge As \\ &= (\mathbf{d}A + A \wedge A)s, \end{aligned}$$

and furthermore that for a vector-bundle valued form $s = s_J \ \mathbf{d}u^J \in \Omega^k_M(\eta)$ that

$$F^\nabla s = (\mathbf{d}A + A \wedge A) \wedge s,$$

and so it is in practice common to then write

$$\begin{aligned} F^\nabla = F(A) &= \mathbf{d}A + A \wedge A \\ &= \sum_{j,k} \partial_k(A_j)\mathbf{d}u^k \wedge \mathbf{d}u^j + A_k A_j \ \mathbf{d}u^k \wedge \mathbf{d}u \\ &= \sum_{j<k} (\partial_k A_j - \partial_j A_k) \ \mathbf{d}u^k \wedge \mathbf{d}u^j + [A_k, A_j] \ \mathbf{d}u^k \wedge \mathbf{d}u^j. \end{aligned}$$

Some authors then use alternative notation $F(A) = \mathbf{d}A + \frac{1}{2}[A \wedge A]$.

Noting that $F^\nabla \in \Omega^2_M(End(\eta))$, one may show the Bianchi identity for $\nabla = \mathbf{d} + A$ of

$$\mathbf{d}F^\nabla = F^\nabla \wedge A - A \wedge F^\nabla,$$

and using the extension $\mathbf{d}^{\tilde{\nabla}} : \Omega^2_M(End(\eta)) \to \Omega^3_M(End(\eta))$ given by $\mathbf{d}^{\tilde{\nabla}} = \mathbf{d} + [A \wedge \cdot]$, one can further show using the Bianchi identity that

$$\mathbf{d}^{\tilde{\nabla}} F^\nabla = \mathbf{d}F + A \wedge F^\nabla - F^\nabla \wedge A = \mathbf{0}.$$

Controls on the curvature field F^∇ then has implications on the behavior of systems modeling A, and we see Uhlenbeck [4] and Donaldson [6][5] prove local existence theorems under assumptions of scale-invariant controls on the curvature field's $L^p(\mathbb{B}^m)$ norm for $p = m/2$.

2.2 Gauge transformations

For our trivial bundle $M \times N$, where we typically take $N = F^n$ with the field $F = \mathbb{R}$ or $F = \mathbb{C}$, we have that connection matrices in general take the form $A \in \Omega^1_M(End(\eta))$ where the endomorphism bundle $End(\eta)$ has fibers $End(\eta)_x \cong \mathfrak{gl}(n, F)$ for each $x \in M$.

Given a connection $\nabla = \mathbf{d} + A$ acting on the trivial vector bundle $\eta = M \times N$, we then say that ∇ is compatible with some gauge group $\mathscr{G}_\eta \subset Aut(\eta)$ provided that $A \in \Omega^1_M(\mathfrak{G}_\eta)$. Here, the gauge group $\mathscr{G}_\eta$ has fibers $(\mathscr{G}_\eta)_x \cong G \subset GL(n, F)$, and the fibers of of the bundle $\mathfrak{G}_\eta \subset Ad(\eta)$ satisfy $(\mathfrak{G}_\eta)_x \cong \mathfrak{G} \subset \mathfrak{gl}(n, F)$ for each $x \in M$.

In practice, we often abbreviate $\mathscr{G}_\eta = G$ and $\mathfrak{G}_\eta = \mathfrak{G}$, where G must be a Lie group associated with the Lie algebra $\mathfrak{G} = \mathfrak{Lie}(G)$. Furthermore, for our interests in partial differential equations, it is often important to focus on *compact* Lie groups G. As our fibers of interest are $N = \mathbb{R}^n$ and $N = \mathbb{C}^n$, we end up focusing on the Lie groups $G \subset SO(n)$ and $G \subset SU(n)$, respectively associated with the Lie algebras $\mathfrak{G} \subset \mathfrak{so}(n)$ and $\mathfrak{G} \subset \mathfrak{su}(n)$. Hence, it is common for us to operate under the assumption that our connections of interest are taken to be represented by a connection matrices

$$A = A_j \ \mathbf{d}u^j \in \Omega^1_M(\mathfrak{G}) = C^\infty(M, \mathfrak{G} \otimes T^*M).$$

The action of the gauge group G on the covariant G-compatible connections $\mathfrak{A}$ is given by conjugation. That is, for $\sigma \in C^\infty(M, G)$,

$$\nabla \to \nabla^\sigma := \sigma^{-1} \circ \nabla \circ \sigma.$$

For $\nabla = \mathbf{d} + A$ and $\sigma = e^\chi$ for some $\chi \in C^\infty(M, \mathfrak{G})$, we have that

$$\begin{aligned}\nabla^\sigma &:= \sigma^{-1} \circ (\mathbf{d} + A) \circ \sigma \\ &= \mathbf{d} + \sigma^{-1}\mathbf{d}\sigma + \sigma^{-1}A\sigma \\ &= \mathbf{d} + \mathbf{d}\chi + e^{-\chi}Ae^\chi.\end{aligned}$$

For $B = \mathbf{d}\chi + e^{-\chi}Ae^\chi$, it holds $B \in \Omega^1_M(\mathfrak{G})$. We would then say that the connections $\nabla = \mathbf{d} + A$ and $\nabla^\sigma = \mathbf{d} + B$ are "gauge-equivalent" via the transformation given by σ.

The action of the gauge group G on the G-compatible connections $\mathfrak{A}$ then partitions the connections into classes of gauge-equivalent connections. Of particular interest becomes the study of "gauge invariance," where the structure of equations is invariant with respect to choice of gauge-equivalent connections.

Examples of models that can be written in a gauge-invariant form include Maxwell's equations and the Yang-Mills equations, as discussed by Donaldson in *Mathematical uses* *of gauge theory.*[1] The choice of connection is equivalent to choice of connection matrix A, or "potential," and the most elegant results then are independent of choice of gauge-equivalent representative potential.

Furthermore, the action of a change of gauge on the curvature field is tensorial. That is, when a change of gauge is given by $\sigma \in C^\infty(M, G)$, for a given connection $\nabla = \mathbf{d} + A$, when we consider the effect on the curvature field

$$F^\nabla = \mathbf{d}A + A \wedge A \in C^\infty\left(M, \mathfrak{G} \otimes \bigwedge^2 T^* M\right),$$

we have that the gauge equivalent connection $\nabla^\sigma = \sigma^{-1} \circ \nabla \circ \sigma$ has curvature field

$$F^{\nabla^\sigma} = F^{\sigma^{-1} \circ \nabla \circ \sigma} = \sigma^{-1}\left(F^\nabla\right)\sigma.$$

One of the reasons that the study of changes of gauge from the perspective of partial differential equations so often restricts consideration to compact gauge (Lie) groups G and compact base manifold domains M is then that every σ is then universally bounded in L^∞, and hence the L^p norms of the curvature fields for gauge equivalent connections are comparable. That is, for all $\sigma \in \mathscr{D}$,

$$\|F^{\nabla^\sigma}\|_{L^p(M)} \lesssim \|F^\nabla\|_{L^p(M)}.$$

This allows authors, including Uhlenbeck [4] and Donaldson [6][5] to use smallness assumptions on the curvature field to be retained under a change of gauge (up to a constant multiple) hence allowing estimates involving control through the curvature field to be more readily proven.

2.3 Coulomb gauges

2.3.1 Hodge $\star$ theory

Basic Hodge theory establishes a duality between forms.

We start with an inner product $(\cdot,\cdot)_g$ on M and volume form $\mathbf{vol}_g \in \Omega_M^m$ respective of g, where $(\mathbf{vol}_g, \mathbf{vol}_g) = (-1)^s$. It can be noted that a Riemannian M will have $s = 0$, and a Lorentzian M will have $s = 1$.

The L^2 inner product on forms $\phi, \psi \in \Omega_M^k$ is then given by

$$(\phi, \psi)_{L^2} := \int_M (\phi, \varphi)_g \mathbf{vol}_g.$$

Then standard derivative $\mathbf{d} : \Omega_M^k \to \Omega_M^{k+1}$ has an L^2 adjoint $\delta = \mathbf{d}^*$ satisfying

$$\delta \ : \ \Omega_M^k \to \Omega_M^{k-1},$$

where for all compactly supported scalar forms $\omega \in \Omega_M^{k-1}$ and $\lambda \in \Omega_M^k$, it holds

$$(\mathbf{d}\omega, \lambda)_{L^2} = (\omega, \delta\lambda)_{L^2}.$$

The development of the Hodge duality allows $\delta = \mathbf{d}^*$ to then be expressed in terms of $\mathbf{d}$ and the Hodge $\star = \star_{g,\mathbf{vol}_g}$. On $M \subset \mathbb{R}^m$, and $k \in \{0, 1, \cdots, m\}$, $\star$ is an isomorphism

$$\star \ : \ \Omega_M^k \to \Omega_M^{m-k}.$$

For all forms $\omega \in \Omega_M^k$ and $\lambda \in \Omega_M^{m-k}$, the Hodge $\star$ is uniquely defined by

$$\omega \wedge \lambda = (\star\omega, \lambda)_g \mathbf{vol}_g,$$

On Riemannian $M \subset \mathbb{R}^m$, where $(\mathbf{vol}_g, \mathbf{vol}_g)_g = 1$, then $\star$ satisfies properties of

i.) $\star 1 = \mathbf{vol}_g$,

ii.) when $\star\ :\ \Omega^k_M \to \Omega^{m-k}_M$, it holds $\star\star = (-1)^{k(m-k)}$,

iii.) for all $\phi, \psi \in \Omega^k_M$, it holds $\phi \wedge \star\psi = (\star\phi, \star\psi)_g \mathbf{vol}_g = (\phi, \psi)_g \mathbf{vol}_g = \psi \wedge \star\phi$,

iv.) and for all $\omega \in \Omega^k_M$ and $\lambda \in \Omega^{m-k}_M$, $(\omega \wedge \lambda, \mathbf{vol}_g)_g = (\star\omega, \lambda)_g$.

On non-Riemannian $M \subset \mathbb{R}^m$, where $(\mathbf{vol}_g, \mathbf{vol}_g)_g = (-1)^s$, then $\star$ satisfies properties of

i.) $\star 1 = (-1)^s \mathbf{vol}_g$,

ii.) when $\star\ :\ \Omega^k_M \to \Omega^{m-k}_M$, it holds $\star\star = (-1)^{k(m-k)+s}$,

iii.) for all $\phi, \psi \in \Omega^k_M$, it holds $\phi \wedge \star\psi = (\star\phi, \star\psi)_g \mathbf{vol}_g = (-1)^s (\phi, \psi)_g \mathbf{vol}_g = \psi \wedge \star\phi$,

iv.) and for all $\omega \in \Omega^k_M$ and $\lambda \in \Omega^{m-k}_M$, $(\omega \wedge \lambda, \mathbf{vol}_g)_g = (-1)^s (\star\omega, \lambda)_g$.

It can then be shown that that $\delta = \mathbf{d}^*\ :\ \Omega^k_M \to \Omega^{k-1}_M$ is given by

$$\delta = \mathbf{d}^* = (-1)^{m(k-1)-1+s} \star \mathbf{d} \star . \tag{2.3}$$

The reader may see A.1 for a verification of Hodge theory on tensors, and we merely summarize the results obtained. For the bundle $\eta = M \times N$ and canonical extensions of $\mathbf{d}$ to tensor-bundle valued forms

$$\begin{aligned} \mathbf{d}\ &: \Omega^k_M(\eta) \to \Omega^{k+1}_M(\eta), \\ \mathbf{d}\ &: \Omega^k_M(Ad(\eta)) \to \Omega^{k+1}_M(Ad(\eta)), \end{aligned}$$

the formula (2.3) still holds, noting $Ad(\eta) \subset End(\eta)$.

For a connection $\nabla = \mathbf{d} + A \in \mathfrak{A}$, with $\nabla : \Omega^0_M(\eta) \to \Omega^1_M(\eta)$ given the extensions

$$\mathbf{d}^{\nabla} \;:\; \Omega^k_M(\eta) \to \Omega^{k+1}_M(\eta),$$
$$\mathbf{d}^{\tilde{\nabla}} \;:\; \Omega^k_M(Ad(\eta)) \to \Omega^{k+1}_M(Ad(\eta)),$$

as given in (2.1) and (2.2), we can then also establish L^2 adjoints $\delta^{\nabla} = (\mathbf{d}^{\nabla})^*$ and $\delta^{\tilde{\nabla}} = (\mathbf{d}^{\tilde{\nabla}})^*$ satisfying

$$\delta^{\nabla} \;: \Omega^k_M(\eta) \to \Omega^{k-1}_M(\eta),$$
$$\delta^{\tilde{\nabla}} \;: \Omega^k_M(Ad(\eta)) \to \Omega^{k-1}_M(Ad(\eta)),$$

and furthermore show that

$$\delta^{\nabla} = (-1)^{m(k-1)-1+s} \star \mathbf{d}^{\nabla} \star,$$
$$\delta^{\tilde{\nabla}} = (-1)^{m(k-1)-1+s} \star \mathbf{d}^{\tilde{\nabla}} \star .$$

When one takes the connection ∇ to be the base connection $\nabla = \mathbf{d} + \mathbf{0}$, we recover

$$\delta \;:\; \Omega^k_M(\eta) \to \Omega^{k-1}_M(\eta),$$
$$\delta \;:\; \Omega^k_M(Ad(\eta)) \to \Omega^{k-1}_M(Ad(\eta)),$$
$$\delta = \mathbf{d}^* = (-1)^{m(k-1)-1+s} \star \mathbf{d} \star$$

with the same formulation given by basic Hodge theory on the scalar forms of

$$\delta \;:\; \Omega^k_M \to \Omega^{k-1}_M,$$
$$\delta = \mathbf{d}^* = (-1)^{m(k-1)-1+s} \star \mathbf{d} \star .$$

Furthermore, one may study the elliptic systems given by the Hodge-Laplacian

$$\Delta_H := \mathbf{d}^*\mathbf{d} + \mathbf{d}\mathbf{d}^*.$$

When applied to sections $\Gamma(\eta) = \Omega^0_M(\eta)$, we have that $\Delta_H = \Delta$, the traditional Laplacian, as $\mathbf{d}^*$ maps all 0-forms to zero. This can be understood in terms of the Hodge-duality, where $\mathbf{d}$ maps all m-forms to zero, and $\mathbf{d}^* = (-1)^{m(k-1)-1+s} \star \mathbf{d} \star$.

2.3.2 Coulomb connections

Having established the L^2 adjoints for the extensions of a connection ∇ to the vector-bundle and adjoint-bundle valued forms, we can then consider the action of such adjoints on a connection matrix $B = B_j \; \mathbf{d}u^j \in C^\infty(M, \mathfrak{G} \otimes T^*M)$.

Definition 1. *We say that a connection* $\mathbf{d} + B \in \mathfrak{A}$ *is* Coulomb *provided*

$$\mathbf{d}^* B = \mathbf{0},$$

and we say that $\mathbf{d} + B$ *is* Coulomb relative to A *for a connection* $\nabla = \mathbf{d} + A$ *provided*

$$\delta^{\tilde{\nabla}} B = \mathbf{0}.$$

Definition 2. *We say that a connection* $\nabla = \mathbf{d} + A \in \mathfrak{A}$ *has a* Coulomb gauge *provided there exists* $\sigma \in \mathscr{D}$ *such that the gauge equivalent connection* $\mathbf{d} + B := \nabla^\sigma$ *is Coulomb.*

The work of Uhlenbeck in [4] shows the local existence of Coulomb gauges, and the works of Rivieré [3],[2] use what amounts to such gauge transformations for computational purposes.

The study of Coulomb connections and gauges inspires interest, not only for physical interpretations, but as well as the computational power afforded by such results.

We are able to show that the connection $\nabla = \mathbf{d}+A$ being Coulomb may be interpreted as a divergence-free condition for the $1-$form $A = A_j\ \mathbf{d}w^j$. That is, we show in A.2 that

$$\mathbf{d}^*A = \partial_j A_j,$$

and hence a connection $\mathbf{d} + A$ being Coulomb with $\mathbf{d}^*A = \mathbf{0}$ is analogous to a scalar valued function f satisfying $div(f) = 0$.

Furthermore, applications of connections in physics often interpret the connection matrix A as a potential, lending to a physical interpretation of Uhlenbeck's results as establishing the local existence of a change of coordinate system, that is a change of gauge, such that under the new coordinate system, the potential A is divergence free.

The computational advantages in integration afforded by divergence-free scalar maps then have analogous advantages for Coulomb connection matrices. Though the advantages in local existence of divergence-free objects within the study of partial differential equations is recognized, we also seek applications of gauge theory that may afford other such computational advantages. This led us to pursue Poincaré inequalities relative to gauge transformations.

Chapter 3

Poincaré Inequalities Involving Gauges

3.1 Main Results

Recognizing the potential for applications in analysis of changes of gauge, as evidenced in multiple works, such as those by Uhlenbeck [4], Donaldson [6][5], Rivieré [3][2], and reviews of such material as by Lamm [12], we then sought additional applications of changes of gauge to analytical arguments.

The works with Coulomb gauges given by Uhlenbeck and Donaldson heavily used boundary value problem formulations, with Dirichlet and Neumann-like conditions being imposed to establish existence of special gauges. Our efforts then settled on seeking Poincaré inequalities under a change of gauge analogous to the classical Poincaré inequalities on bounded convex domains $M \subset \mathbb{R}^m$. One may reference in Gilbarg and Trudinger's inequality remark (7.45) [13] or Evans' Theorem 1 in §5.8 [14] for statements of such classical results, where for $f \in L^p_1$ and $p \geq 1$,

$$\|f - \overline{f}\|_{L^p} \lesssim \|\mathbf{D}f\|_{L^p}.$$

We additionally sought results analogous to the classical Poincaré-Sobolev inequalities for $1 \leq p < m = dim(M)$ and $1 \leq q \leq p^*$, taking the form

$$\|f - \overline{f}\|_{L^q} \lesssim \|\mathbf{D}f\|_{L^p}.$$

A statement of this classical result may be found in Lieb and Loss' Theorems 8.11-8.12 [15].

Towards establishing analogous Poincaré inequalities for connection matrices, we sought conditions under which a change of gauge given by some σ exists that establishes gauge equivalence between connections $\nabla = \mathbf{d} + A$ and $\nabla^\sigma = \mathbf{d} + A - B$, where B is a constant connection matrix (1-form). With a change of gauge given by the equation

$$\nabla^\sigma := \sigma^{-1} \circ \nabla \circ \sigma = \mathbf{d} + \sigma^{-1}\mathbf{d}\sigma + \sigma^{-1}A\sigma,$$

solving for such a gauge amounts to solving for σ in the equation

$$A - B = \sigma^{-1}\mathbf{d}\sigma + \sigma^{-1}A\sigma.$$

We are able to then show two main conditions under which what we refer to as "translation gauges" exist.

The existence of translation gauges then allows us to establish Poincaré inequalities for connections $\nabla = \mathbf{d} + A$ of the form

$$\|\sigma^{-1}\mathbf{d}\sigma + \sigma^{-1}A\sigma\|_{L^p} \lesssim \|\mathbf{D}A\|_{L^p},$$

and for $1 \le q \le p^*$ when $1 \le p < m$, Poincaré-Sobolev inequalities of the form

$$\|\sigma^{-1}\mathbf{d}\sigma + \sigma^{-1}A\sigma\|_{L^q} \lesssim \|\mathbf{D}A\|_{L^p}$$

where a change of gauge given by $\sigma \in \mathscr{D}$ solves

$$\mathbf{d} + A - \overline{A} = \sigma^{-1} \circ \nabla \circ \sigma,$$
$$A - \overline{A} = \sigma^{-1}\mathbf{d}\sigma + \sigma^{-1}A\sigma.$$

The first condition under which a translation gauge exists is proved as Theorem 9 in section 3.2.1, which is stated for reference here as Theorem 1.

Theorem 1. *Let domain $M \subset \mathbb{R}^1$ be compact and connected. Let $N = \mathbb{R}^n$ (or $N = \mathbb{C}^n$) and compact Lie group $G \subset SO(n)$ (respectively $G \subset SU(n)$) act on the fibers of trivial vector bundle $\eta = M \times N$. Denote Lie algebra $\mathfrak{Lie}(G) = \mathfrak{G}$, noting $\mathfrak{G} \subset \mathfrak{so}(n)$ (respectively $\mathfrak{G} \subset \mathfrak{su}(n)$).*

For any connection matrix $A = A_1 \ \mathbf{d}u^1 \in C^\infty(M, \mathfrak{G} \otimes T^*M)$, and any constant connection matrix $B = B_1 \ \mathbf{d}u^1$ with $B_1 \in \mathfrak{G}$, there exists a change of gauge from the connection $\nabla = \mathbf{d} + A \in \mathfrak{A}$ to the connection $\mathbf{d} + A - B \in \mathfrak{A}$ given by some $\sigma \in \mathscr{D}$. That *is,*

$$\begin{aligned}\nabla^{\sigma} &:= \sigma^{-1} \circ A \circ \sigma \\ &= \mathbf{d} + \sigma^{-1}\mathbf{d}\sigma + \sigma^{-1}A\sigma \\ &= \mathbf{d} + A - B.\end{aligned}$$

The second condition under which a translation gauge exists is proved in Theorem 12 in section 3.2.4, which is stated for reference here as Theorem 2.

Theorem 2. *Let domain $M \subset \mathbb{R}^m$ be compact and connected. Let $N = \mathbb{R}^n$ (or $N = \mathbb{C}^n$) and compact Lie group $G \subset SO(n)$ (respectively $G \subset SU(n)$) act on the fibers of trivial vector bundle $\eta = M \times N$. Denote Lie algebra $\mathfrak{Lie}(G) = \mathfrak{G}$, noting $\mathfrak{G} \subset \mathfrak{so}(n)$ (respectively $\mathfrak{G} \subset \mathfrak{su}(n)$).*

Assume the connected compact Lie group G is also abelian.

Then for any connection matrix $A = A_\alpha\, \mathbf{d}u^\alpha \in C^\infty(M, \mathfrak{G} \otimes T^*M)$, and any constant connection matrix $B = B_\alpha\, \mathbf{d}u^\alpha$ with each $B_\alpha \in \mathfrak{G}$, there exists a change of gauge from the connection $\nabla = \mathbf{d} + A \in \mathfrak{A}$ to the connection $\mathbf{d} + A - B \in \mathfrak{A}$ given by some $\sigma = e^\chi \in \mathscr{D}$ with $\chi \in C^\infty(M, \mathfrak{G})$. That is,

$$\begin{aligned}\mathbf{d} + A - B = \nabla^{\sigma} &= \sigma^{-1} \circ \nabla \circ \sigma \\ &= \nabla^{e^\chi} = e^{-\chi} \circ \nabla \circ e^\chi.\end{aligned}$$

The requirement that Lie groups of consideration must be abelian is to fulfill a technical requirement for our constructive proof. For our approach to work, we require that the adjoint action of the Lie group on the Lie algebra is the identity. That is, we require for all $\varphi \in G$ and $\mathfrak{g} \in \mathfrak{G}$ that $\varphi^{-1}\mathfrak{g}\varphi = \mathfrak{g}$. In turn, this identity allows important structures in the equations governing the gauge transformation to simplify, avoiding potential nonlinear coupling.

For purposes of the analysis of partial differential equations, we also restrict the vast majority of this work to consideration of connected *compact* Lie groups. Then it is enlightening to consider the implications of the classification of all connected compact abelian Lie groups. Bröcker and Dieck [16] present such classification up to isomorphism; we discuss this in section 3.2.5.

We note that the existence of translation gauges more generally fails. The translation gauge equation we consider represents an *overdetermined* system of partial differential equations when applied on general vector bundles $M \times N$ with multidimensional base manifold M. This contrasts the works by Uhlenbeck [4] and Donaldson [6][5] on Coulomb gauges, for which the Coulomb gauge equations collapse down to be representative of the same number of equations as unknowns.

To then show that the compatibility conditions we propose in Theorems 9 and 12 are reasonable, we contrive data A that results in the translation gauge's governing overdetermined system having no solution, and present this as Theorem 14 in section 3.2.7, which is stated for reference here as Theorem 3.

Theorem 3. *Let connected compact domain $M \subset \mathbb{R}^m$ with $m \geq 2$. Let M be the base manifold for trivial vector bundle $\eta := M \times N$, where $N \cong \mathbb{R}^n$ with $n \geq 3$. Let non-abelian compact Lie group G act on the fibers of η, and have Lie algebra $\mathfrak{Lie}(G) = \mathfrak{G}$.*

There exist connections $\nabla = \mathbf{d} + A \in \mathfrak{A}$ such that for every nonzero constant *connection matrix $B = B_\alpha \, \mathbf{d}u^\alpha$ with each $B_\alpha \in \mathfrak{G}$, the translation gauge equation given by*

$$\begin{aligned} \mathbf{d} + A - B &= \sigma^{-1} \circ \nabla \circ \sigma \\ &= \mathbf{d} + \mathbf{d}\chi + e^{-\chi} A e^{\chi} \end{aligned}$$

yields no solutions $\sigma = e^{\chi} \in \mathscr{D}$ *with* $\chi \in \mathbb{C}^{\infty}(M, \mathfrak{G})$.

Having an understanding of the conditions under which a translation gauge is possible, we then use these results to establish our analogous Poincaré inequalities in Theorem 15 in section 3.3.1, which is stated for reference here as Theorem 4.

Theorem 4. *Let connected domain* $M \subset \mathbb{R}^m$ *be compact. Let* $N = \mathbb{R}^n$ *(or* $N = \mathbb{C}^n$*) and compact Lie group* $G \subset SO(n)$ *(respectively* $G \subset SU(n)$*) act on the fibers of trivial vector bundle* $\eta = M \times N$. *Denote Lie algebra* $\mathfrak{Lie}(G) = \mathfrak{G}$, *noting* $\mathfrak{G} \subset \mathfrak{so}(n)$ *(respectively* $\mathfrak{G} \subset \mathfrak{su}(n)$*).*

Let $\nabla = \mathbf{d} + A \in \mathfrak{A}$ *be any finitely integrable connection, and assume at least one of the following holds.*

$I.)$ *The base manifold domain* $M \subset \mathbb{R}^1$.

$II.)$ *The connected compact Lie group* G *is also abelian.*

Let $p \geq 1$ *and convex subdomain* $M' \subset M$. *There exists some constant* $C = C(M', N, p)$ *such that for the connection matrix* $\tilde{A} = A - \overline{A}$, *there exists a change of gauge given by some* $\sigma \in \mathscr{D}$ *between connections* ∇ *and* $\nabla^{\sigma} = \sigma^{-1} \circ \nabla \circ \sigma = \mathbf{d} + \tilde{A}$ *satisfying*

$$\tilde{A} = \sigma^{-1}\mathbf{d}\sigma + \sigma^{-1}A\sigma,$$

$$\|\tilde{A}\|_{L^p(M')} \leq C\|\mathbf{D}A\|_{L^p(M')}.$$

Furthermore, we have Poincaré-Sobolev inequalities available under our translation gauge results, proved as Theorem 17 in section 3.3.1, which is stated for reference here as Theorem 5.

Theorem 5. *Let domain* $M \subset \mathbb{R}^m, N, \eta = M \times N,\ G,$ *and* $\mathfrak{G}$ *be as in Theorem 4.*

Assume $m \geq 2$ and the connected compact Lie group G is also abelian.

Let $1 \leq p < m$, $1 \leq q \leq p^* := \frac{mp}{m-p}$, and convex subdomain $M' \subset M$.

There exists $C = C(M', N, p, q) \in (0, \infty)$ such that for any smooth Sobolev connection $\nabla = \mathbf{d} + A \in \mathfrak{A}_1^p$, there exists a change of gauge given by some $\sigma \in \mathscr{D}$ between connections ∇ and $\nabla^\sigma = \mathbf{d} + \tilde{A}$ satisfying

$$\tilde{A} = \sigma^{-1}\mathbf{d}\sigma + \sigma^{-1}A\sigma,$$
$$\|\tilde{A}\|_{L^q(M')} \leq C \, \|\mathbf{D}A\|_{L^p(M')} \, .$$

For our definition of Sobolev connections $\mathfrak{A}_k^p$, you may refer to Definition 3 in section 3.3.1.

We may further consider these results under assumptions complementary to the study of Coulomb gauges due to the ellipticity afforded in such systems. For a more thorough discussion of Coulomb systems' ellipticity, one may refer to A.3.

Briefly motivated, one may consider the Hodge-Laplacian $\Delta_H := \mathbf{d}^*\mathbf{d} + \mathbf{d}\mathbf{d}^*$ acting on Coulomb A, which results in similar elliptic estimates as those afforded by a Poisson equation of the form $\Delta f = div(g)$.

Hence, as a corollary to Theorem 15 in section 3.3.1, we obtain Corollary 18 in section 3.3.2, which is stated for reference here as Corollary 6.

Corollary 6. *Let connected compact convex domain $M \subset \mathbb{R}^m, N, \eta = M \times N,\ G,$ and $\mathfrak{G}$ be as in Theorem 4.*

Let compactly supported Coulomb connection $\nabla = \mathbf{d} + A \in \mathfrak{A}$, where $supp(A) \subset\subset M$, *and assume at least one of the following holds.*

I.) The base manifold domain $M \subset \mathbb{R}^1$.

II.) The connected compact Lie group G is also abelian.

Let $p \geq 1$. There exists some constant $C = C(M, N, p)$ such that for the connection matrix $B = A - \overline{A}$, there exists a change of gauge given by some $\sigma \in \mathscr{D}$ such that $\mathbf{d} + \tilde{A} = \nabla^\sigma = \sigma^{-1} \circ \nabla \circ \sigma$ satisfying

$$\tilde{A} = \sigma^{-1}\mathbf{d}\sigma + \sigma^{-1}A\sigma,$$

$$\|\tilde{A}\|_{L^p(M)} \leq C\|\mathbf{d}A\|_{L^p(M)}.$$

Finally, as a corollary to Theorem 17 in section 3.3.1, we obtain Corollary 19 in section 3.3.2, which is stated for reference here as Corollary 7.

Corollary 7. *Let connected compact convex domain $M \subset \mathbb{R}^m, N, \eta = M \times N,\ G$, and $\mathfrak{G}$ be as in Theorem 5.*

Assume $m \geq 2$ and the Lie group G is abelian, connected, and compact.

Let $1 \leq p < m$, and $1 \leq q \leq p^* := \frac{mp}{m-p}$.

There exists $C = C(M, N, p, q) \in (0, \infty)$ such that for any smooth compactly supported Sobolev connection $\nabla = \mathbf{d} + A \in \mathfrak{A}_1^p$, where supp$(A) \subset\subset M$, there exists a change of gauge given by some $\sigma \in \mathscr{D}$ between connections ∇ and $\nabla^\sigma = \mathbf{d} + \tilde{A}$ satisfying

$$\tilde{A} = \sigma^{-1}\mathbf{d}\sigma + \sigma^{-1}A\sigma,$$

$$\|\tilde{A}\|_{L^q(M)} \leq C\,\|\mathbf{d}A\|_{L^p(M)}.$$

In studying local divergence-free Coulomb gauges, we also considered a 2000 joint paper by Gustavo Ponce, T. Kato, M. Mitrea, and M. Taylor [17] on extensions of divergence-free fields, though ultimately no conclusions were incorporated into this work.

Beyond the results in this dissertation, due to the identity

$$\mathbf{d}A = F^{\nabla} - A \wedge A,$$

additional assumptions on the norms of the curvature field F^{∇} and potential A allows one to use Hölder inequalities on the quadratic term $A \wedge A$ to further manipulate inequalities controlled by $\mathbf{d}A$. This style of using Sobolev norm control of the curvature field F^{∇} and quadratic structure $A \wedge A$ is common in the works of Uhlenbeck and Donaldson, where Sobolev control of the quadratic structure is attributed to Palais [18] (Chapter 9); we leave this as a remark only, as ultimately such techniques were not incorporated into this dissertation.

3.2 Translation gauges on bundles $M \times N$

3.2.1 Base manifold domain $M \subset \mathbb{R}^1$

For the sake of thoroughness, we consider 1-dimensional connected compact base manifold domains M (i.e. intervals), and trivial bundles $\eta = M \times N$. In the following, we see that the existence of translation gauges in this simple setting amounts to a result in ordinary differential equations on manifolds.

We refer to a classical result regarding the existence of global flow on compact manifolds. One presentation of this classical result is given by Lee [19], which we reword here as Lemma 8 for reference.

Lemma 8. *On a compact smooth manifold G, every smooth vector field v is complete.*

That is, v generates a global flow.

Proof. We refer to Lee's *Smooth Manifolds* [19], Corollary 9.17, page 216. □

When considering compactly supported vector fields, one may alternatively refer to Arnold's presentation of this classical result in *Ordinary Differential Equations*[9], §35 - The Phase Flow Determined by a Vector Field. In Arnold's approach, remark 35.4 on page 252 addresses extension of previous non-autonomous results by identifying $M \times G$ as a subset of a projective space $\mathbb{RP}^k$. We mention this, but choose to apply Lee's corollary instead.

Using Lemma 8, we are able to prove the existence of translation gauges on trivial bundles $M \times N$ with 1-dimensional base manifolds, and establish the following Theorem 9.

Theorem 9. *Let domain $M \subset \mathbb{R}^1$ be compact and connected. Let $N = \mathbb{R}^n$ (or $N = \mathbb{C}^n$) and compact Lie group $G \subset SO(n)$ (respectively $G \subset SU(n)$) act on the fibers of trivial vector bundle $\eta = M \times N$. Denote Lie algebra $\mathfrak{Lie}(G) = \mathfrak{G}$, noting $\mathfrak{G} \subset \mathfrak{so}(n)$ (respectively $\mathfrak{G} \subset \mathfrak{su}(n)$).*

For any connection matrix $A = A_1 \ \mathbf{d}u^1 \in C^\infty(M, \mathfrak{G} \otimes T^*M)$, and any constant connection matrix $B = B_1 \ \mathbf{d}u^1$ with $B_1 \in \mathfrak{G}$, there exists a change of gauge from the connection $\nabla = \mathbf{d} + A \in \mathfrak{A}$ to the connection $\mathbf{d} + A - B \in \mathfrak{A}$ given by some $\sigma \in \mathscr{D}$. That is,

$$\begin{aligned}\nabla^\sigma &:= \sigma^{-1} \circ A \circ \sigma \\ &= \mathbf{d} + \sigma^{-1}\mathbf{d}\sigma + \sigma^{-1}A\sigma \\ &= \mathbf{d} + A - B. \end{aligned} \tag{3.1}$$

Proof. Since domain $M \subset \mathbb{R}^1$, then for any section $s \in \Gamma(\eta)$, $\mathbf{d}s = \partial_1 s \ \mathbf{d}u^1 = s' \ \mathbf{d}u^1$, the equation (3.1) is equivalent to

$$(A_1 - B_1)\ \mathbf{d}u^1 = \sigma^{-1}\partial_1\sigma\ \mathbf{d}u^1 + \sigma^{-1}(A_1\ \mathbf{d}u^1)\sigma.$$

By isolating the coefficients of the 1-form $\mathbf{d}u^1$, and recalling all objects are maps on the base 1-dimensional manifold domain M, by notating $\partial_1\sigma(u) = \sigma'(u)$, this yields the (non-autonomous) ordinary differential equation,

$$\begin{aligned}\sigma'(u) &= A_1(u)\sigma(u) - \sigma(u)A_1(u) - \sigma(u)B_1\\ &= v(\sigma(u), u).\end{aligned}$$

Given smoothness of the data, for $u \in M$ and $\sigma(u) \in G$, $v(\sigma, u) := [A_1(u), \sigma] - B_1\sigma$ is a smooth non-autonomous vector field on the smooth compact manifold G taking values in $\mathfrak{G}$. The gauge group $\mathscr{G}$ is then also smooth and compact, and the classical technique of reducing non-autonomous ordinary differential equations to the autonomous case applies, and and thus by Lemma 8, there exists a global flow σ for v with $\sigma(u) \in G$ for all $u \in M$. Then σ solves (3.1). □

Corollary 10. *Let domain $M \subset \mathbb{R}^1, N, \eta = M \times N,\ G,$ and $\mathfrak{G}$ be as in Theorem 9. For any finitely integrable connection matrix $A = A_1\ \mathbf{d}u^1 \in C^\infty(M, \mathfrak{G} \otimes T^*M),$ define*

$$\overline{A} := \overline{A}_1\ \mathbf{d}u^1 := \left(\frac{1}{|M|}\int_M A(x)\ \mathbf{d}x\right)\ \mathbf{d}u^1.$$

Then there exists a change of gauge given by some $\sigma \in \mathscr{D}$ taking $\nabla = \mathbf{d} + A \in \mathfrak{A}$ to

$$\begin{aligned}\nabla^\sigma &:= \sigma^{-1} \circ A \circ \sigma \\ &= \mathbf{d} + \sigma^{-1}\mathbf{d}\sigma + \sigma^{-1}A\sigma \\ &= \mathbf{d} + A - \overline{A}.\end{aligned}$$

Proof. We set $B = \overline{A} \in \mathfrak{G}$, and apply Theorem 9. □

Having established the existence of translation gauges over compact base manifold domains $M \subset \mathbb{R}^1$, we are then able to add this class of trivial vector bundles $\eta = M \times N$ to those that a Poincaré style inequality may hold under a proper change of gauge.

3.2.2 Example: fiber $N = \mathbb{R}^2$

Having established the existence of translation gauges when the base manifold domain is 1-dimensional, we then move on to consider when the base manifold domain $M \subset \mathbb{R}^m$ where we intend $m \geq 2$. To help motivate some of the results in this context, we consider the following simple example with fibers isomorphic to $\mathbb{R}^2$, and computationally establish a translation gauge from $\nabla = \mathbf{d} + A$ to $\nabla^\sigma = \mathbf{d} + A - \overline{A}$, which represents the technique for more general translation gauges on the bundle $\eta = M \times \mathbb{R}^2$.

Let compact domain $M \subset \mathbb{R}^m$ be the base manifold for trivial vector bundle $\eta := M \times \mathbb{R}^2 \cong M \times \mathbb{C}$, with Lie group $G = SO(2) \cong \mathbb{S}^1$ and Lie algebra $\mathfrak{G} = \mathfrak{so}(2) \cong \imath\mathbb{R}$. We then take adjoint bundle $Ad(\eta)$ and gauge group $\mathscr{G} \subset Aut(\eta)$ to have fibers $Ad(\eta)_x \cong \mathfrak{so}(2)$ and $\mathscr{G}_x \cong SO(2)$ for $x \in M$.

For finitely integrable connection matrix A and connection $\nabla = \mathbf{d} + A$, represent $A \in \mathbb{C}^\infty(M, \mathfrak{so}(2) \otimes T^*M)$ by $A = A_\alpha \ \mathbf{d}u^\alpha$, and define $\overline{A}, \overline{A}_\alpha, \theta_\alpha, \overline{\theta}_\alpha$ as follows:

$$A_\alpha = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \theta_\alpha, \qquad \overline{\theta}_\alpha = \frac{1}{|M|} \int\limits_M \theta_\alpha(x) \mathbf{d}x \in \mathbb{R},$$

$$\overline{A}_\alpha = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \overline{\theta}_\alpha = \frac{1}{|M|} \int\limits_M A_\alpha(x) \mathbf{d}x \in \mathfrak{so}(2), \qquad \overline{A} = \overline{A}_\alpha \ \mathbf{d}u^\alpha.$$

We then are able computationally prove gauge equivalence of connections $\mathbf{d} + A$ and $\mathbf{d} + (A - \overline{A})$.

We solve for $\mathfrak{G}$-valued map $\mathcal{Y} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} y \in \mathbb{C}^\infty(M, \mathfrak{so}(2))$ such that for G-valued map $\sigma = e^{\mathcal{Y}} = \begin{bmatrix} \cos(y) & -\sin(y) \\ \sin(y) & \cos(y) \end{bmatrix} \in C^\infty(M, SO(2))$, the gauge equivalence equation

$$\mathbf{d} + (A - \overline{A}) = \sigma^{-1} \circ (\mathbf{d} + A) \circ \sigma$$

holds. Equivalently, we may compute

$$\begin{aligned} A - \overline{A} &= \sigma^{-1} \mathbf{d}\sigma + \sigma^{-1} A \sigma \\ &= e^{-\mathcal{Y}} \mathbf{d} e^{\mathcal{Y}} + e^{-\mathcal{Y}} A e^{\mathcal{Y}} \\ &= e^{-\mathcal{Y}} e^{\mathcal{Y}} \mathbf{d}\mathcal{Y} + \begin{bmatrix} \cos(y) & \sin(y) \\ -\sin(y) & \cos(y) \end{bmatrix} \left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \theta_\alpha \ \mathbf{d}u^\alpha \right) \begin{bmatrix} \cos(y) & -\sin(y) \\ \sin(y) & \cos(y) \end{bmatrix} \\ &= \mathbf{d} \left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} y \right) + \begin{bmatrix} \cos(y) & \sin(y) \\ -\sin(y) & \cos(y) \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \cos(y) & -\sin(y) \\ \sin(y) & \cos(y) \end{bmatrix} \theta_\alpha \ \mathbf{d}u^\alpha. \end{aligned}$$

Then, due to the simplification of $\sigma^{-1}(u)A_\alpha(u)\sigma(u) = A_\alpha(u)$ for $u \in M$, $\sigma(u) \in SO(2)$, and $A_\alpha(u) \in \mathfrak{so}(2)$, we may calculate

$$\begin{aligned} A - \overline{A} &= \mathbf{d}\left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} y\right) + \sigma^{-1}A\sigma \\ &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \partial_\alpha y \ \mathbf{d}u^\alpha + \begin{bmatrix} \cos(y) & \sin(y) \\ -\sin(y) & \cos(y) \end{bmatrix} \begin{bmatrix} -\sin(y) & -\cos(y) \\ \cos(y) & -\sin(y) \end{bmatrix} \theta_\alpha \ \mathbf{d}u^\alpha \\ &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \partial_\alpha y \ \mathbf{d}u^\alpha + \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \theta_\alpha \ \mathbf{d}u^\alpha. \end{aligned}$$

Note that

$$-\overline{A} + A = -\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \overline{\theta}_\alpha \ \mathbf{d}u^\alpha + \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \theta_\alpha \ \mathbf{d}u^\alpha,$$

which leads to the equivalent system indexed by $\alpha \in \{1, \cdots, m\}$ of

$$\partial_\alpha y := -\overline{\theta}_\alpha. \tag{3.2}$$

Taking $\vec{\theta} := \left(\overline{\theta}_\alpha\right)_{\alpha=1}^m$ and coordinate representation $u = (u^\alpha)_{\alpha=1}^m \in M$, then (3.2) is solved by

$$\begin{aligned} y(u) &:= -\vec{\theta} \cdot u + C \\ &= -\overline{\theta}_\alpha u^\alpha + C \end{aligned}$$

where $C \in \mathbb{R}$ may be arbitrarily chosen. This then yields a family of gauge transformations $\sigma(u)$ with

$$\sigma(u) := e^{\mathcal{Y}(u)} = \begin{bmatrix} \cos(y(u)) & -\sin(y(u)) \\ \sin(y(u)) & \cos(y(u)) \end{bmatrix} = \begin{bmatrix} \cos(-\vec{\theta}\cdot u + C) & -\sin(-\vec{\theta}\cdot u + C) \\ \sin(-\vec{\theta}\cdot u + C) & \cos(-\vec{\theta}\cdot u + C) \end{bmatrix}$$

satisfying

$$\mathbf{d} + (A - \overline{A}) = \sigma^{-1} \circ (\mathbf{d} + A) \circ \sigma.$$

We note that this computation is dependent on the extreme simplifications afforded from the conjugation by $\sigma \in SO(2)$ acting on $\mathfrak{so}(2)$ being the identity map.

Furthermore, the gauge equivalence $\mathbf{d}+A \sim \mathbf{d}+(A-B)$ is achievable for any constant connection matrix $B \in \mathfrak{so}(2) \otimes T^*M$, and the choice of $B = \overline{A}$ is not required.

3.2.3 Example: fiber $N = \mathbb{C}$

In the previous section, we computationally show the existence of a representative translation gauge for connections acting on a bundle with fibers isomorphic to $\mathbb{R}^2$. With the fiber $\mathbb{C}$ being used in the description of electromagnetism, we will briefly address this case explicitly. However, due to the standard isomorphism $M \times \mathbb{R}^2 \cong M \times \mathbb{C}$, we will see that this example follows quite similarly to the previous example. As such, we will abbreviate the discussion heavily, and adjust our notation to more closely follow approaches in physics, such as discussed by Donaldson in *Mathematical uses of gauge theory.* [1].

Let compact domain $M \subset \mathbb{R}^m$ be the base manifold for trivial vector bundle $\eta := M \times \mathbb{C}$, with Lie group $G = U(1) \cong \mathbb{S}^1$ and Lie Algebra $\mathfrak{G} = \mathfrak{u}(1) \cong \imath\mathbb{R}$. We then take adjoint bundle $Ad(\eta)$ and gauge group $\mathscr{G} \subset Aut(\eta)$ to have fibers $Ad(\eta)_x \cong \mathfrak{u}(1)$ and

$\mathscr{G}_x \cong U(1)$ for $x \in M$.

Given a finitely integrable potential

$$A = \hat{\imath}a = \hat{\imath}a_\alpha \ \mathbf{d}u^\alpha \in \mathbb{C}^\infty(M, \mathfrak{u}(1) \otimes T^*M)$$

where each a_α is an integrable $\mathbb{R}$-valued map, consider the connection $\nabla = \mathbf{d} + \hat{\imath}a$. Define $\overline{a}_\alpha$ and constant potential $\widehat{\imath a}$ by

$$\overline{a}_\alpha := \frac{1}{|M|} \int_M a_\alpha(x) \mathbf{d}x \in \mathbb{R}, \qquad \widehat{\imath a} := \widehat{\imath a}_\alpha \ \mathbf{d}u^\alpha \in \mathfrak{u}(1) \otimes T^*M.$$

We then are able computationally prove gauge equivalence of the connections $\mathbf{d} + \hat{\imath}a$ and $\mathbf{d} + \hat{\imath}(a - \overline{a})$ by solving for $\mathbb{R}$-valued map y such that the $U(1)$-valued map $\sigma = e^{\hat{\imath}y}$ satisfies the gauge equivalence equation

$$\mathbf{d} + \hat{\imath}(a - \overline{a}) = \sigma^{-1} \circ (\mathbf{d} + \hat{\imath}a) \circ \sigma. \tag{3.3}$$

As $\mathbb{C}$ is abelian, the gauge transformation given by $\sigma = e^{\hat{\imath}y}$ simplifies to

$$\sigma^{-1} \circ (\mathbf{d} + \hat{\imath}a) \circ \sigma = \hat{\imath}\mathbf{d}y + \hat{\imath}a,$$

resulting in the gauge equivalence equation (3.3) being equivalent to the real system indexed by $\alpha \in \{1, \cdots, m\}$ of

$$\partial_\alpha y = -\overline{a}_\alpha. \tag{3.4}$$

Using coordinates $u = (u^\alpha)_{\alpha=1}^m \in M$, this system in turn has the obvious family of solutions for arbitrary $C \in \mathbb{R}$ of

$$y(u) := -\overline{a}_\alpha u^\alpha + C,$$

leading to the family of gauge transformations

$$\sigma(u) := e^{\hat{\imath}y(u)} = \cos(-\overline{a}_\alpha u^\alpha + C) + \hat{\imath}\sin(-\overline{a}_\alpha u^\alpha + C)$$

that solve the gauge equivalence equation (3.3).

As in the previous example, the choice of constant potential $\hat{\imath}\overline{a} \in U(1) \otimes T^*M$ held no special value, and any constant potential has a similarly computed translation gauge.

3.2.4 Abelian Lie groups G acting on fibers N

For a base manifold domain $M \subset \mathbb{R}^m$ with $m \geq 2$, the system of partial differential equations governing the existence of a translation gauge on a general bundle $\eta = M \times N$ proves to be in general an overdetermined system. However, when the adjoint action of the gauge group on the Lie algebra $\mathfrak{G}$, given by conjugation by objects in the Lie group G, proves to be the identity map, the otherwise overdetermined system collapses into a well posed system.

It is sufficient that $\mathfrak{G}$ be a 1-dimensional vector space over $\mathbb{R}$ for this to hold, as seen in the case of $\mathfrak{G} = \mathfrak{so}(2)$ with fiber $\mathbb{R}^2$, as well as $\mathfrak{G} = \mathfrak{u}(1)$ with fiber $\mathbb{C}$. More generally, it is sufficient that G be abelian, or the Lie bracket on $\mathfrak{G}$ be trivially $\mathbf{0}$, in which case the exponential map satisfies $e^{-\chi}\mathfrak{g}e^{\chi} = \mathfrak{g}$ for all $\mathfrak{g}, \chi \in \mathfrak{G}$, a key identity in precluding data A which causes the translation gauge equation to yield no solution due to nonlinearity and coupling. We chose not present the following in terms of the Lie bracket, though a reader wishing to do so may reference *Naive Lie Theory*, by Stillwell [20].

Lemma 11. *For connected Lie group G with Lie algebra $\mathfrak{G} = \mathfrak{Lie}(G)$, the conjugation by $\varphi \in G$ on $\mathfrak{G}$ acts universally as the identity map on $\mathfrak{G}$ if and only if G is abelian.*

Proof. Let $\mathfrak{g} \in \mathfrak{G}$, so the parametrized path $e^{t\mathfrak{g}} \subset G$.

If G is abelian, for arbitrary $\varphi \in G$ and $\mathfrak{g} \in \mathfrak{G}$, we then have that

$$\mathfrak{g} = \left.\frac{\mathbf{d}}{\mathbf{d}t}\right|_{t=0} e^{t\mathfrak{g}} = \left.\frac{\mathbf{d}}{\mathbf{d}t}\right|_{t=0} \varphi^{-1} e^{t\mathfrak{g}} \varphi = \varphi^{-1} \left(\left.\frac{\mathbf{d}}{\mathbf{d}t}\right|_{t=0} e^{t\mathfrak{g}} \right) \varphi = \varphi^{-1} \mathfrak{g} \varphi,$$

hence the action on $\mathfrak{G}$ by conjugation from any $\varphi \in G$ is the identity on $\mathfrak{G}$.

If the action on $\mathfrak{G}$ by conjugation from any $\varphi \in G$ is the identity on $\mathfrak{G}$, then for arbitrary $\mathfrak{g} \in \mathfrak{G}$ and $\psi = e^{\mathfrak{g}} \in G$,

$$\psi = e^{\mathfrak{g}} = e^{\varphi^{-1}\mathfrak{g}\varphi} = \varphi^{-1} e^{\mathfrak{g}} \varphi = \varphi^{-1} \psi \varphi,$$

hence $\psi\varphi = \varphi\psi$ for arbitrary $\psi = e^{\mathfrak{g}}, \varphi \in G$, and thus G must be abelian. □

The well known result from the previous lemma then allows the assumption of abelian Lie group G to yield a vital computational step in the following theorem.

Theorem 12. *Let domain $M \subset \mathbb{R}^m$ be compact and connected. Let $N = \mathbb{R}^n$ (or $N = \mathbb{C}^n$) and compact Lie group $G \subset SO(n)$ (respectively $G \subset SU(n)$) act on the fibers of trivial vector bundle $\eta = M \times N$. Denote Lie algebra $\mathfrak{Lie}(G) = \mathfrak{G}$, noting $\mathfrak{G} \subset \mathfrak{so}(n)$ (respectively $\mathfrak{G} \subset \mathfrak{su}(n)$).*

Assume the connected compact Lie group G is also abelian.

Then for any connection matrix $A = A_\alpha \, \mathbf{d}u^\alpha \in C^\infty(M, \mathfrak{G} \otimes T^*M)$, and any constant connection matrix $B = B_\alpha \, \mathbf{d}u^\alpha$ with each $B_\alpha \in \mathfrak{G}$, there exists a change of gauge from the connection $\nabla = \mathbf{d} + A \in \mathfrak{A}$ to the connection $\mathbf{d} + A - B \in \mathfrak{A}$ given by some $\sigma = e^{\chi} \in \mathscr{D}$

with $\chi \in C^\infty(M, \mathfrak{G})$. *That is,*

$$\begin{aligned} \mathbf{d} + A - B = \nabla^\sigma &= \sigma^{-1} \circ \nabla \circ \sigma \\ &= \nabla^{e^\chi} = e^{-\chi} \circ \nabla \circ e^\chi. \end{aligned} \tag{3.5}$$

Proof. By the assumption that G is abelian, we may apply Lemma 11. Hence, we have that for all $\sigma = e^\chi \in G$ and $\mathfrak{g} \in \mathfrak{G}$, it holds $\sigma^{-1}\mathfrak{g}\sigma = e^{-\chi}\mathfrak{g}e^\chi = \mathfrak{g}$.

Hence, we have that (3.5) is equivalent to

$$\begin{aligned} \mathbf{d} + (A_\alpha - B_\alpha)\ \mathbf{d}u^\alpha &= \mathbf{d} + e^{-\chi}\mathbf{d}e^\chi + e^{-\chi}A_\alpha e^\chi\ \mathbf{d}u^\alpha \\ &= \mathbf{d} + \mathbf{d}\chi + A_\alpha\ \mathbf{d}u^\alpha \end{aligned}$$

or simply the system indexed by $\alpha \in \{1, \cdots, m\}$ of

$$\partial_\alpha \chi = -B_\alpha. \tag{3.6}$$

Given that each B_α is assumed constant in $\mathfrak{G}$ over the manifold M, there exists a family of solutions $\chi \in C^\infty(M, \mathfrak{G})$. Given standard coordinate system $u = (u^\alpha)_{\alpha=1}^m \in M \subset \mathbb{R}^m$, for any constant $C \in \mathfrak{G}$ we may take

$$\chi(u) = -B_\alpha u^\alpha + C,$$

and note that $\chi(u)$ as a solution to (3.6) implies $\sigma(u) := e^{\chi(u)}$ solves (3.5). $\square$

Corollary 13. *Let domain* $M \subset \mathbb{R}^m, N, \eta = M \times N,\ G,$ *and* $\mathfrak{G}$ *be as in Theorem 12. Assume the connected compact Lie group* G *is also abelian. For any finitely integrable* *connection matrix* $A = A_\alpha \, \mathbf{d}u^\alpha \in C^\infty(M, \mathfrak{G} \otimes T^*M)$, *define*

$$\overline{A} := \overline{A}_\alpha \, \mathbf{d}u^\alpha,$$
$$\overline{A}_\alpha := \frac{1}{|M|} \int_M A_\alpha(x) \, \mathbf{d}x.$$

Then there exists a change of gauge given by some $\sigma = e^\chi \in \mathscr{D}$ *taking* $\nabla = \mathbf{d} + A \in \mathfrak{A}$ *to* $\nabla^\sigma = \mathbf{d} + A - \overline{A} \in \mathfrak{A}$.

Proof. We merely apply Theorem 12 with $B = \overline{A}$. □

3.2.5 Scope: compact abelian Lie groups $\mathbb{T}^n$

Our results regarding the existence of translation gauges on multidimensional base manifold domains M are restricted to compact abelian Lie groups G acting on the fibers of the bundle $M \times N$, and we now discuss the scope of this result.

Bröcker and Dieck [16] classify compact abelian Lie groups. Their Corollary 3.7 then implies that every connected compact abelian Lie group G is isomorphic to some torus $\mathbb{T}^n \cong (S^1)^n$ associated with Lie algebra $(\imath\mathbb{R})^n$.

We saw examples of this in sections 3.2.2 - 3.2.3, when we addressed cases of the connected compact abelian Lie groups $SO(2) \cong U(1) \cong S^1 \cong \mathbb{T}^1$ with 1-dimensional Lie algebras $\mathfrak{so}(2) \cong \mathfrak{u}(1) \cong \imath\mathbb{R}$.

Furthermore, in the upcoming section 3.2.6, we will consider the Lie subgroups of $SU(2)$ acting on the bundle $M \times \mathbb{C}^2$, noting that 1-dimensional Lie sub-algebras, such as $\mathfrak{G}_y := span_{\mathbb{R}}\{\imath\sigma_y\} \subset SU(2)$, actually imply $SO(2) \cong G_y := e^{\mathfrak{G}_y} \subset SU(2)$, and such G_y will be an abelian Lie group isomorphic to $\mathbb{T}^1$ as well.

Beyond 1-dimensional Lie algebras, we then have a whole class of nontrivial compact abelian Lie groups $G \cong \mathbb{T}^n$ with *multidimensional* Lie algebras $(\imath\mathbb{R})^n$ for which Theorem

12 applies. This is most notable when domain $M \subset \mathbb{R}^m$ with $m \geq 2$, and Theorem 9 does not apply in this case.

When we take the Lie algebra $\mathfrak{G}$, with

$$\begin{aligned}\mathfrak{Lie}(\mathbb{T}^n) \cong (\hat{\imath}\mathbb{R})^n \cong \mathfrak{G} &:= \left\{ \sum_{j=1}^{n} \hat{\imath} x_j \; e_j \otimes \varepsilon^j \;\middle|\; \text{each } x_j \in \mathbb{R} \right\} \\ &= \left\{ \begin{bmatrix} \hat{\imath}x_1 & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \hat{\imath}x_n \end{bmatrix} \; : \; \text{each } x_j \in \mathbb{R} \right\},\end{aligned}$$

which is associated with the connected compact abelian Lie group G with

$$\begin{aligned}\mathbb{T}^n \cong G &:= \left\{ \sum_{j=1}^{n} (\cos(x_j) + \hat{\imath}\sin(x_j)) \; e_j \otimes \varepsilon^j \;\middle|\; \text{each } x_j \in \mathbb{R} \right\} \\ &= \left\{ \begin{bmatrix} \cos(x_1) + \hat{\imath}\sin(x_1) & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \cos(x_n) + \hat{\imath}\sin(x_n) \end{bmatrix} \; : \; \text{each } x_j \in \mathbb{R} \right\},\end{aligned}$$

we then see that solving the translation gauge equation for connection $\nabla = \mathbf{d} + A$ with $A = A_\alpha \, \mathbf{d}u^\alpha \in C^\infty(M, \mathfrak{G} \otimes T^*M)$ acting on the vector bundle $\eta = M \times \mathbb{C}^n$ has solutions. Indeed, a translation by any constant connection matrix $B = B_\alpha \, \mathbf{d}u^\alpha$ with each $B_\alpha \in \mathfrak{G}$ yields the identification

$$\begin{aligned}\nabla^{e^\chi} &:= e^{-\chi} \circ (\mathbf{d} + A) \circ e^\chi \\ &= \mathbf{d} + \mathbf{d}\chi + e^{-\chi} A e^\chi \\ &= \mathbf{d} + \mathbf{d}\chi + A \\ &= \mathbf{d} + A - B.\end{aligned}$$

As in our other cases, $\mathbf{d}\chi = -B$ is readily solvable by $\chi(u) = -B_\alpha u^\alpha + C$ for any $C \in \mathfrak{G}$, where we use standard coordinates $u = (u^\alpha)_{\alpha=1}^m \in M \subset \mathbb{R}^m$ as we did previously.

Of course, one may note that this merely amounts to solving along the diagonal n copies of the 1-dimensional systems already discussed in detail in section 3.2.3.

3.2.6 Counter example: bundle $M \times \mathbb{C}^2$

Having established the behavior of translation gauges on bundles equipped with abelian Lie groups acting on the fibers, we now consider an important non-abelian Lie group as an example, and show that in general the system representing the translation gauge equation is not only overdetermined but fails to have a solution. We reference Fuchs and Schweigert's Chapter 9.6 in *Symmetries, Lie Algebras and Representations: A Graduate Course for Physicists* [21] for some contextual motivation and notational choice for this section.

Of particular interest to the study of spin and fermions in quantum mechanics are bundles $M \times \mathbb{C}^2$ with the gauge group $G = SU(2)$ acting on the fibers. It should be noted that Lie group $SU(2)$ is locally congruent to the Lie group $SO(3)$, and $SU(2)$ acts as a double cover of $SO(3)$, topologically speaking. The canonical isomorphism $\mathfrak{su}(2) \cong \mathfrak{so}(3)$ between their Lie algebras allows much of the following discussion to motivate similar results were one to model the bundle $M \times \mathbb{R}^3$, as is done when studying angular momentum.

As the bundle $M \times \mathbb{C}^2$ is of great interest in quantum physics, we will also explicitly discuss some computational results in the study of changes of gauge in this setting, as it highlights the challenges of solving for translation gauges on general bundles.

Let connected compact domain $M \subset \mathbb{R}^m$ be the base manifold for trivial vector bundle $\eta := M \times \mathbb{C}^2$. Let compact non-abelian Lie group $G = SU(2)$ act on the fibers of η. Then associated with G is Lie algebra $\mathfrak{G} = \mathfrak{su}(2)$, which has nontrivial Lie bracket. We then take adjoint bundle $Ad(\eta)$ and automorphism bundle $Aut(\eta)$ to have fibers $Ad(\eta)_u \cong \mathfrak{su}(2)$ and $Aut(\eta)_u \cong SU(2)$ for each $u \in M$.

It is convenient to represent $\mathfrak{su}(2)$ and $SU(2)$ in terms of the Pauli matrices, as which are given as follows:

$$\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad \sigma_y = \begin{bmatrix} 0 & -\hat{\imath} \\ \hat{\imath} & 0 \end{bmatrix}, \qquad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The Pauli matrices satisfy the following non-commutative multiplication identities

$$\mathbf{I} = \sigma_x^2 = \sigma_y^2 = \sigma_z^2, \quad \hat{\imath}\sigma_z = \sigma_x\sigma_y = -\sigma_y\sigma_x, \quad \hat{\imath}\sigma_x = \sigma_y\sigma_z = -\sigma_z\sigma_y, \quad \hat{\imath}\sigma_y = \sigma_z\sigma_x = -\sigma_x\sigma_z.$$

The spaces of Hermitian and traceless skew-Hermitian matrices in $\mathbb{C}^2$ are able to be represented in terms of the Pauli matrices. The space of Hermitian matrices in $\mathbb{C}^2$ has basis $\{\mathbf{I}, \sigma_x, \sigma_y, \sigma_z\}$ over $\mathbb{R}$ and is then given by

$$\left\{ h \;\middle|\; h = \begin{bmatrix} a+d & b-\hat{\imath}c \\ b+\hat{\imath}c & a-d \end{bmatrix} = a\mathbf{I} + b\sigma_x + c\sigma_y + d\sigma_z \;:\; a,b,c,d \in \mathbb{R} \right\}.$$

The space $\mathfrak{su}(2)$ of traceless skew-Hermitian matrices has basis $\{\hat{\imath}\sigma_x, \hat{\imath}\sigma_y, \hat{\imath}\sigma_z\}$ over $\mathbb{R}$ and is given by

$$\mathfrak{su}(2) = \left\{ u \;\middle|\; u = \begin{bmatrix} \hat{\imath}z & \hat{\imath}x + y \\ \hat{\imath}x - y & -\hat{\imath}z \end{bmatrix} = x\hat{\imath}\sigma_x + y\hat{\imath}\sigma_y + z\hat{\imath}\sigma_z \;:\; x,y,z \in \mathbb{R} \right\}.$$

Furthermore, the special unitary group $SU(2)$ is given by

$$SU(2) = \left\{ s \;\middle|\; s = \begin{bmatrix} a + \hat{\imath}d & \hat{\imath}b + c \\ \hat{\imath}b - c & a - \hat{\imath}d \end{bmatrix} = a\mathbf{I} + b\hat{\imath}\sigma_x + c\hat{\imath}\sigma_y + d\hat{\imath}\sigma_z \;:\; a^2 + b^2 + c^2 + d^2 = 1 \right\}.$$

For $s = a\mathbf{I} + b\hat{\imath}\sigma_x + c\hat{\imath}\sigma_y + d\hat{\imath}\sigma_z \in SU(2)$, it is also straightforward to calculate the conjugate-transpose, as it holds that

$$\begin{aligned} s^* &= s^{-1} = \overline{s^T} \\ &= a\mathbf{I} - b\hat{\imath}\sigma_x - c\hat{\imath}\sigma_y - d\hat{\imath}\sigma_z. \end{aligned}$$

It is known that the exponential map $\exp(\cdot) = e^{\cdot}$ maps $\mathfrak{su}(2)$ onto $SU(2)$. Let us consider some examples in terms of the Pauli matrices.

For $z\hat{\imath}\sigma_z \in \mathfrak{su}(2)$, we then compute

$$\begin{aligned} e^{z\hat{\imath}\sigma_z} &= \exp \begin{bmatrix} \hat{\imath}z & 0 \\ 0 & -\hat{\imath}z \end{bmatrix} = \begin{bmatrix} \cos(z) + \hat{\imath}\sin(z) & 0 \\ 0 & \cos(z) - \hat{\imath}\sin(z) \end{bmatrix} \\ &= \cos(z)\ \mathbf{I} + \sin(z)\ \hat{\imath}\sigma_z \in SU(2). \end{aligned}$$

For $y\hat{\imath}\sigma_y \in \mathfrak{su}(2)$, we then compute

$$e^{y\hat{\imath}\sigma_y} = \exp \begin{bmatrix} 0 & y \\ -y & 0 \end{bmatrix} = \begin{bmatrix} \cos(y) & \sin(y) \\ -\sin(y) & \cos(y) \end{bmatrix} = \cos(y)\ \mathbf{I} + \sin(y)\ \hat{\imath}\sigma_y \in SU(2).$$

For $x\hat{\imath}\sigma_x \in \mathfrak{su}(2)$, we then compute

$$e^{x\hat{\imath}\sigma_x} = \exp\begin{bmatrix} 0 & \hat{\imath}x \\ \hat{\imath}x & 0 \end{bmatrix} = \begin{bmatrix} \cos(x) & \hat{\imath}\sin(x) \\ \hat{\imath}\sin(x) & \cos(x) \end{bmatrix} = \cos(x)\ \mathbf{I} + \sin(x)\ \hat{\imath}\sigma_x \in SU(2).$$

For arbitrary $\chi = x\hat{\imath}\sigma_x + y\hat{\imath}\sigma_y + z\hat{\imath}\sigma_z \in \mathfrak{su}(2)$, we then are able to use Mathematica [22] to compute

$$\begin{aligned} e^{\chi} &= e^{x\hat{\imath}\sigma_x + y\hat{\imath}\sigma_y + z\hat{\imath}\sigma_z} \\ &= \exp\begin{bmatrix} \hat{\imath}z & \hat{\imath}x + y \\ \hat{\imath}x - y & -\hat{\imath}z \end{bmatrix} \\ &= \begin{bmatrix} \cos\left(\sqrt{x^2+y^2+z^2}\right) + \frac{\hat{\imath}z\sin\left(\sqrt{x^2+y^2+z^2}\right)}{\sqrt{x^2+y^2+z^2}} & \frac{(\hat{\imath}x+y)\sin\left(\sqrt{x^2+y^2+z^2}\right)}{\sqrt{x^2+y^2+z^2}} \\ \frac{(\hat{\imath}x-y)\sin\left(\sqrt{x^2+y^2+z^2}\right)}{\sqrt{x^2+y^2+z^2}} & \cos\left(\sqrt{x^2+y^2+z^2}\right) - \frac{\hat{\imath}z\sin\left(\sqrt{x^2+y^2+z^2}\right)}{\sqrt{x^2+y^2+z^2}} \end{bmatrix}. \end{aligned}$$

We set $\rho = \sqrt{x^2+y^2+z^2}$ to more clearly see that

$$\begin{aligned} e^{\chi} &= \begin{bmatrix} \cos(\rho) + \frac{\hat{\imath}z\sin(\rho)}{\rho} & \frac{(\hat{\imath}x+y)\sin(\rho)}{\rho} \\ \frac{(\hat{\imath}x-y)\sin(\rho)}{\rho} & \cos(\rho) - \frac{\hat{\imath}z\sin(\rho)}{\rho} \end{bmatrix} \\ &= \cos(\rho)\ \mathbf{I} + \frac{x\sin(\rho)}{\rho}\ \hat{\imath}\sigma_x + \frac{y\sin(\rho)}{\rho}\ \hat{\imath}\sigma_y + \frac{z\sin(\rho)}{\rho}\ \hat{\imath}\sigma_z. \end{aligned}$$

Naturally identifying coordinates $e^{\chi} = a\ \mathbf{I} + b\ \hat{\imath}\sigma_x + c\ \hat{\imath}\sigma_y + d\ \hat{\imath}\sigma_z$ to match our previous representation of $SU(2)$, we indeed see $e^{\chi} \in SU(2)$, as $\rho^2 = x^2 + y^2 + z^2$ implies

$$\begin{aligned} 1 &= \cos^2(\rho) + \frac{x^2\sin^2(\rho)}{\rho^2} + \frac{y^2\sin^2(\rho)}{\rho^2} + \frac{z^2\sin^2(\rho)}{\rho^2} \\ &= a^2 + b^2 + c^2 + d^2. \end{aligned}$$

Take $A = j\hat{\imath}\sigma_x + k\hat{\imath}\sigma_y + l\hat{\imath}\sigma_z \in \mathfrak{su}(2)$. Using the "Pauli" Mathematica package authored by Heiko Feldmann, we may then calculate the representation of $e^{-\chi}Ae^{\chi} \in \mathfrak{su}(2)$ in terms of real coordinates $\tilde{j}, \tilde{k}, \tilde{l}$, with respect to the basis $\{\hat{\imath}\sigma_x, \hat{\imath}\sigma_y, \hat{\imath}\sigma_z\}$ as follows, again taking $\rho = \sqrt{x^2+y^2+z^2}$. Given $e^{-\chi}Ae^{\chi} = \tilde{j}\hat{\imath}\sigma_x + \tilde{k}\hat{\imath}\sigma_y + \tilde{l}\hat{\imath}\sigma_z$, then

$$\tilde{j} = \frac{x(jx+ky+lz) + (j(y^2+z^2) - x(ky+lz))\cos(2\rho) - \rho(-ly+kz)\sin(2\rho)}{x^2+y^2+z^2},$$

$$\tilde{k} = \frac{y(jx+ky+lz) + (k(x^2+z^2) - y(jx+lz))\cos(2\rho) - \rho(lx-jz)\sin(2\rho)}{x^2+y^2+z^2},$$

$$\tilde{l} = \frac{z(jx+ky+lz) + (l(x^2+y^2) - z(jx+ky))\cos(2\rho) - \rho(-kx+jy)\sin(2\rho)}{x^2+y^2+z^2}. \tag{3.7}$$

This computation is primarily to emphasize the coupled nonlinearities in the system, which will lead to our overdetermined system representing a translation gauge lacking a solution for certain choices in data.

At this point, we may instance previous work by α, ranging over the coordinates of the base domain $M \subset \mathbb{R}^m$, such that for a connection matrix (1-form) A,

$$A = A_\alpha \, \mathbf{d}u^\alpha = (j_\alpha\hat{\imath}\sigma_x + k_\alpha\hat{\imath}\sigma_y + l_\alpha\hat{\imath}\sigma_z) \, \mathbf{d}u^\alpha \in C^\infty(M, \mathfrak{su}(2) \otimes T^*M).$$

We then consider a gauge transformation given by $\sigma = e^{\chi} \in C^\infty(M, SU(2))$ taking the connection $\nabla = \mathbf{d} + A$ to some connection $\nabla^\sigma = \mathbf{d} + B$, where $B = B_\alpha \, \mathbf{d}u^\alpha$. Unlike in previous sections, where B was assumed to be a constant connection matrix, to reduce notational clutter in the forthcoming calculations, we intend to have $B - A$ represent a constant connection matrix in the rest of this section.

The gauge transformation computations

$$\begin{aligned} \mathbf{d} + B_\alpha \ \mathbf{d}u^\alpha &= e^{-\chi}(\mathbf{d} + A_\alpha \ \mathbf{d}u^\alpha)e^\chi \\ &= \mathbf{d} + e^{-\chi} \ \mathbf{d}e^\chi + e^{-\chi}(A_\alpha)e^\chi \ \mathbf{d}u^\alpha \end{aligned} \tag{3.8}$$

may then be represented by instancing the above computations for $\tilde{j}, \tilde{k}, \tilde{l}$ by α, yielding $\tilde{j}_\alpha, \tilde{k}_\alpha, \tilde{l}_\alpha$, and where isolating coefficients of the basis of 1-forms $\{\mathbf{d}u^\alpha\}_{1\leq\alpha\leq m}$ yields

$$\begin{aligned} B_\alpha &= e^{-\chi}\partial_\alpha[e^\chi] + e^{-\chi}A_\alpha e^\chi \\ &= \partial_\alpha \chi + e^{-\chi}A_\alpha e^\chi \\ &= \partial_\alpha[x\hat{\imath}\sigma_x + y\hat{\imath}\sigma_y + z\hat{\imath}\sigma_z] + \tilde{j}_\alpha \hat{\imath}\sigma_x + \tilde{k}_\alpha \hat{\imath}\sigma_y + \tilde{l}_\alpha \hat{\imath}\sigma_z \\ &= \Big(\partial_\alpha(x, y, z) + (\tilde{j}_\alpha, \tilde{k}_\alpha, \tilde{l}_\alpha)\Big) \cdot (\hat{\imath}\sigma_x, \hat{\imath}\sigma_y, \hat{\imath}\sigma_z). \end{aligned}$$

Here, we emphasize representing $\mathfrak{su}(2)$-valued objects in terms of coordinates of the basis $\{\hat{\imath}\sigma_x, \hat{\imath}\sigma_y, \hat{\imath}\sigma_z\}$ via dot-product notation for mere convenience.

We now arrive at a point where we can discuss the existence of translation gauges on the vector bundle $M \times \mathbb{C}^2$. By this, we of course mean that we consider gauge equivalent connection matrices $B = B_\alpha \ \mathbf{d}u^\alpha$, relative to some fixed $A = A_\alpha \ \mathbf{d}u^\alpha$, such that each $B_\alpha - A_\alpha$ is constant over the manifold M.

Moreover, we are interested in the existence of a translation gauge such that $B = A - \overline{A}$ is gauge equivalent to A, where by $\overline{A}$ we mean the obvious interpretation of the constant 1-form taking on complex values in each coordinate equal to the average over M of the respective coordinates of A. That is, given

$$A = (j_\alpha \hat{\imath}\sigma_x + k_\alpha \hat{\imath}\sigma_y + l_\alpha \hat{\imath}\sigma_z) \ \mathbf{d}u^\alpha \in C^\infty(M, \mathfrak{su}(2) \otimes T^*M),$$

we define the constant 1-form $\overline{A} \in C^\infty(M, \mathfrak{su}(2) \otimes T^*M)$ by

$$\overline{A} = \overline{A}_\alpha \ \mathbf{d}u^\alpha := (\bar{\jmath}_\alpha \hat{\imath}\sigma_x + \overline{k}_\alpha \hat{\imath}\sigma_y + \bar{l}_\alpha \hat{\imath}\sigma_z) \ \mathbf{d}u^\alpha,$$
$$\bar{\jmath}_\alpha := \frac{1}{|M|}\int_M j_\alpha, \quad \overline{k}_\alpha := \frac{1}{|M|}\int_M k_\alpha, \quad \bar{l}_\alpha := \frac{1}{|M|}\int_M l_\alpha.$$

We will focus on the case of a gauge taking A to $A - \overline{A}$, only to maintain notation, recognizing identical arguments hold for the discussion of existence of general translation gauges.

Solving for a translation gauge given by e^χ and taking A to $B = A - \overline{A}$ is equivalent to simultaneously solving for $\chi = x\hat{\imath}\sigma_x + y\hat{\imath}\sigma_y + z\hat{\imath}\sigma_z$, or simply the real coordinates $\{x, y, z\}$, in the systems indexed by α given by

$$\begin{aligned} A_\alpha - \overline{A}_\alpha &= (j_\alpha - \bar{\jmath}_\alpha, k_\alpha - \overline{k}_\alpha, l_\alpha - \bar{l}_\alpha) \cdot (\hat{\imath}\sigma_x, \hat{\imath}\sigma_y, \hat{\imath}\sigma_z) \\ &= \Big(\partial_\alpha(x, y, z) + (\tilde{\jmath}_\alpha, \tilde{k}_\alpha, \tilde{l}_\alpha)\Big) \cdot (\hat{\imath}\sigma_x, \hat{\imath}\sigma_y, \hat{\imath}\sigma_z). \end{aligned}$$

Alternatively written in terms of a three-equation system of Pauli matrix coordinate functions, we have m distinct real scalar three-equation systems indexed by $\alpha \in \{1, \cdots, m\}$ of

$$\begin{aligned} \partial_\alpha x &= j_\alpha - \bar{\jmath}_\alpha - \tilde{\jmath}_\alpha, \\ \partial_\alpha y &= k_\alpha - \overline{k}_\alpha - \tilde{k}_\alpha, \\ \partial_\alpha z &= l_\alpha - \bar{l}_\alpha - \tilde{l}_\alpha, \end{aligned} \tag{3.9}$$

where $j_\alpha, k_\alpha, l_\alpha$ are real scalar functions given as data, where $\bar{\jmath}_\alpha, \overline{k}_\alpha, \bar{l}_\alpha$ are prescribed constant data. As $\alpha \in \{1, \cdots, m\}$ ranges over its values, we see that we have only 3 unknowns for which to simultaneously solve, those being $\{x, y, z\}$, but we have $3m$

equations. Whenever $m > 1$, we have an overdetermined system of equations, and do not expect solutions to exist in general.

Furthermore, $\tilde{j}_\alpha, \tilde{k}_\alpha, \tilde{l}_\alpha$ are the highly nonlinear functions of x, y, z, and the data, as similarly given above in (3.7), neglecting indexing by α. As such, for a general base manifold domain $M \subset \mathbb{R}^m$ with $m \geq 2$, we have an overdetermined nonlinear coupled system of partial differential equations that cannot be expected to have solutions for general data.

In the case where domain $M \subset \mathbb{R}^1$, the indexing by α is trivial, where $\alpha \in \{1\}$. In this case, suppressing subscripts α and emphasizing the nonlinear dependence for readability, the entirety of our system amounts to

$$\begin{aligned} x' &= j - \overline{j} - \tilde{j}(x, y, z, j, k, l) \\ y' &= k - \overline{k} - \tilde{k}(x, y, z, j, k, l) \\ z' &= l - \overline{l} - \tilde{l}(x, y, z, j, k, l) \end{aligned}$$

and is no more than a nonlinear coupled system of ordinary differential equations, and most importantly no longer overdetermined. In this special case, we instead model the problem in terms of solving for a flow $\sigma = e^\chi$, as in section 3.2.1 given the equation

$$(e^\chi)' = [e^\chi, A_1] - e^\chi \overline{A}_1,$$

on the compact smooth manifold and Lie group $G = SU(2)$, where we use the standard matrix commutator for $[e^\chi, A_1] = e^\chi A_1 - A_1 e^\chi$. Now, we simply appeal to our previous results for domain $M \subset \mathbb{R}^1$ in Theorem 9 and Corollary 10, establishing existence of a solution to (3.8) with $B_1 = A_1 - \overline{A}_1$ of some $\sigma = e^\chi \in C^\infty(M, G) = C^\infty(M, SU(2))$.

Accepting the existence of translation gauges when domain $M \subset \mathbb{R}^1$, we turn our attention back to more complicated base manifolds $M \subset \mathbb{R}^m$ with $m \geq 2$, when the system given in (3.9) is indeed overdetermined as $\alpha \in \{1, \cdots, m\}$. Let us represent the base manifold's coordinate system in a standard fashion by $u = (u^\alpha)_{\alpha=1}^m \in M$, with $\partial_\alpha = \frac{\partial}{\partial u^\alpha}$.

Considering prescribing data

$$A = A_1(u^1)\ \mathbf{d}u^1 + A_2(u^2)\ \mathbf{d}u^2 + \cdots + A_m(u^m)\ \mathbf{d}u^m.$$

Then choosing data for A by making the collection of coordinate functions $j_\alpha, k_\alpha, l_\alpha$ dependent respectively and solely on the base manifold variable u^α, and constant with respect to all other base manifold variables u^β with $\beta \neq \alpha$, results in the indexed systems given in (3.9) degenerating from collectively being coupled systems of partial differential equations to each index $\alpha' s$ subsystem essentially representing a coupled ordinary differential equation in terms of solely the variable u^α.

Based on our discussion of 1-dimensional base manifolds, we expect a unique "solution" $\chi = \chi_\alpha(u^\alpha)$ to exist for an individual $\alpha' s$ subsystem of ordinary differential equations, but by no means will $\chi_\alpha = \chi_\beta$ for all $\alpha, \beta \in \{1, \cdots, m\}$ as data $j_\alpha, k_\alpha, l_\alpha$ (dependent only on u^α) is chosen independently from $j_\beta, k_\beta, l_\beta$ (dependent only on u^β).

In conclusion, on a bundle $M \times \mathbb{C}^2$ with Lie algebra $\mathfrak{su}(2)$ and Lie group $SU(2)$ acting on the fibers, the existence of translation gauges may only exist in general under the restriction $M \subset \mathbb{R}^1$.

We may contrast this vector bundle case with the vector bundles $M \times \mathbb{R}^2$, and $M \times \mathbb{C}$, where more general base manifolds M still admit translation gauges due to the abelian nature of the respective Lie groups $SO(2)$ and $U(1)$.

This suggests it may be possible to avoid the problems arising from the nonlinearity in the overdetermined system governing the existence of translation gauges for connections on $M \times \mathbb{C}^2$, provided one restricts the values of the connection matrix to a 1-dimensional Lie sub-algebra, and furthermore the change of gauge to be given by a map taking values in the respective Lie subgroup.

For instance, we may consider taking $\mathfrak{G}_y \subset \mathfrak{G} = \mathfrak{su}(2)$ given by $\mathfrak{G}_y := span_{\mathbb{R}}\{\hat{\imath}\sigma_y\}$, respectively paired with the Lie subgroup $G_y := e^{\mathfrak{G}_y} \subset G = SU(2)$. Of course, as shown in our earlier calculations, this amounts to $G_y \cong SO(2)$, and so little surprise is found at this restricted system admitting translation gauges for connection matrices $A \in \mathbb{C}^\infty(M, \mathfrak{G}_y \otimes T^*M)$ when the translation itself is restricted to taking values in $\mathfrak{G}_y$. Furthermore, there is some convenience in that the translation we are most interested in is by $\overline{A}$, which naturally would be restricted in value to the same Lie sub-algebra as imposed upon A.

3.2.7 Nonexistence of translation gauges on general bundles

Inspired by the non-commutative challenges introduced by the Lie group $SU(2)$ acting on the bundle $M \times \mathbb{C}^2$, we prove the existence of connection matrix data A on a more general bundle such that the overdetermined system representing the translation gauge equation has no solution.

To do so, it is necessary for our proof to have the Lie algebra $\mathfrak{G}$ to be greater than 1-dimensional over $\mathbb{R}$, and allow for multiplicatively non-commutative elements of a basis for matrix Lie algebra $\mathfrak{G}$ to be used. We additionally require that the Lie group G acting on the fibers of the bundle remain compact, so we must preclude the simple fibers $\mathbb{R}^2$ and $\mathbb{C}$, as their only respective compact Lie groups $SO(2) \cong U(1)$ have 1-dimensional abelian Lie algebras $\mathfrak{so}(2) \cong \mathfrak{u}(1)$. Furthermore, the base domain M must be of dimension 2 or larger, which then precludes the application ordinary differential equations to arrive at a translation gauge by Theorem 9.

We further remark that the forthcoming theorem merely justifies the intuition that the translation gauge equation, such as in (3.5), is overdetermined and fails to have solutions in general. This intuition follows from simply comparing the number of equations in the system to the unknowns and realizing the system indeed has more data independent equations than unknowns. Taking the place of the "unknowns" in this interpretation are then the coefficient maps of χ with respect to the basis of $\mathfrak{G}$ previously mentioned.

For a compact Lie group G acting on the fibers of a bundle $\eta = M \times N$, if the associated Lie algebra $\mathfrak{G}$ is q-dimensional with basis $\{\xi_j\}_{j=1}^q$, then solving for a gauge given by $\sigma = e^\chi \in C^\infty(M, G)$ amounts to solving for scalar $\{\chi^j\}_{j=1}^q$ such that $\chi = \xi_j \chi^j$. From this perspective, we have q unknown scalar maps on M.

In comparison, when we separate the translation gauge equation into (tensor-valued) coordinate equations with respect to the basis $\{\mathbf{d}u^\alpha\}_{\alpha=1}^m$ of $\Omega_M^1(\mathfrak{G})$, we end up with m distinct subsystems indexed by α. If we further separate these subsystems into scalar coordinate functions with respect to the basis $\{\xi_j\}_{j=1}^q$, we see that our entire gauge translation equation amounts to a system of mq equations.

Hence, unless $m = 1$, and we are merely in the case addressed by Theorem 9, then $mq > q$ and the translation gauge equation is overdetermined with more equations than unknowns. This being the case, then we expect solutions fail to exist in general without some sort of compatibility condition on the system. It is in this overdetermined case that violation of the assumptions of Theorem 12 then result in the translation gauge equation lacking a solution for general data, which we address in this theorem.

Theorem 14. *Let connected compact domain $M \subset \mathbb{R}^m$ with $m \geq 2$. Let M be the base manifold for trivial vector bundle $\eta := M \times N$, where $N \cong \mathbb{R}^n$ with $n \geq 3$. Let non-abelian compact Lie group G act on the fibers of η, and have Lie algebra $\mathfrak{Lie}(G) = \mathfrak{G}$.*

There exist connections $\nabla = \mathbf{d} + A \in \mathfrak{A}$ such that for every nonzero constant *con-*

nection matrix $B = B_\alpha\ \mathbf{d}u^\alpha$ *with each* $B_\alpha \in \mathfrak{G}$*, the translation gauge equation given by*

$$\begin{aligned}\mathbf{d} + A - B &= \sigma^{-1} \circ \nabla \circ \sigma \\ &= \mathbf{d} + \mathbf{d}\chi + e^{-\chi} A e^{\chi}\end{aligned}$$

yields no solutions $\sigma = e^\chi \in \mathscr{D}$ *with* $\chi \in \mathbb{C}^\infty(M, \mathfrak{G})$.

Proof. We begin by noting that the adjoint bundle $Ad(\eta)$ and gauge group bundle $\mathscr{G} \subset Aut(\eta)$ then have fibers $Ad(\eta)_u \cong \mathfrak{G}$ and $\mathscr{G}_u \cong G$ for each $u \in M$.

Let $\{\xi_j\}_{j=1}^q$ be a multiplicatively non-commutative basis over $\mathbb{R}$ for $\mathfrak{G}$. That is, $e^{\xi_i + \xi_j} \neq e^{\xi_i} e^{\xi_j}$ for some $i \neq j$. For the base manifold M, represent in standard coordinates $u = (u^\alpha)_{\alpha=1}^m \in M \subset \mathbb{R}^m$, hence $\partial_\alpha = \frac{\partial}{\partial u^\alpha}$. Let $\pi_j : \mathfrak{G} \to \mathbb{R}$ be coordinate projection maps with respect to this basis.

Represent each constant $B_\alpha = \xi_j \pi_j(B_\alpha) = \xi_j b^j_\alpha$ with respect to the basis $\{\xi_j\}$.

For the connection $\nabla = \mathbf{d} + A$, we denote the connection matrix $A = A_\alpha\ \mathbf{d}u^\alpha$. Let each A_α be a map of a single coordinate of M into $\mathfrak{G}$ and constant with respect to all other coordinates of M. Specifically, let each $A_\alpha = \sum_j \xi_j a^j_\alpha(u^\alpha)$ for some nonzero smooth $\mathbb{R}$-valued $a^j_\alpha(\cdot) \in \mathbb{C}^\infty$. Hence we set data $A(u) = \xi_j a^j_\alpha(u^\alpha)\ \mathbf{d}u^\alpha$.

For every $\sigma \in C^\infty(M, G)$, there are some $\chi^j \in C^\infty(M, \mathbb{R})$ where $\chi = \xi_j \chi^j \in C^\infty(M, \mathfrak{G})$ satisfies $\sigma = e^\chi$. Given this, we then have that

$$\mathbf{d} + A - B = \mathbf{d} + A_\alpha\ \mathbf{d}u^\alpha - B_\alpha\ \mathbf{d}u^\alpha$$

and

$$\sigma^{-1} \circ \nabla \circ \sigma = \mathbf{d} + \partial_\alpha \chi\ \mathbf{d}u^\alpha + e^{-\chi} A_\alpha e^\chi\ \mathbf{d}u^\alpha.$$

Towards contradiction, we assume $\mathbf{d} + A - B = \sigma^{-1} \circ \nabla \circ \sigma$.

Isolating the coefficient of each $\mathbf{d}u^\alpha$, we then have m distinct subsystems

$$\begin{aligned} \partial_\alpha \chi = \frac{\partial}{\partial u^\alpha}\left[\xi_j \chi^j\right] &= A_\alpha - e^{-\chi} A_\alpha e^\chi - B_\alpha \\ &= \left[\xi_j - e^{-\chi}\xi_j e^\chi\right] a^j_\alpha(u^\alpha) - \xi_j b^j_\alpha. \end{aligned} \tag{3.10}$$

As $\{\xi_j\}$ is a non-commutative basis for $\mathfrak{G}$, then for $\chi = \xi_j \chi^j$, we have that for some j, it holds $\mathbf{0} \neq \xi_j - e^{-\chi}\xi_j e^\chi$ provided $\chi^i \neq \mathbf{0}$ for some $i \neq j$. This in turn implies $\pi_i\left[\xi_j - e^{-\chi}\xi_j e^\chi\right]$ is a nonlinear scalar map dependent on $\{\chi^j\}_{j=1}^q$, recalling $q = dim(\mathfrak{G})$.

As we assumed B was a nonzero connection matrix, some $B_{\alpha_0} \neq \mathbf{0}$, and therefore some $b^j_{\alpha_0} \neq 0$. This implies that any solution χ cannot be identically zero on M. As such, the nonlinearity of (3.10) and independently chosen data $a^j_\alpha(u^\alpha)$ forces the overdetermined system to be contradictory. That is, though for the single α_0 a nonzero solution χ may solve the subsystem associated with α_0 given some data $a^j_{\alpha_0}(u^{\alpha_0})$, we may choose data for $a^j_{\alpha_1}(u^{\alpha_1})$ that causes the subsystem associated with α_1 to fail to hold, and moreover that

$$\partial_{\alpha_1}\chi - A_{\alpha_1} + e^{-\chi} A_{\alpha_1} e^\chi$$

fails to even be constant over M, let alone is not equal to the desired constant $B_{\alpha_1} \in \mathfrak{G}$. This then contradicts the assumption $\mathbf{d} + A - B = e^{-\chi} \circ \nabla \circ e^\chi$, completing the proof. □

Again, we reiterate that $B = \mathbf{0}$ always affords the trivial solution $\chi = \mathbf{0}$ and $e^\chi = I$ to (3.10). Furthermore, were data for A and B carefully chosen to take values in a restricted Lie sub-algebra $\mathfrak{G}' \subset \mathfrak{G}$ associated with *abelian* Lie subgroup $G' = e^{\mathfrak{G}'}$, then a solution χ exists that also takes values in $\mathfrak{G}'$ by Theorem 12. For the proof of Theorem 14, we then chose data for A that forced the full nonlinearity of the conjugation $e^{-\chi} A e^\chi$

by having *nonzero* coordinate contributions a^j_α for each basis element ξ_j of $\mathfrak{G}$, but of course a contradiction may also arise without requiring all be nonzero. Furthermore, our choice to have data A_α to be dependent only on the base manifold coordinate u^α was to guarantee the subsystems indeed were distinct in behavior from the discussion for base manifold domain $M \subset \mathbb{R}^1$, in that the nonlinearity in the overdetermined system was forced to be dependent on more than one u^α.

3.3 Poincaré inequalities on select bundles $M \times N$

3.3.1 Poincaré and Poincaré-Sobolev inequalities

When a compact Lie group G acts on a vector bundle that has translation gauges for all connections, then we have Poincaré-like inequalities under a change of gauge.

Before stating the next theorem, we note that for any l-tensor valued k-form $A = A_J\ \mathbf{d}u^J$ with ordered multi-indices J, by convention we have for the full derivative $\mathbf{D}$ that $\mathbf{D}A$ is taken to be the $(l+1)$-tensor valued k-form given by $\mathbf{D}A = (\mathbf{D}A_J)\ \mathbf{d}u^J$. This convention amounts to identifying the space of tensor-valued k-forms on domain $M \subset \mathbb{R}^m$ with vectors of length $\binom{k}{m}$ that have tensors for the vector coordinates. This is done so that l-tensor valued forms are essentially treated no differently than a standard $(l+1)$-tensor for purposes of calculating Sobolev norms.

Theorem 15. *Let connected domain $M \subset \mathbb{R}^m$ be compact. Let $N = \mathbb{R}^n$ (or $N = \mathbb{C}^n$) and compact Lie group $G \subset SO(n)$ (respectively $G \subset SU(n)$) act on the fibers of trivial vector bundle $\eta = M \times N$. Denote Lie algebra $\mathfrak{Lie}(G) = \mathfrak{G}$, noting $\mathfrak{G} \subset \mathfrak{so}(n)$ (respectively $\mathfrak{G} \subset \mathfrak{su}(n)$).*

Let $\nabla = \mathbf{d} + A \in \mathfrak{A}$ be any finitely integrable connection, and assume at least one of *the following holds.*

$$I.)\ \textit{The base manifold domain } M \subset \mathbb{R}^1. \tag{3.11}$$

$$II.)\ \textit{The connected compact Lie group } G \textit{ is also abelian.} \tag{3.12}$$

Let $p \geq 1$ and convex subdomain $M' \subset M$. There exists some constant $C = C(M', N, p)$ such that for the connection matrix $\tilde{A} = A - \overline{A}$, there exists a change of gauge given by some $\sigma \in \mathscr{D}$ between connections ∇ and $\nabla^\sigma = \sigma^{-1} \circ \nabla \circ \sigma = \mathbf{d} + \tilde{A}$ satisfying

$$\tilde{A} = \sigma^{-1}\mathbf{d}\sigma + \sigma^{-1}A\sigma,$$
$$\|\tilde{A}\|_{L^p(M')} \leq C\|\mathbf{D}A\|_{L^p(M')}.$$

Proof. We begin by setting connection matrix $\tilde{A} = A - \overline{A}$ for

$$\overline{A} = \overline{A}_\alpha\ \mathbf{d}u^\alpha = \left(\frac{1}{|M'|} \int\limits_{M'} A_\alpha \right) \mathbf{d}u^\alpha,$$

and solving for σ, which amounts to solving for an appropriate translation gauge. If (3.11) holds, we may apply Theorem 9 to obtain $\sigma \in \mathscr{D}$. If (3.12) holds, we may apply Theorem 12 to obtain $\sigma \in \mathscr{D}$.

Denoting the connection matrices $A = A_\alpha\ \mathbf{d}u^\alpha$, $\tilde{A} = \tilde{A}_\alpha\ \mathbf{d}u^\alpha$, and the constant connection matrix $\overline{A} = \overline{A}_\alpha\ \mathbf{d}u^\alpha$, we then have that

$$\|\tilde{A}\|_{L^p} = \|\tilde{A}_\alpha\ \mathbf{d}u^\alpha\|_{L^p} = \sum_{1 \leq \alpha \leq m} \|\tilde{A}_\alpha\|_{L^p} = \sum_{1 \leq \alpha \leq m} \|A_\alpha - \overline{A}_\alpha\|_{L^p}.$$

We represent the 2-tensor (matrix) valued maps in terms of standard coordinates with respect to the basis $\{e^i \otimes \varepsilon_j\}_{1 \leq i,j \leq n}$. That is, for each α we denote $A_\alpha = (a_\alpha)^i_j\ e^i \otimes \varepsilon_j$ and

$\overline{A}_\alpha = (\overline{a}_\alpha)^i_j \; e^i \otimes \varepsilon_j$. We then have for each $1 \le i, j \le n$

$$\|A_\alpha - \overline{A}_\alpha\|_{L^p} = \sum_{1 \le i,j \le n} \|(a_\alpha)^i_j - (\overline{a}_\alpha)^i_j\|_{L^p}.$$

By construction and assumption, each $(a_\alpha)^i_j \in C^\infty(M)$ satisfies $(\overline{a}_\alpha)^i_j = \frac{1}{|M'|} \int\limits_{M'} (a_\alpha)^i_j(x)\, \mathbf{d}x$, so the classical Poincaré inequality, such as presented by Gilbarg and Trudinger [13] on page 164, then holds for some $C = C(M,p)$, yielding

$$\|(a_\alpha)^i_j - (\overline{a}_\alpha)^i_j\|_{L^p(M')} \le C\|\mathbf{D}(a_\alpha)^i_j\|_{L^p(M')},$$

where we use the convention $\|\mathbf{D}(a_\alpha)^i_j\|_{L^p} := \sum\limits_{|\beta|=1} \|\mathbf{D}^\beta(a_\alpha)^i_j\|_{L^p}$ for multi-indices β. We then have that

$$\|\tilde{A}\|_{L^p} \le C \sum_{\alpha,i,j} \|\mathbf{D}(a_\alpha)^i_j\|_{L^p} = C \sum_{\alpha,i,j} \sum_{|\beta|=1} \|\mathbf{D}^\beta(a_\alpha)^i_j\|_{L^p}.$$

Then our choice of Sobolev norm convention on tensor-valued forms implies

$$\begin{aligned}\|\mathbf{D}A\|_{L^p} &:= \sum_{\alpha} \left\|(\mathbf{D}^\beta A_\alpha)_{|\beta|=1}\right\|_{L^p} \\ &= \sum_{\alpha,|\beta|=1} \|\mathbf{D}^\beta A_\alpha\|_{L^p} = \sum_{\alpha,|\beta|=1} \sum_{i,j} \|\mathbf{D}^\beta(a_\alpha)^i_j\|_{L^p},\end{aligned}$$

and we are then able to conclude that

$$\|\tilde{A}\|_{L^p(M')} \le C\|\mathbf{D}A\|_{L^p(M')}. \tag{3.13}$$

□

One may note that by our Sobolev norm convention on tensor valued forms that we do not actually have any dependence on N in the constant $C = C(M', N, p)$. We choose to leave it as is, as the most common alternative Sobolev norm convention on tensors, such as used by Uhlenbeck [4] and Donaldson [6], would have been equivalent to our Sobolev norm convention by a scalar multiple dependent on the fiber N, so the statement of our inequality holds under the alternative norm convention.

We may also use the existence of a translation gauge to establish a Poincaré-Sobolev-style inequality, comparable to the classical inequality. Lieb and Loss [15] Theorem 8.12 is one presentation of this classical inequality, and we state a simplified version of their theorem here, for reference.

Theorem 16. *Let $M \subset \mathbb{R}^m$ be a bounded, connected, open domain with the cone property. Let $1 \leq p < m$ and $1 \leq q \leq p^* := \frac{mp}{m-p}$. Let $h \in L^{p^*}(M)$ be a function such that $\int\limits_M h = 1$. Then there exists $C = C(M, h, p, q) \in (0, \infty)$ such that for any $f \in L^p_1(M)$, it holds*

$$\left\| f - \int\limits_M fh \right\|_{L^q(M)} \leq C \left\| \mathbf{D} f \right\|_{L^p(M)}. \tag{3.14}$$

Proof. We refer the reader to Lieb and Loss [15] Theorem 8.12 on page 221. □

For the next result, we introduce notation to depict connections of finite Sobolev norm. This is similar to the notation used by Uhlenbeck [4] for potentially non-smooth connections with finite Sobolev norms, but as Donaldson [6] does, we still focus on smooth connections.

Definition 3. *For $p \geq 1, k \in \mathbb{N} \cup \{0\}$, we define the affine space of Sobolev connections $\mathfrak{A}^p_k = \mathfrak{A}^p_k(\eta)$ on a trivial vector bundle $\eta = M \times N$ by*

$$\mathfrak{A}^p_k := \left\{ \nabla \in \mathfrak{A} \;\middle|\; \nabla = \mathbf{d} + A, \;\; \|A\|_{L^p_k(M)} < \infty \right\}.$$

Theorem 17. *Let domain* $M \subset \mathbb{R}^m, N, \eta = M \times N,\ G,$ *and* $\mathfrak{G}$ *be as in Theorem 15.*

Assume $m \geq 2$ *and the connected compact Lie group* G *is also abelian.*

Let $1 \leq p < m$, $1 \leq q \leq p^* := \frac{mp}{m-p}$, *and convex subdomain* $M' \subset M$.

There exists $C = C(M', N, p, q) \in (0, \infty)$ *such that for any smooth Sobolev connection* $\nabla = \mathbf{d} + A \in \mathfrak{A}_1^p$, *there exists a change of gauge given by some* $\sigma \in \mathscr{D}$ *between connections* ∇ *and* $\nabla^\sigma = \mathbf{d} + \tilde{A}$ *satisfying*

$$\tilde{A} = \sigma^{-1}\mathbf{d}\sigma + \sigma^{-1}A\sigma,$$

$$\|\tilde{A}\|_{L^q(M')} \leq C \left\|\mathbf{D}A\right\|_{L^p(M')}. \tag{3.15}$$

Proof. We begin by noting that $\mathbf{d} + A \in \mathfrak{A}_1^p$ implies for bounded M and $p, q \geq 1$ as given in the hypotheses, we have that

$$\|A\|_{L^1(M)} \lesssim \|A\|_{L^q(M)} \lesssim \|A\|_{L_1^p(M)} < \infty$$

by Hölder and Sobolev inequalities. Therefore, for $A = A_\alpha\ \mathbf{d}u^\alpha$, we have each $A_\alpha \in L^1$ and $\overline{A}_\alpha := \frac{1}{|M'|} \int\limits_{M'} A_\alpha \in \mathfrak{G}$ is well defined, so the constant connection matrix $\overline{A} := \overline{A}_\alpha\ \mathbf{d}u^\alpha$ is well defined.

We then use Theorem 12 to obtain a change of gauge given by some $\sigma \in \mathscr{D}$ which solves

$$A - \overline{A} = \sigma^{-1}\mathbf{d}\sigma + \sigma^{-1}A\sigma,$$

and set $\tilde{A} := A - \overline{A}$, noting $\nabla^\sigma = \mathbf{d} + \tilde{A}$ by construction.

As in the proof of Theorem 15, if we decompose

$$
\begin{aligned}
A &= A_\alpha\ \mathbf{d}u^\alpha = (a_\alpha)^i_j\ e_i \otimes \varepsilon^j \otimes \mathbf{d}u^\alpha, \\
\overline{A} &= \overline{A}_\alpha\ \mathbf{d}u^\alpha = (\overline{a}_\alpha)^i_j\ e_i \otimes \varepsilon^j \otimes \mathbf{d}u^\alpha \\
&= \left(\frac{1}{|M'|} \int\limits_{M'} (a_\alpha)^i_j(x)\mathbf{d}x \right)\ e_i \otimes \varepsilon^j \otimes \mathbf{d}u^\alpha,
\end{aligned}
$$

then we may apply the classical Poincaré-Sobolev inequality. We then use 3.14, such as presented by Lieb and Loss [15], using the constant function $h = \frac{1}{|M'|}$ and applying to the scalar maps $(a_\alpha)^i_j \in L^p_1$ to obtain

$$
\|(a_\alpha)^i_j - (\overline{a}_\alpha)^i_j\|_{L^q(M')} \leq C\|\mathbf{D}(a_\alpha)^i_j\|_{L^p(M')},
$$

hence

$$
\begin{aligned}
\|\tilde{A}\|_{L^q} &= \sum_{\alpha,i,j} \|(a_\alpha)^i_j - (\overline{a}_\alpha)^i_j\|_{L^q} \\
&\leq C \sum_{\alpha,i,j} \|\mathbf{D}(a_\alpha)^i_j\|_{L^p}.
\end{aligned} \tag{3.16}
$$

Again as before in Theorem 15, since for each α, i, j we have

$$
\|\mathbf{D}(a_\alpha)^i_j\|_{L^p} = \sum_{|\beta|=1} \left\|\mathbf{D}^\beta(a_\alpha)^i_j\right\|_{L^p},
$$

and by our choice of Sobolev norm conventions on tensor valued forms we have

$$
\|\mathbf{D}A\|_{L^p} = \sum_{\alpha,i,j,|\beta|=1} \left\|\mathbf{D}^\beta(a_\alpha)^i_j\right\|_{L^p},
$$

so we may combine these with (3.16) to conclude

$$\|\tilde{A}\|_{L^q(M')} \leq C\|\mathbf{D}A\|_{L^p(M')},$$

which concludes the proof. □

3.3.2 Poincaré inequalities with Coulomb data

We may further consider these results under assumptions complementary to the study of Coulomb gauges due to the ellipticity afforded in such systems. For a more thorough discussion of Coulomb systems' ellipticity, one may refer to A.3.

If we additionally have compactly supported Coulomb data A, then our Poincaré inequality may be controlled by the 2-form $\mathbf{d}A$ rather than the full derivative $\mathbf{D}A$.

Hence, as a corollary to Theorem 15 in section 3.3.1, we obtain Corollary 18 as follows.

Corollary 18. *Let connected compact convex domain $M \subset \mathbb{R}^m, N, \eta = M \times N,\ G,$ and $\mathfrak{G}$ be as in Theorem 15.*

Let compactly supported Coulomb connection $\nabla = \mathbf{d} + A \in \mathfrak{A}$, where supp$(A) \subset\subset M$, and assume at least one of the following holds.

I.) The base manifold domain $M \subset \mathbb{R}^1$.

II.) The connected compact Lie group G is also abelian.

Let $p \geq 1$. There exists some constant $C = C(M, N, p)$ such that for the connection matrix $B = A - \overline{A}$, there exists a change of gauge given by some $\sigma \in \mathscr{D}$ such that

$\mathbf{d} + \tilde{A} = \nabla^\sigma = \sigma^{-1} \circ \nabla \circ \sigma$ *satisfying*

$$\tilde{A} = \sigma^{-1}\mathbf{d}\sigma + \sigma^{-1}A\sigma,$$
$$\|\tilde{A}\|_{L^p(M)} \leq C\|\mathbf{d}A\|_{L^p(M)}.$$

Proof. Under these assumptions we may apply Theorem 15. Furthermore, the elliptic regularity results for compactly supported Coulomb A discussed in A.3 allow the bounding

$$\|\mathbf{D}A\|_{L^p(M)} \lesssim \|\mathbf{d}A\|_{L^p(M)},$$

which combined with (3.13) completes the proof. □

Similarly, as a corollary to Theorem 17 in section 3.3.1, we obtain Corollary 19 as follows.

Corollary 19. *Let connected compact convex domain* $M \subset \mathbb{R}^m, N, \eta = M \times N,$ G, *and* $\mathfrak{G}$ *be as in Theorem 17.*

Assume $m \geq 2$ *and the Lie group* G *is abelian, connected, and compact.*

Let $1 \leq p < m$, *and* $1 \leq q \leq p^* := \frac{mp}{m-p}$.

There exists $C = C(M, N, p, q) \in (0, \infty)$ *such that for any smooth compactly supported Sobolev connection* $\nabla = \mathbf{d}+A \in \mathfrak{A}_1^p$, *where supp*$(A) \subset\subset M$, *there exists a change of gauge given by some* $\sigma \in \mathscr{D}$ *between connections* ∇ *and* $\nabla^\sigma = \mathbf{d} + \tilde{A}$ *satisfying*

$$\tilde{A} = \sigma^{-1}\mathbf{d}\sigma + \sigma^{-1}A\sigma,$$
$$\|\tilde{A}\|_{L^q(M)} \leq C \left\|\mathbf{d}A\right\|_{L^p(M)}.$$

Proof. Under these assumptions we may apply Theorem 17. Furthermore, the elliptic regularity results discussed in A.3 allow the bounding

$$\|\mathbf{D}A\|_{L^p(M)} \lesssim \|\mathbf{d}A\|_{L^p(M)},$$

which combined with (3.15) completes the proof. □

As we previously discussed in conclusion to the "Main Results" section 3.1, beyond these results, due to the identity

$$\mathbf{d}A = F^{\nabla} - A \wedge A,$$

additional assumptions on the norms of the curvature field F^{∇} and potential A allows one to use Hölder inequalities on the quadratic term $A \wedge A$ to further manipulate inequalities controlled by $\mathbf{d}A$. This style of using Sobolev norm control of the curvature field F^{∇} and quadratic structure $A \wedge A$ is common in the works of Uhlenbeck and Donaldson, where Sobolev control of the quadratic structure is attributed to Palais [18] (Chapter 9); we leave this as a remark only, as ultimately such techniques were not incorporated into this dissertation.

Appendix A

Appendices

A.1 Hodge theory on tensors

We have classic definition of the L^2-adjoint of d denoted $\delta = \mathbf{d}^*$ via Hodge-$\star$. That is,

$$\mathbf{d} : \Omega^p_M \to \Omega^{p+1}_M \qquad \text{with } (\mathbf{vol}_g, \mathbf{vol}_g) = (-1)^s \text{ implies}$$

$$\delta : \Omega^p_M \to \Omega^{p-1}_M, \qquad \delta = \mathbf{d}^* = (-1)^{m(p-1)-1+s} \star \mathbf{d} \star .$$

We then seek to express using Hodge-$\star$ the induced L^2-adjoints of the extension of the covariant derivative

$$\nabla = \mathbf{d} + A, \text{ with } A \in \Omega^1_M(\eta), \text{ acting on } \Omega^0_M(\eta)$$

denoted $\mathbf{d}^\nabla$ which acts on vector-valued p-forms, and the extension of $\tilde{\nabla} = \mathbf{d} + \mathcal{A}$, $\mathcal{A} = [A, \cdot] \in \Omega^1_M(Ad(\eta))$, denoted $\mathbf{d}^{\tilde{\nabla}}$ which acts on matrix-valued p-forms. We will respectively denote the adjoints of the extensions $\delta^\nabla := (\mathbf{d}^\nabla)^*$ and $\delta^{\tilde{\nabla}} := (\mathbf{d}^{\tilde{\nabla}})^*$. That is, we use knowledge of

$$\mathbf{d}^{\nabla} := \mathbf{d} + A \wedge \cdot \qquad \mathbf{d}^{\nabla} : \Omega^p_M(\eta) \to \Omega^{p+1}_M(\eta),$$
$$\mathbf{d}^{\tilde{\nabla}} := \mathbf{d} + [A \wedge \cdot] \qquad \mathbf{d}^{\tilde{\nabla}} : \Omega^p_M(Ad(\eta)) \to \Omega^{p+1}_M(Ad(\eta)),$$

to then describe

$$\delta^{\nabla} : \Omega^p_M(\eta) \to \Omega^{p-1}_M(\eta),$$
$$\delta^{\tilde{\nabla}} : \Omega^p_M(Ad(\eta)) \to \Omega^{p-1}_M(Ad(\eta)).$$

We then establish how Hodge-$\star$ exhibits on bundle-valued forms. We use the orthonormal basis $\{e_\alpha\}$ of $\mathbb{R}^l$ and its dual basis $\{\varepsilon^\beta\}$ when referencing in coordinates.

For $s = s^\alpha e_\alpha = e_\alpha \otimes s^\alpha \in \Omega^p_M(\eta)$, with $s^\alpha \in \Omega^p_M, e_\alpha \in \mathbb{R}^l$, and similarly for $R = r^\alpha_\beta \; e_\alpha \otimes \varepsilon_\beta = e_\alpha \otimes \varepsilon^\beta \otimes r^\alpha_\beta \in \Omega^p_M(Ad(\eta))$, with $r^\alpha_\beta \in \Omega^p_M, e_\alpha \otimes \varepsilon^\beta \in End(\mathbb{R}^l)$, it holds that linear Hodge-$\star$ affects only the forms, not the tensor structure, hence

$$\star s = \star(s^\alpha e_\alpha) = (\star s^\alpha) e_\alpha = e_\alpha \otimes (\star s^\alpha),$$
$$\star R = \star(r^\alpha_\beta \; e_\alpha \otimes \varepsilon^\beta) = e_\alpha \otimes \varepsilon^\beta \otimes (\star r^\alpha_\beta).$$

Additionally, recall that for compactly supported scalar-valued forms $\omega \in \Omega^{p-1}_M, \lambda \in \Omega^p_M$, we have the following computations based on the definition of the L^2-norm of forms. For $\phi, \psi \in \Omega^k_M$ and $\phi \wedge \theta \in \Omega^m_M$, we use identities

$$(\phi, \psi)_g = (-1)^s (\star\phi, \star\psi)_g,$$
$$\phi \wedge \theta = (\star\phi, \theta)_g \mathbf{vol}_g,$$
$$\star\star = (-1)^{k(m-k)+s} \text{ on } \Omega^k_M$$

while using $k = m - p + 1$, implying

$$
\begin{aligned}
(\mathbf{d}\omega, \lambda)_{L^2} &:= \int\limits_M (\mathbf{d}\omega, \lambda)_g \mathbf{vol}_g = \int\limits_M (-1)^s (\star\mathbf{d}\omega, \star\lambda)_g \mathbf{vol}_g = \int\limits_M (-1)^s \mathbf{d}\omega \wedge (\star\lambda) \qquad \text{(A.1)} \\
&= (-1)^s \int\limits_M \mathbf{d}[\omega \wedge (\star\lambda)] - (-1)^{p-1}\omega \wedge \mathbf{d}(\star\lambda) \\
&= 0 + (-1)^{s+p} \int\limits_M \omega \wedge \left[(-1)^{(m-p+1)(p-1)+s} \star\star\mathbf{d}(\star\lambda)\right] \\
&= (-1)^{s+p+m(p-1)-p(p-1)+1(p-1)} \int\limits_M (-1)^s \omega \wedge [\star(\star\mathbf{d}\star\lambda)] \\
&= (-1)^{s+m(p-1)-p(p-1)+2p-1} \int\limits_M (-1)^s \omega \wedge [\star(\star\mathbf{d}\star\lambda)] \\
&= (-1)^{s+m(p-1)-1} \int\limits_M (\omega, \star\mathbf{d}\star\lambda)_g \, \mathbf{vol}_g \\
&= \left(\omega, (-1)^{s+m(p-1)-1} \star\mathbf{d}\star\lambda\right)_{L^2} = (\omega, \delta\lambda)_{L^2} .
\end{aligned}
$$

Not only does

$$\mathbf{d}^* := \delta \equiv (-1)^{s+m(p-1)-1} \star\mathbf{d}\star \text{ on } \Omega^p_M,$$

but this relationship holds on $\Omega^p_M(\eta)$ and $\Omega^p_M(Ad(\eta))$ respectively for $\mathbf{d}^\nabla$ and $\mathbf{d}^{\tilde{\nabla}}$.

We use $\{e_\alpha\}$ an orthonormal basis of $\mathbb{R}^l$ with orthonormal dual basis $\{\varepsilon^\beta\}$. Let compactly supported $R = e_\alpha \otimes \varepsilon^\beta \otimes r^\alpha_\beta \in \Omega^{p-1}_M(Ad(\eta))$ and $S = e_\alpha \otimes \varepsilon^\beta \otimes s^\alpha_\beta \in \Omega^p_M(Ad(\eta))$, so $r^\alpha_\beta \in \Omega^{p-1}_M$ and $s^\alpha_\beta \in \Omega^p_M$.

We note that the inner product $(\cdot,\cdot)_g$ on Ω^p_M (induced by the inner product g on M) combined with the inner products on $\mathbb{R}^l$ and its dual then induces an inner product on $\Omega^p_M(Ad(\eta))$ through a standard argument for inherited inner products on tensor spaces. Hence, for $R \in \Omega^{p-1}_M(Ad(\eta))$ and $S \in \Omega^p_M(Ad(\eta))$ as denoted above, due to orthonormality,

$$\begin{aligned}
(\mathbf{d}R, S)_{L^2} &= \left(e_\alpha \otimes \varepsilon^\beta \otimes \mathbf{d}r^\alpha_\beta, e_a \otimes \varepsilon^b \otimes s^a_b\right)_{L^2} \\
&:= \int_M (e_\alpha, e_a)_{\mathbb{R}^l} (\varepsilon^\beta, \varepsilon^b)_{\mathbb{R}^{l*}} (\mathbf{d}r^\alpha_\beta, s^a_b)_g \mathbf{vol}_g \\
&= \int_M (\mathbf{d}r^\alpha_\beta, s^\alpha_\beta)_g \mathbf{vol}_g \\
&= \int_M (r^\alpha_\beta, (-1)^{s+m(p-1)-1} \star \mathbf{d} \star s^\alpha_\beta)_g \mathbf{vol}_g \\
&= \int_M (e_\alpha, e_a)_{\mathbb{R}^l} (\varepsilon^\beta, \varepsilon^b)_{\mathbb{R}^{l*}} (r^\alpha_\beta, (-1)^{s+m(p-1)-1} \star \mathbf{d} \star s^a_b)_g \mathbf{vol}_g \\
&= \left(R, (-1)^{s+m(p-1)-1} \star \mathbf{d} \star S\right)_{L^2} = (R, \delta S)_{L^2}
\end{aligned}$$

So $\delta = (-1)^{s+m(p-1)-1} \star \mathbf{d}\star$ on $\Omega^p_M(Ad(\eta))$, and by similar arguments also on $\Omega^p_M(\eta)$.

With the L^2 adjoint of $\mathbf{d}$ understood on Ω^p_M, $\Omega^p_M(\eta)$, and $\Omega^p_M(Ad(\eta))$, we now consider the L^2 adjoints of $\mathbf{d}^\nabla = \mathbf{d} + A \wedge \cdot$ and $\mathbf{d}^{\tilde{\nabla}} = \mathbf{d} + [A \wedge \cdot]$ respectively on the latter spaces, respectively denoted δ^∇ and $\delta^{\tilde{\nabla}}$.

Let $r = r^\alpha e_\alpha = e_\alpha \otimes r^\alpha \in \Omega^{p-1}_M(\eta)$ and $s = s^\alpha e_\alpha = e_\alpha \otimes s^\alpha \in \Omega^p_M(\eta)$. For $A = \omega^\alpha_\beta \; e_\alpha \otimes \varepsilon^\beta \in \Omega^1_M(Ad(\eta))$ with $\nabla = \mathbf{d} + A$ on $\Omega^0_M(\eta)$. By using the same identities to establish (A.1), we then have

$$\begin{aligned}
(A \wedge r, s)_{L^2} &= \left(\left(\omega^\alpha_\beta \; e_\alpha \otimes \varepsilon^\beta\right) \wedge \left(r^\beta e_\beta\right), s\right)_{L^2} = \left(e_\alpha \otimes \varepsilon^\beta \otimes \omega^\alpha_\beta \wedge r^\beta e_\beta, s\right)_{L^2} \\
&= \left(e_\alpha \otimes \omega^\alpha_\beta \wedge r^\beta, e_a \otimes s^a\right)_{L^2} = \int_M (e_\alpha, e_a)_{\mathbb{R}^l} \left(\omega^\alpha_\beta \wedge r^\beta, s^a\right)_g \mathbf{vol}_g \\
&= \int_M \left(\omega^\alpha_\beta \wedge r^\beta, s^\alpha\right)_g \mathbf{vol}_g = \int_M (-1)^s \left(\star(\omega^\alpha_\beta \wedge r^\beta), \star s^\alpha\right)_g \mathbf{vol}_g \\
&= \int_M (-1)^s (\omega^\alpha_\beta \wedge r^\beta) \wedge (\star s^\alpha) = \int_M (-1)^s (-1)^{p-1} (r^\beta \wedge \omega^\alpha_\beta) \wedge (\star s^\alpha).
\end{aligned}$$

Then by the antisymmetry of $A \in \Omega^1_M(Ad(\eta))$, which allows compatibility of the covariant derivative with the inner product, and the identity for $\star\star$ on $\Omega^k_M \ni \omega^\alpha_\beta \wedge \star s^\alpha$, where as before we use $k = m - p + 1$, we then may conclude that

$$\begin{aligned}
(A \wedge r, s)_{L^2} &= (-1)^{p-1} \int\limits_M (-1)^s r^\beta \wedge (-1)^{(m-p+1)(p-1)+s} \star\star(-\omega^\beta_\alpha \wedge \star s^\alpha) \\
&= (-1)^{s+m(p-1)-p(p-1)+2(p-1)} \int\limits_M (-1)^s r^\beta \wedge \star\left(\star(-\omega^\beta_\alpha \wedge \star s^\alpha)\right) \\
&= (-1)^{s+m(p-1)} \int\limits_M \left(r^\beta, \star(-\omega^\beta_\alpha \wedge \star s^\alpha)\right)_g \mathbf{vol}_g \\
&= (-1)^{s+m(p-1)-1} \int\limits_M (e_b, e_\beta)_{\mathbb{R}^l} \left(r^b, \star(\omega^\beta_\alpha \wedge \star s^\alpha)\right)_g \mathbf{vol}_g \\
&= (-1)^{s+m(p-1)-1} \left(e_b \otimes r^b, e_\beta \otimes \star(\omega^\beta_\alpha \wedge \star s^\alpha)\right)_{L^2} \\
&= (-1)^{s+m(p-1)-1} \left(e_b \otimes r^b, e_\beta \otimes \varepsilon^\alpha \otimes \star(\omega^\beta_\alpha \wedge \star s^\alpha) e_\alpha\right)_{L^2} \\
&= (-1)^{s+m(p-1)-1} (r, \star A \wedge \star s)_{L^2} = (r, (A\wedge)^* s)_{L^2}\,.
\end{aligned}$$

Hence, on $\Omega^p_M(\eta)$, we have that

$$\begin{aligned}
\delta^\nabla &:= \left(\mathbf{d}^\nabla\right)^* = (\mathbf{d} + A \wedge \cdot)^* = \mathbf{d}^* + (A \wedge \cdot)^* \\
&= \delta + (-1)^{s+m(p-1)-1} \star A \wedge \star = (-1)^{s+m(p-1)-1}(\star\mathbf{d}\star + \star A \wedge \star) \\
&= (-1)^{s+m(p-1)-1} \star \mathbf{d}^\nabla \star\,.
\end{aligned}$$

We then claim on $\Omega^p_M(Ad(\eta))$ we also have that

$$\delta^{\tilde\nabla} := \left(\mathbf{d}^{\tilde\nabla}\right)^* = (\mathbf{d} + \mathcal{A})^* = (\mathbf{d} + [A \wedge \cdot])^* = (-1)^{s+m(p-1)-1} \star \mathbf{d}^{\tilde\nabla} \star\,.$$

Let compactly supported $R = R_J\, \mathbf{d}u^J \in \Omega^{p-1}_M(Ad(\eta))$ and $S = S_K\, \mathbf{d}u^K \in \Omega^p_M(Ad(\eta))$. Furthermore represent $A = A_I\ \mathbf{d}u^I \in \Omega^1_M(Ad(\eta))$ so $\mathbf{d}^{\tilde\nabla} = \mathbf{d} + [A_I\ \mathbf{d}u^I \wedge \cdot]$. Hence, for each ordered multi-index I, J, K, we have lengths $|I| = 1$, $|J| = p-1$ and $|K| = p$, with antisymmetric $A_I, R_J, S_K \in Ad(\eta)$. We then have that

$$
\begin{aligned}
([A \wedge R], S)_{L^2} &= ([A_I \ \mathbf{d}u^I \wedge R_J \ \mathbf{d}u^J], S_K \ \mathbf{d}u^K)_{L^2} \\
&= ([A_I, R_J] \ \mathbf{d}u^I \wedge \mathbf{d}u^J, S_K \ \mathbf{d}u^K)_{L^2} \\
&= \int_M ([A_I, R_J], S_K)_{\mathbb{R}^{l^2}} \, (\mathbf{d}u^I \wedge \mathbf{d}u^J, \mathbf{d}u^K)_g \mathbf{vol}_g \\
&= \int_M ([A_I, R_J], S_K)_{\mathbb{R}^{l^2}} \, (-1)^s \left(\star(\mathbf{d}u^I \wedge \mathbf{d}u^J), \star\mathbf{d}u^K\right)_g \mathbf{vol}_g \\
&= \int_M ([A_I, R_J], S_K)_{\mathbb{R}^{l^2}} \, (-1)^s \mathbf{d}u^I \wedge \mathbf{d}u^J \wedge \star\mathbf{d}u^K \\
&= \int_M ([A_I, R_J], S_K)_{\mathbb{R}^{l^2}} \, (-1)^s (-1)^{p-1} \mathbf{d}u^J \wedge \mathbf{d}u^I \wedge \star\mathbf{d}u^K
\end{aligned}
$$

Consider then by antisymmetry of A_I that

$$
\begin{aligned}
([A_I, R_J], S_K)_{\mathbb{R}^{l^2}} &= ((A_I R_J - R_J A_I), S_K)_{\mathbb{R}^{l^2}} \\
&= \left({A_I}^i_k {R_J}^k_j - {R_J}^i_k {A_I}^k_j\right) {S_K}^i_j \\
&= {A_I}^i_k {R_J}^k_j {S_K}^i_j - {R_J}^i_k {A_I}^k_j {S_K}^i_j \\
&= {R_J}^k_j (-{A_I}^k_i) {S_K}^i_j - {R_J}^i_k {S_K}^i_j (-{A_I}^j_k) \\
&= (R_J, -A_I S_K)_{\mathbb{R}^{l^2}} + (R_J, S_K A_I)_{\mathbb{R}^{l^2}} \\
&= (R_J, S_K A_I - A_I S_K)_{\mathbb{R}^{l^2}} \qquad = -(R_J, [A_I, S_K])_{\mathbb{R}^{l^2}}
\end{aligned}
$$

Hence as before, since $\mathbf{d}u^I \wedge \star\mathbf{d}u^K \in \Omega_M^{m-p+1}$ for each index,

$$
\begin{aligned}
&([A \wedge R], S)_{L^2} \\
&= \int_M ([A_I, R_J], S_K)_{\mathbb{R}^{l^2}} \; (-1)^s (-1)^{p-1} \mathbf{d}u^J \wedge \mathbf{d}u^I \wedge \star \mathbf{d}u^K \\
&= \int_M -(R_J, [A_I, S_K])_{\mathbb{R}^{l^2}} \; (-1)^{s+p-1} \mathbf{d}u^J \wedge (-1)^{(m-p+1)(p-1)+s} \star\star \left(\mathbf{d}u^I \wedge \star \mathbf{d}u^K\right) \\
&= (-1)^p (-1)^{(m-p+1)(p-1)+s} \int_M (R_J, [A_I, S_K])_{\mathbb{R}^{l^2}} \; (-1)^s \mathbf{d}u^J \wedge \star \left(\star \left(\mathbf{d}u^I \wedge \star \mathbf{d}u^K\right)\right) \\
&= (-1)^{m(p-1)-p(p-1)+2p-1+s} \int_M (R_J, [A_I, S_K])_{\mathbb{R}^{l^2}} \left(\mathbf{d}u^J, \star \left(\mathbf{d}u^I \wedge \star \mathbf{d}u^K\right)\right)_g \mathbf{vol}_g \\
&= (-1)^{m(p-1)-1+s} \int_M (R_J, [A_I, S_K])_{\mathbb{R}^{l^2}} \left(\mathbf{d}u^J, \star \left(\mathbf{d}u^I \wedge \star \mathbf{d}u^K\right)\right)_g \mathbf{vol}_g \\
&= (-1)^{m(p-1)-1+s} \left(R_J \; \mathbf{d}u^J, [A_I, S_K] \star \left(\mathbf{d}u^I \wedge \star \mathbf{d}u^K\right)\right)_{L^2} \\
&= (-1)^{m(p-1)-1+s} \left(R_J \; \mathbf{d}u^J, \star [A_I \; \mathbf{d}u^I \wedge \star S_K \; \mathbf{d}u^K]\right)_{L^2} \\
&= (-1)^{m(p-1)-1+s} \left(R, \star [A \wedge \star S]\right)_{L^2} = (-1)^{m(p-1)-1+s} \left(R, [A \wedge \cdot]^* S\right)_{L^2}
\end{aligned}
$$

As claimed, then on $\Omega^p_M(Ad(\eta))$ we also have that the L^2-adjoint of $\mathbf{d}^{\tilde{\nabla}} = \mathbf{d} + [A \wedge \cdot]$ is indeed

$$
\begin{aligned}
\delta^{\tilde{\nabla}} &= \mathbf{d}^* + [A \wedge \cdot]^* = (-1)^{s+m(p-1)-1} \star \mathbf{d} \star + (-1)^{s+m(p-1)-1} \star [A \wedge \star \cdot] \\
&= (-1)^{s+m(p-1)-1} \star \mathbf{d}^{\tilde{\nabla}} \star .
\end{aligned}
$$

and we have established that

$$
\begin{aligned}
&\delta^{\nabla} := (\mathbf{d}^{\nabla})^* = (-1)^{s+m(p-1)-1} \star \mathbf{d}^{\nabla} \star \qquad && \delta^{\nabla} : \Omega^p_M(\eta) \to \Omega^{p-1}_M(\eta) \\
&\delta^{\tilde{\nabla}} := (\mathbf{d}^{\tilde{\nabla}})^* = (-1)^{s+m(p-1)-1} \star \mathbf{d}^{\tilde{\nabla}} \star \qquad && \delta^{\tilde{\nabla}} : \Omega^p_M(Ad(\eta)) \to \Omega^{p-1}_M(Ad(\eta)).
\end{aligned}
$$

A.2 Calculation of $\mathbf{d}^* A = \mathbf{0}$

Let us clarify the condition $\mathbf{d}^* A = \mathbf{0}$. Suppose the connection $\nabla = \mathbf{d} + A$ acting on trivial vector bundle $\eta = M \times N$ is given by connection matrix

$$A = A_j \ \mathbf{d}u^j = e_\alpha \otimes \varepsilon^\beta \otimes \omega^\alpha_\beta = e_\alpha \otimes \varepsilon^\beta \otimes \omega^\alpha_{\beta j} \ \mathbf{d}u^j,$$

with each A_j taking values in fibers $Ad(\eta)_x \cong \mathfrak{G}$, $1 \leq \alpha, \beta \leq n$ with antisymmetric $\omega^\alpha_\beta \in C^\infty(M, Ad(\eta) \otimes T^*M)$, and antisymmetric scalars $\omega^\alpha_{\beta j} \in C^\infty(M)$, that is, $\omega^\alpha_{\beta j} = -\omega^\beta_{\alpha j}$.

Recall $\mathbf{d}^* = (-1)^{s+m(q-1)-1} \star \mathbf{d} \star$ on $\Omega^q_M(Ad(\eta))$. Using the exclusion notation

$$\mathbf{d}u^1 \wedge \ldots \wedge \widehat{\mathbf{d}u^j} \wedge \ldots \wedge \mathbf{d}u^m := \mathbf{d}u^1 \wedge \ldots \wedge \mathbf{d}u^{j-1} \wedge \mathbf{d}u^{j+1} \wedge \ldots \wedge \mathbf{d}u^m,$$

we calculate

$$\begin{aligned}
(-1)^{s-1}\mathbf{d}^* A &= \star\mathbf{d} \star A \\
&= \star\mathbf{d} \star (A_j \ \mathbf{d}u^j) = \star\mathbf{d} \star \left(e_\alpha \otimes \varepsilon^\beta \otimes \omega^\alpha_{\beta j} \ \mathbf{d}u^j\right) \\
&= e_\alpha \otimes \varepsilon^\beta \otimes \left(\star\mathbf{d} \star \omega^\alpha_{\beta j} \ \mathbf{d}u^j\right) \\
&= e_\alpha \otimes \varepsilon^\beta \otimes \left(\star\mathbf{d}\left((-1)^{j+1}\omega^\alpha_{\beta j}\mathbf{d}u^1 \wedge \ldots \wedge \widehat{\mathbf{d}u^j} \wedge \ldots \wedge \mathbf{d}u^m\right)\right) \\
&= e_\alpha \otimes \varepsilon^\beta \otimes \left(\star(-1)^{j+1}\left(\partial_k \omega^\alpha_{\beta j}\mathbf{d}u^k\right) \wedge \mathbf{d}u^1 \wedge \ldots \wedge \widehat{\mathbf{d}u^j} \wedge \ldots \wedge \mathbf{d}u^m\right) \\
&= e_\alpha \otimes \varepsilon^\beta \otimes \left(\star(-1)^{j+1}\partial_j \omega^\alpha_{\beta j}(-1)^{j-1}\mathbf{d}u^1 \wedge \ldots \wedge \mathbf{d}u^j \wedge \ldots \wedge \mathbf{d}u^m\right) \\
&= e_\alpha \otimes \varepsilon^\beta \otimes \left(\star\partial_j \omega^\alpha_{\beta j}\mathbf{vol}_g\right) = e_\alpha \otimes \varepsilon^\beta \otimes \partial_j \omega^\alpha_{\beta j}\left(\star\mathbf{vol}_g\right) \\
&= e_\alpha \otimes \varepsilon^\beta \otimes \partial_j \omega^\alpha_{\beta j}\,(1) \\
&= \partial_j A_j.
\end{aligned}$$

Furthermore recall that for Riemannian M we have $s = 1$ in the above formulation.

A.3 Coulomb ellipticity

The following definition is given by R. Nara [23] in his text as definition 3.3.14.

Definition 4. *Let $P = p(x, D)$ be a linear differential operator of order k between vector bundles E, F over manifold M, respectively of rank r, s. With multi-indices α and partial differential operators $\mathbf{D}^\alpha$, for some $r \times s$ matrices $a_\alpha(x)$, we have representation for $u \in C^\infty(M, E)$*

$$Pu = p(x, D)u = \sum_{|\alpha| \leq k} a_\alpha(x)\mathbf{D}^\alpha u.$$

We then have the principal symbol $\sigma_P(\xi)$ *given by*

$$\sigma_P(\xi) = \sum_{|\alpha|=k} a_\alpha(x)\xi^\alpha.$$

If for each $x \in M, \xi \neq \vec{0}$, the linear map between fibers $\sigma_P(\xi)_x : E_x \to F_x$ is injective, *commonly* invertible, *then we call the operator between bundles E, F* (weakly, overdetermined) **elliptic**.

For matrix-valued 1-forms $A = A_j \, \mathbf{d}u^j$, the differential system

$$\begin{cases} \mathbf{d}^* A &= g \\ \mathbf{d}A &= f \end{cases}$$

is (overdetermined) elliptic. Let us clarify this statement.

The first equation is equivalent to $\partial_j A_j = g$. Taking $f = f_{i,j} \, \mathbf{d}u^i \wedge du^j$ with $i < j$, the second equation is equivalent to the system of $\frac{m(m-1)}{2}$ equations

$$\partial_i A_j - \partial_j A_i = f_{i,j}.$$

Expressing the differential operator in the block matrix form

$$P(D)A = \sum_{1\le k\le m} B_k\,\partial_k \begin{bmatrix} A_1 \\ \vdots \\ A_m \end{bmatrix} = \begin{bmatrix} \partial_j A_j \\ \hline \uparrow \\ \partial_i A_j - \partial_j A_i \\ \downarrow \end{bmatrix} \cong \begin{bmatrix} \mathbf{d}^* A \\ \hline \mathbf{d} A \end{bmatrix}$$

leads to each B_k being a matrix of size $1+\frac{m(m-1)}{2}$ rows and m columns.

That is, to let $P(D) = \sum_{1\le k\le m} B_k\,\partial_k$ we need

$$P(D) = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ \hline 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ \hline & & \mathbf{0} & & \\ \hline & & \vdots & & \\ \hline & & \mathbf{0} & & \end{bmatrix} \partial_1 + \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ \hline -1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \\ \hline 0 & 0 & 1 & \dots & 0 \\ 0 & 0 & 0 & 1\dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ \hline & & \mathbf{0} & & \\ \hline & & \vdots & & \\ \hline & & \mathbf{0} & & \end{bmatrix} \partial_2 + \cdots$$

$$
\cdots + \left[\begin{array}{ccccccc}
0 & 0 & & \cdots & & 0 & 1 \\
\hline
0 & 0 & & \cdots & & 0 & 0 \\
0 & 0 & & \cdots & & 0 & 0 \\
\vdots & \vdots & & \ddots & & \vdots & \vdots \\
-1 & 0 & & \cdots & & 0 & 0 \\
0 & 0 & & \cdots & & 0 & 0 \\
\hline
\multicolumn{7}{c}{\mathbf{0}} \\
\hdashline
0 & -1 & 0 & \cdots & \cdots & \cdots & 0 \\
0 & 0 & 0 & \cdots & \cdots & \cdots & 0 \\
\hline
\multicolumn{7}{c}{\mathbf{0}} \\
\hdashline
0 & 0 & -1 & 0 & \cdots & \cdots & 0 \\
0 & 0 & 0 & 0 & \cdots & \cdots & 0 \\
\hline
\multicolumn{7}{c}{\vdots} \\
\hline
0 & \cdots & \cdots & 0 & -1 & 0 & 0 \\
0 & \cdots & \cdots & \cdots & 0 & 0 & 0 \\
\hline
0 & \cdots & \cdots & \cdots & 0 & 0 & 1
\end{array}\right] \partial_{m-1} + \left[\begin{array}{ccccccc}
0 & 0 & & \cdots & & 0 & 1 \\
\hline
0 & 0 & & \cdots & & 0 & 0 \\
0 & 0 & & \cdots & & 0 & 0 \\
\vdots & \vdots & & \ddots & & \vdots & \vdots \\
0 & 0 & & \cdots & & 0 & 0 \\
-1 & 0 & & \cdots & & 0 & 0 \\
\hline
\multicolumn{7}{c}{\mathbf{0}} \\
\hdashline
0 & 0 & 0 & 0 & \cdots & \cdots & 0 \\
0 & -1 & 0 & \cdots & \cdots & \cdots & 0 \\
\hline
\multicolumn{7}{c}{\mathbf{0}} \\
\hdashline
0 & 0 & 0 & 0 & \cdots & \cdots & 0 \\
0 & 0 & -1 & 0 & \cdots & \cdots & 0 \\
\hline
\multicolumn{7}{c}{\vdots} \\
\hline
0 & \cdots & \cdots & \cdots & 0 & 0 & 0 \\
0 & \cdots & \cdots & \cdots & -1 & 0 & 0 \\
\hline
0 & \cdots & \cdots & \cdots & 0 & -1 & 0
\end{array}\right] \partial_m,
$$

hence the principal symbol is the block matrix

$$\sigma_P(\xi) = \left[\begin{array}{cccccc}
\xi^1 & \xi^2 & \xi^3 & \xi^4 & \dots & \xi^m \\
\hline
\multicolumn{1}{c|}{-\xi^2} & & & & & \\
\multicolumn{1}{c|}{-\xi^3} & & & & & \\
\multicolumn{1}{c|}{-\xi^4} & & & & & \\
\multicolumn{1}{c|}{\vdots} & & & & \xi^1 I_{m-1} & \\
\multicolumn{1}{c|}{-\xi^m} & & & & & \\
\hline
0 & \multicolumn{1}{c|}{-\xi^3} & & & & \\
0 & \multicolumn{1}{c|}{-\xi^4} & & & & \\
\vdots & \multicolumn{1}{c|}{\vdots} & & & \xi^2 I_{m-2} & \\
0 & \multicolumn{1}{c|}{-\xi^m} & & & & \\
\hline
0 & 0 & \multicolumn{1}{c|}{-\xi^4} & & & \\
\vdots & \vdots & \multicolumn{1}{c|}{\vdots} & & \xi^3 I_{m-3} & \\
0 & 0 & \multicolumn{1}{c|}{-\xi^m} & & & \\
\hline
 & \vdots & & & \vdots & \\
\hline
0 & 0 & 0 & \dots \quad \multicolumn{1}{c|}{-\xi^{m-1}} & \xi^{m-2} & 0 \\
0 & 0 & 0 & \dots \quad \multicolumn{1}{c|}{-\xi^m} & 0 & \xi^{m-2} \\
\hline
0 & 0 & 0 & \dots \quad 0 & \multicolumn{1}{c|}{-\xi^m} & \xi^{m-1}
\end{array}\right]$$

of size $(1 + m(m-1)/2) \times m$. For nonzero ξ, we have nonzero ξ^k for some least k with $1 \leq k \leq m$. Then the symbol $\sigma_P(\xi)$ is row-equivalent to some block matrix $\begin{bmatrix} \mathbb{M} \\ \mathbb{E} \end{bmatrix}$ where

$\mathbb{M}$ is a block matrix of the form

$$\mathbb{M} = \left[\begin{array}{c|c} \xi^k I_k & \begin{matrix} \mathbf{0} \\ \xi^{k+1} \quad \dots \quad \xi^m \end{matrix} \\ \hline \begin{matrix} & -\xi^{k+1} \\ \mathbf{0} & \vdots \\ & -\xi^m \end{matrix} & \xi^k I_{m-k} \end{array}\right].$$

Then $\mathbb{M}$ has a nonzero determinant

$$\begin{aligned} det(\mathbb{M}) &= (\xi^k)^{k-1} \; det \left[\begin{array}{c|ccc} \xi^k & \xi^{k+1} & \dots & \xi^m \\ \hline -\xi^{k+1} & \xi^k & & \mathbf{0} \\ \vdots & & \ddots & \\ -\xi^m & \mathbf{0} & & \xi^k \end{array}\right] \\ &= (\xi^k)^{k-1} \left(\sum_{j=k}^{m} (\xi^j)^2 \; det(\xi^k I_{m-k-1}) \right) = (\xi^k)^{m-2} \sum_{j=k}^{m} (\xi^j)^2 \geq (\xi^k)^m > 0, \end{aligned}$$

therefore $\mathbb{M}$ is invertible, and the kernel of $\sigma_P(\xi)$ for nonzero ξ is trivial, making $\sigma_P(\xi)$ an injective mapping.

Therefore, as for nonzero ξ the symbol $\sigma_P(\xi)$ is injective, we conclude that the differential operator $P(D)$ is (overdetermined) elliptic by Definition 4, and the system

$$\begin{cases} \mathbf{d}^* A &= g \\ \mathbf{d} A &= f \end{cases}$$

admits appropriate elliptic regularity, such as $\|\mathbf{D}A\|_{L^p} \lesssim \|g\|_{L^p} + \|f\|_{L^p}$.

When one imposes further that A is Coulomb, with $\mathbf{d}^* A = \mathbf{0} = g$, then we have the equivalent system written in terms of the Hodge-Laplacian $\Delta_H := \mathbf{d}^*\mathbf{d} + \mathbf{d}\mathbf{d}^*$ of

$$\Delta_H A = \mathbf{d}^* f,$$

which admits elliptic regularity for $A \in L_1^p(M)$ of the form

$$\|\mathbf{D}A\|_{L^p(M)} \lesssim \|f\|_{L^p(M)} = \|\mathbf{d}A\|_{L^p(M)}$$

for bounded convex M with Dirichlet or Neumann boundary conditions for A on the Lipschitz ∂M, as a special case of elliptic regularity. This type of elliptic regularity result used by Uhlenbeck [4] in her Lemma 2.5.

Furthermore, there are additional regularity results such as

$$\|\mathbf{D}A\|_{L^p(M')} \lesssim \|f\|_{L^p(M')} + \|A\|_{L^1(M')} = \|\mathbf{d}A\|_{L^p(M')} + \|A\|_{L^1(M')}$$

for convex subdomains $M' \subset M$ with no boundary conditions on $\partial M'$ for A, and

$$\|\mathbf{D}A\|_{L^p(\mathbb{R}^m)} \lesssim \|f\|_{L^p(\mathbb{R}^m)} = \|\mathbf{d}A\|_{L^p(\mathbb{R}^m)}$$

when $M = \mathbb{R}^m$. Estimates such as these are discussed in the appendices of Donaldson and Kronheimer's text [6], statements (A7)-(A8) and Remark (iii) on pages 422-423.

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