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Stochastic Flocking and Its Application to Systemic Risk with Jumps

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Abstract

Stochastic Flocking and Its Application to Systemic Risk with Jumps

Yi-Tai Chiu

In this dissertation, we first study the effect of jumps on a stochastic flocking model and discuss characteristics of this model. We investigate its application to understand systemic risk by proposing an interbank lending model with jump diffusions and further show that there will be a higher systemic risk with jumps (i.e., sudden increases or decreases in reserves) in our model. Then, to examine how the systemic risk will be affected when each bank is acting toward their best self-interest, we integrate a game feature with jumps where each bank controls its rate of borrowing/lending to a central bank. We then solve Nash equilibria with finitely many players in this game with jumps, within which the central bank acts as a clearing house and adds liquidity to the system. The result indicates that the linear growth contributed by jumps to the system does not affect the systemic risk in the model. Finally, we propose another model with a central bank as well as peripheral banks and investigate the impact of interaction between all banks on systemic risk. The systemic risk might be reduced if the central bank is allowed to monitor liquidity by solving an optimal control problem. We also provide a mean-field game approach to approximate the equilibria for finitely many players.

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Chapter 1

Introduction

1.1 Stochastic flocking with jumps

Flocking is a natural behavior exhibited by a group of birds traveling in unison during flight. It can also be used to describe similar group behavior of other species such as insects or fish and is recently discussed in several papers [Aoki, 1982], [Partridge, 1982], [Toner and Tu, 1998], [Milewski and Yang, 2008] and [Degond and Motsch, 2008]. Flocking behavior is based on two simple rules: velocity alignment and group formation. Velocity alignment suggests that individuals in a group are moving towards the weighted average speed of their neighbors according to the interaction in the group. Group formation shows that individuals are traveling within a certain range. In other words, a group of individuals that demonstrates flocking behavior will be steering towards the same direction within a parallel timeframe.

Cucker and Smale proposed a deterministic model that can capture the flocking behavior without the stochastic noise term in [Cucker and Smale, 2007]. In [Ha et al., 2009], the authors introduce the stochastic Cucker-Smale (SCS) model and study the time-asymptotic flocking in the SCS model where the noise term is driven by Brownian motions. Previous studies that model flocking behavior such as [Ha et al., 2009] do not deal with the case that allows sudden dramatic increase or decrease in the velocity of particles. In this study, we apply a stochastic model to illustrate flocking behavior where we treat each bird as a random particle in space. We denote the coordinates (x_t^i, v_t^i), to be the position and velocity that describe the flocking behavior mathematically (see Chapter 2). We expect that the flocking criteria will hold when we replace the noise term Brownian motions with Levy processes (i.e. the velocity of particles is allowed to decrease or increase dramatically).

1.2 Systemic risk with jumps

Since the 2008 financial crisis, in particular, after the Lehman Brothers Bankruptcy, modeling risk in banking system and preventing the systemic risk has become an important topic. Many researchers have investigated systemic risk and its impact on the banking market [Carmona et al., 2014], [Garnier et al., 2013], [Fouque and Ichiba, 2013], [Ichiba dn Shkolnikov, 2013], [Capponi and Chen, 2013] and [Bo and Capponi, 2015]. There are two major approaches for modeling systemic risk. One approach to model systemic risk is by describing the system failures due to contagion of counterparty risk in a financial network as stated in [Acemoglu et al., 2013]. Another approach is through diffusion based models as studied in [Fouque and Ichiba, 2013] and [Garnier et al., 2013]. More topics about systemic risk has also been studied in [Fouque and Langsam, 2013]. In this study, we will discuss systemic risk via diffusion based modeling.

Systemic risk is defined in [Fouque and Sun, 2013] as the risk that all banks will bankrupt simultaneously for a given horizon time. It is a rare event that may occur under certain circumstances and sometimes lead to catastrophic consequences. In this study, our goal is to establish a mathematical model to characterize the banking system and the effect on systemic risk when each bank has instantaneous supply and demand shocks, sudden bankruptcy, or positive or negative news in the market. For example, the event that the central bank of China greatly reduces the bank reserve requirements might increase the systemic risk of China's entire banking system. To get a more in-depth understanding of systemic risk that may help us to navigate and further prevent such events, we propose an interbank borrowing and lending model that allows sudden increases or decreases in individual bank reserves to illustrate systemic risk by using a stochastic flocking system. In addition, to explore if systemic risk will be affected when each bank is acting solely in their best interest, we create another model based on the assumption that each bank controls its rate of borrowing/lending to a central bank. The result of the research aims to inform future practice in understanding and modeling systemic risk.

This study is built on the research result of [Carmona et al., 2014]. The authors propose a mathematical model to characterize systemic risk and further discuss and analyze systemic risk using game features and mean field games. However, the above research does not discuss the impact of jumps to systemic risk. In this study, we first illustrate the characteristics of systemic risk by applying a dynamic of stochastic flocking system with jumps. Then, we extend the previous research on systematic risk and investigate the effect on systemic risk when each bank is allowed to have instantaneous shocks or sudden bankruptcy.

In a fixed time period, a systemic risk is characterized by a large number of banks reaching a certain critical level. Here we anticipate a small probability that every bank will bankrupt simultaneously. In order to obtain this probability, we need an explicit formula to calculate the distribution of the first passage time for a stochastic process with jumps. However, such explicit formula is hard to obtain. Therefore, we use the Laplace transform approach and the inversion formula to calculate this probability. Our model integrates a game feature with jumps where each bank controls its rate of borrowing/lending to a central bank. We use game theory and stochastic optimal control with jump processes to analyze the impact of jumps to our interbank borrowing and lending model.

1.3 Outline of the dissertation

In Chapter 2, we briefly introduce the Cucker-Smale model as stated in [Ha et al., 2009] and review the background of Lévy processes needed for stochastic flocking with jumps. We then propose a stochastic Cucker-Smale (SCS) model with jumps and further investigate the time-asymptotic flocking in both linear and radially symmetric communication rate cases. We provide an estimate of fluctuation of all particles for characterizing the flocking behavior in a jump phenomenon. We also give some simulation results of the SCS model to illustrate flocking behavior. At the end of this Chapter 2, we propose a new flocking model with a specified central particle that lays the foundation for building an interbank system with a central bank that we will discuss in Chapter 5.

In Chapter 3, we start with a systemic risk model with N banks similar to [Fouque and Sun, 2013] and further investigate its mean field limit. Then, we study the systemic risk with jumps through simulating the reserve processes and obtain a formula for the systemic risk through an inversion Laplace transform of the distribution of the first passage time.

In Chapter 4, we investigate the stochastic differential game with jumps as well as its impact on systemic risk. In section 4.1, we give the needed background material on stochastic optimal control with jump diffusions for obtaining the Nash equilibrium in a stochastic differential game. In section 4.2, we search for a feedback equilibria by using both a forward-backward stochastic differential equation

(FBSDE) and HJB approach. We then provide some numerical results and discuss the effect of jumps on systemic risk with combining a game feature.

In Chapter 5, we propose a new model with a central bank and peripheral banks. We study their interactions between each other and how this interactions affect systemic risk. Then, we solve a control problem that the central bank can now control its rate of interactions and investigate the result with a solved optimal response. We also give a possible approach to approximate the Nash equilibrium in a finite player game with a central bank.

Chapter 2

Stochastic Flocking Model

2.1 Introduction

There has been much research about *flocking, schools or swarming* [Aoki, 1982], [Aoki, 1982], [Partridge, 1982] and [Toner and Tu, 1998], yet only a few of them discuss stochastic flocking. In nature, flocking is a phenomenon demonstrated by a group of birds flying and swooping in a highly coordinated union. In this chapter, we consider stochastic flocking as a behavior exhibited by a group of particles in which each particle mirrors or shadows each other's movement while interacting with dramatic, random environmental factors that we identify as jumps.

In [Cucker and Smale, 2007], the authors proposed a deterministic flocking model that considers a group of birds in which each bird alters its velocity towards the average of its neighbors' velocities. Let $(x_t^i, v_t^i) \in R^{2d}$ for $i = 1, ..., N$ be the position and the velocity of birds, respectively. The Cucker-Smale (C-S) model is described as:

$$\begin{aligned} dv_t^i &= \frac{\alpha}{N}\sum_{j=1}^{N}\phi\left(x_t^j, x_t^i\right)\left(v_t^j - v_t^i\right)dt \\ dx_t^i &= v_t^i dt, \quad i = 1, ..., N, \end{aligned}$$

where $\alpha \geq 0$ is the coupling strength and ϕ is a nonnegative function defined as the communication rate which satisfies the following conditions:

$$\begin{aligned} &(symmetry) && \phi\left(x_t^i, x_t^j\right) = \phi\left(x_t^j, x_t^i\right), \ i = 1, .., N, \\ &(translation\ invariance) && \phi\left(x_t^i + M, x_t^j + M\right) = \phi\left(x_t^i, x_t^j\right), \ \text{for } M \in R^d. \end{aligned}$$

Given the positions of two brids, the communication rate ϕ in [Cucker and Smale, 2007] depends on the distance between two birds in a space ($d = 3$). The closer the physical distance of two birds are, the stronger the communication rate between the two will be. More precisely, ϕ is defined as a non-increasing function of the distance. In [Cucker and Smale, 2007], they show that under certain circumstances, flocking will emerge based on the initial configuration. The C-S model has been applied to different fields, especially in physics [Carrillo et al., 2010], [Ha et al., 2014] and [Li and Xue, 2014].

In [Ha et al., 2009], they extend the C-S model and propose a new stochastic flocking system by adding noise terms into the dynamics of velocity. In this new stochastic flocking system, each bird exhibits interactions with the random environmental factors. In addition, the velocity of each bird is influenced by noise terms that varies over time. However, the noise terms in [Ha et al., 2009] do not include jumps—the primary focus of this Chapter. The velocity of any bird may increase or decrease dramatically in the presence of random jumps, which may further cause the failure of the flocking behavior. Our goal in this chapter is to investigate whether the flocking behavior can be anticipated with random noises including jumps. Jump processes are a type of Lévy process. In the next section, we give some background on Lévy processes.

2.2 Background of Lévy processes

In this section, we review the basic concepts and results of Lévy processes needed to understand stochastic flocking with jumps. A Lévy process is a stochastic process that possesses continuous or discontinuous paths [Applebaum, 2004], [Bertoin, 1998], [Sato, 1999]. They have been widely applied in many fields including finance [Cont and Tankov, 2003] and engineering [Kyprianou, 2013]. The definition of Lévy processes is as follows:

Definition 2.1. *A one dimensional stochastic process* $L = (L_t, t \geq 0)$ *defined on a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *is said to be a Lévy process if it has the following properties:*

1. $L_0 = 0$ *almost surely.*

2. L *has stationary and independent increments.*

3. L is continuous in probability. i.e.,

$\forall \epsilon > 0 \quad and\ s \geq 0, \lim_{t \to s} \mathbb{P}(|L_t - L_s| > \epsilon) = 0.$

To understand more about Lévy processes, we provide the following theorem that illustrates the characteristic function of Lévy processes.

Theorem 2.1. (*Lévy-Khintchine formula for Lévy processes*) *Suppose there exists a triplet* (a, σ, ρ), *where* $a \in R$, $\sigma \geq 0$ *and* ρ *is a measure satisfying*

$$\rho(\{0\}) = 0 \text{ and } \int_R \left(1 \wedge x^2\right) \rho(dx) < \infty.$$

Then there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ *on which a Lévy process has the following characteristic function:*

$$\mathbb{E}\left(e^{i\theta L_t}\right) = e^{-t\Psi(\theta)} \text{ for all } t \geq 0,$$

where Ψ *is defined as*

$$\Psi(\theta) = ia\theta + \frac{1}{2}\sigma^2\theta^2 + \int_R \left(1 - e^{i\theta x} + i\theta x \mathbf{1}_{(|x|<1)}\right) \rho(dx), \text{ for any } \theta \in R.$$

The proof can be found in [Bertoin, 1996]. Measure ρ is called the Lévy measure that describes the sizes and rates for which jumps of Lévy process occurs. We call (a, σ, ρ) the generating triplet of L_t. The quantities a, σ and ρ are the drift, the Gaussian variance and the Lévy measure of L_t^i, respectively. One can observe that a standard Brownian motion is one of the Lévy processes where the triplet is given by $(0, 1, 0)$. The following is a classic example of Lévy processes:

Example 2.1. (*Compound Poisson processes*) *Let*

$$L_t = \sum_{i=1}^{N_t} \xi_i,$$

where N_t *is a Poisson process with intensity* λ, *and jump sizes* $\{\xi_i, i \geq 1\}$ *are i.i.d. random variables with common law* F, *independent from* N_t. *The characteristic function of* L_t, *for* $\theta \in R$, *is given by*

$$\mathbb{E}\left[e^{i\theta \sum_{i=1}^{N_t} \xi_i}\right] = e^{-\lambda t \int_R \left(1-e^{i\theta x}\right) F(dx)} = e^{-t\Psi(\theta)}.$$

where $\Psi(\theta) = \lambda \int_R \left(1 - e^{i\theta x}\right) F(dx)$.

The triplet (a, σ, ρ) is then given by $a = -\lambda \int x \mathbf{1}_{(|x|<1)} F(dx)$, $\sigma = 0$ and $\rho(dx) = \lambda F(dx)$.

For discontinuous Lévy processes at $t \geq 0$, we introduce the jump part of L_t which is defined as

$$\Delta L_t = L_t - L_{t^-}.$$

Let B be the family of Borel sets $U \subset \mathbb{R}$ and $\{0\} \notin \bar{U}$. For any $U \in B$ we define

$$N(t, U) = N(t, U, \omega) = \sum_{0<s<t} \chi_U(\Delta L_s).$$

$N(t, U)$ is called the Poisson random measure which is the number of jumps of size $\Delta L_s \in U$ which occur before or at time t. The following theorem indicates that any Lévy process can be decomposed into the sum of continuous and discontinuous stochastic processes.

Theorem 2.2. (*Lévy-Ito decomposition*) *Let* $\{L_t\}_{t\geq 0}$ *be a Lévy process. Suppose that* $a \in R$, $\sigma \geq 0$ *and* ρ *is a measure satisfying*

$$\rho(\{0\}) = 0 \text{ and } \int_R \left(1 \wedge x^2\right) \rho(dx) < \infty.$$

We then have

$$L_t = at + \sigma W_t + \int_{|x|\leq 1} x\tilde{N}(t, dx) + \int_{|x|>1} xN(t, dx),$$

$$\text{where} \quad \tilde{N}(t, dx) = N(t, dx) - t\rho(dx) \quad \text{is a compensated martingale.}$$

The proof of the above theorem can be found in chapter 2 in [Kyprianou, 2006]. Similar to the Ito formula for stochastic processes with continuous paths, we have a more general formula for Lévy processes in the one dimension case, i.e., when $d = 1$.

Theorem 2.3. (*Ito's formula*) *Let* $\{L_t\}_{t\geq 0}$ *be a Lévy process with triplet* (a, σ, ρ) *and* $f : R \to R$ *a* C^2 *function. Then*

$$\begin{aligned} f(L_t) &= f(0) + \int_0^t \frac{\sigma^2}{2} f''(L_s)\, ds + \int_0^t f'(L_{s^-})\, dL_s \\ &\quad + \sum_{\substack{0\leq s\leq t \\ \Delta L_s \neq 0}} \left[f(L_{s^-} + \Delta L_s) - f(L_{s^-}) - \Delta L_s f'(L_{s^-})\right]. \end{aligned}$$

Proof of this can be found in [Cont and Tankov, 2004]. Note that the Ito's formula for Lévy processes is different from the one for Brownian motion only in the last term that involves the sum of each discontinuity of Lévy processes. In the next section, we will examine further on the stochastic flocking model.

2.3 Stochastic Cucker-Smale (SCS) flocking model

In this section, we will give an estimate of the fluctuation of all particles in the stochastic system with random jumps. We will study the interactions between all particles and the environment with random noise and with jumps. Our main goal is to extend the result of [Ha et al., 2009] which is a stochastic version of C-S model driven by Brownian motions. We replace the noise terms, Brownian motions, in [Ha et al., 2009] with Lévy processes and investigate the time-asymptotic flocking within such stochastic particle systems. Within the nonlinear stochastic system, we only discuss the case of $d = 1$ for the purpose of simplicity. Similar to the C-S model, we consider a nonlinear system with N autonomous particles with the pair $(x_t^i, v_t^i) \in R^2, i = 1, .., N$, where x_t^i and v_t^i are regarded as the position and the velocity of birds, respectively.

The SCS model with jumps states the following:

$$\begin{aligned} dv_t^i &= \frac{\alpha}{N} \sum_{j=1}^{N} \phi \left(x_t^j, x_t^i\right) \left(v_t^j - v_t^i\right) dt + dL_t^i \\ dx_t^i &= v_t^i dt, \quad i = 1, ..., N, \end{aligned} \tag{2.1}$$

where L_t^i is a Lévy process with the triplet (a, σ, ρ^i). In spite of the importance of the initial configuration in the C-S model, we assume that the initial condition is a deterministic value, i.e., $(x_0^i, v_0^i) = (x_0, v_0)$, $i = 1, .., N$, since such conditions will not affect our analysis in time-asymptotic flocking.

Note that for every $t > 0$, L_t^i are i.i.d. random variables. In addition, for $i = 1, ..., N$, the Lévy-Ito decomposition yields

$$L_t^i = at + \sigma W_t^i + \int_{|x|<1} x\tilde{N}^i(t, dx) + \int_{|x|\geq 1} xN^i(t, dx),$$

where $\tilde{N}^i(t, dx) = N^i(t, dx) - t\rho^i(dx)$ is a compensated Poisson random measure and ρ^i is a Lévy measure that satisfies $\int_R (1 \wedge x^2)\,\rho^i(dx) < \infty$.

Remark: In the case studied in [Ha et al., 2009], the noise terms are only Brownian motions. Here, we assume that each particle has distinct jumps based on the Levy measure ρ^i which might lead to the failure of flocking. We will study this situation in the following section.

In the SCS model with jumps, the velocity of each bird is now influenced by the environment. Moreover, some birds may speed up dramatically at some moment in response to the random environmental factors such as the appearance of an attractive target or by other environmental factors. We provide the definition of time-asymptotic flocking in [Ha et al., 2009] here:

Definition 2.2. *We say there is time-asymptotic flocking for a group of birds if the position and velocity processes* (x_t^i, v_t^i), $i = 1, ..., N$, *satisfy the follow conditions:*

$$\begin{aligned} &(\textit{velocity alignment}) \quad \lim_{t\to\infty} \left|\mathbb{E}\left(v_t^i\right) - \mathbb{E}\left(v_t^j\right)\right| = 0 \\ &(\textit{group formation}) \quad \sup_{0\leq t<\infty} \left|\mathbb{E}\left(x_t^i\right) - \mathbb{E}\left(x_t^j\right)\right| < \infty. \end{aligned} \tag{2.2}$$

Note that the SCS model is a nonlinear system and is difficult to analyze without an explicit form of the communication rate ϕ. In the next section, with a simple explicit form of ϕ, such as considering ϕ as a constant, the SCS model can be reduced to an Ornstein-Uhlenbeck type process which then becomes more tractable. Moreover, under some boundedness condition on ϕ, it is also possible to analyze this nonlinear stochastic system which we will discuss in section 2.4.

2.3.1 Linear communication rate

Assuming that $\phi = 1$, without loss of generality, the SCS model is given by

$$\begin{aligned} dv_t^i &= \frac{\alpha}{N}\sum_{j=1}^{N}\left(v_t^j - v_t^i\right) dt + dL_t^i \\ &= \alpha\left(\bar{v}_t - v_t^i\right) dt + dL_t^i \qquad (2.3) \\ dx_t^i &= v_t^i dt, \quad i = 1, ..., N, \end{aligned}$$

where $\bar{v}_t = \sum_{i=1}^{N} v_t^i$ is defined as the ensemble average and the L_t^i are independent Lévy processes for $i = 1, ..., N$. From the dynamics of the velocity v_t^i, one can see that each v_t^i is now attracted to ensemble average $\bar{v}_t$ which is an Ornstein-Uhlenbeck Lévy-type process. In oder to show the stochastic system follows the flocking criteria (2.2), we follow the analysis in [Ha et al., 2009] where the authors define the following two variables: ensemble average (*macro variable*) and the fluctuations with respect to the ensemble average (*micro variable*), named $(\bar{x}_t, \bar{v}_t)$ and $(\hat{x}_t^i, \hat{v}_t^i)$, respectively.

We define

$$\begin{aligned} \text{(Macro variable)} \quad \bar{x}_t &= \frac{1}{N}\sum_{i=1}^{N} x_t^i, \bar{v}_t = \frac{1}{N}\sum_{i=1}^{N} v_t^i, \\ \text{(Micro variable)} \quad \hat{x}_t^i &= x_t^i - \bar{x}_t, \quad \hat{v}_t^i = v_t^i - \bar{v}_t. \end{aligned}$$

Note that the micro variable $(\hat{x}_t^i, \hat{v}_t^i)$ satisfies $\frac{1}{N}\sum_{i=1}^N \hat{x}_t^i = \frac{1}{N}\sum_{i=1}^N (x_t^i - \bar{x}_t) = 0$ and $\frac{1}{N}\sum_{i=1}^N \hat{v}_t^i = 0$. The dynamics for the ensemble average $(\bar{x}_t, \bar{v}_t)$ are then given by

$$\begin{aligned} d\bar{v}_t &= \frac{1}{N}\sum_{i=1}^N dL_t^i, \\ d\bar{x}_t &= \bar{v}_t dt, \end{aligned}$$

and the dynamics for the micro variable $\hat{v}_t^i$ are

$$\begin{aligned} d\hat{v}_t^i &= dv_t^i - d\bar{v}_t \\ &= \frac{\alpha}{N}\sum_{j=1}^N \left(\hat{v}_t^j - \hat{v}_t^i\right) dt + dL_t^i - \frac{1}{N}\sum_{i=1}^N dL_t^i, \\ &= \left[\frac{\alpha}{N}\sum_{j=1}^N \hat{v}_t^j - \alpha\hat{v}_t^i\right] dt + d\left[L_t^i - \frac{1}{N}\sum_{i=1}^N L_t^i\right] \end{aligned}$$

Since $\frac{\alpha}{N}\sum_{j=1}^N \hat{v}_t^j = \frac{\alpha}{N}\sum_{j=1}^N \left(v_t^j - \bar{v}_t\right) = 0$, the dynamics for $\hat{v}_t^i$ can be reduced to

$$\begin{aligned} d\hat{v}_t^i &= -\alpha\hat{v}_t^i dt + d\left[L_t^i - \frac{1}{N}\sum_{i=1}^N L_t^i\right] \\ &= -\alpha\hat{v}_t^i dt + dZ_t^i, \end{aligned}$$

where $Z_t^i = \left(1 - \frac{1}{N}\right) L_t^i - \frac{1}{N}\sum_{j\neq i}^N L_t^j$, which is a Lévy-type OU process. Therefore, the dynamics for the micro variable $(\hat{x}_t^i, \hat{v}_t^i)$ are given by

$$\begin{aligned} d\hat{v}_t^i &= -\alpha\hat{v}_t^i dt + dZ_t^i \\ d\hat{x}_t^i &= \hat{v}_t^i dt. \end{aligned}$$

Note that the processes $\{L_t^i, 1 \leq i \leq N\}$ are independent, but the processes $\{Z_t^i, 1 \leq i \leq N\}$ are now correlated through the ensemble noises.

As investigated in [Sato, 1999], it is easy to check that the solution $\hat{v}_t^i$ with initial condition $\hat{v}_0^i$ is given by

$$\hat{v}_t^i = e^{-\alpha t}\hat{v}_0^i + \int_0^t e^{-\alpha(t-s)} dZ_t^i. \tag{2.4}$$

We now consider the difference defined as $\hat{v}_t^{i,j} = \hat{v}_t^i - \hat{v}_t^j$ and $\hat{x}_t^{i,j} = \hat{x}_t^i - \hat{x}_t^j$ $, \forall i \neq j \in \{1, ..., N\}$. Then, we have

$$\begin{aligned} \hat{v}_t^{i,j} &= e^{-\alpha t}\hat{v}_0^{i,j} + \int_0^t e^{-\alpha(t-s)} dL_s^{i,j}, \\ \hat{x}_t^{i,j} &= \hat{x}_0^{i,j} + \int_0^t \hat{v}_s^{i,j} ds, \end{aligned} \tag{2.5}$$

where $L_s^{i,j} = L_s^i - L_s^j$.

Lemma 2.1. *Assume that there exists a constant $\mu^i \in \mathbb{R}$ for all $i = 1, ..., N$ such that $\int_{|x|\geq 1} x\rho^i(dx) = \mu^i < \infty$, and let $\mu^{i,j} = \mu^i - \mu^j$. We have the following estimates for $\hat{x}_t^{i,j}$ and $\hat{v}_t^{i,j}$:*

$$\begin{aligned} \mathbb{E}\left[\hat{v}_t^{i,j}\right] &= e^{-\alpha t}\mathbb{E}\left[\hat{v}_0^{i,j}\right] + \frac{\mu^{i,j}}{\alpha}\left(1 - e^{-\alpha t}\right) \text{ and} \\ \mathbb{E}\left[\hat{x}_t^{i,j}\right] &= \mathbb{E}\left[\hat{x}_0^{i,j}\right] + \mathbb{E}\left[\hat{v}_0^{i,j}\right]\frac{\left(1 - e^{-\alpha t}\right)}{\alpha} + \mu^{i,j}\left[\frac{t}{\alpha} + \frac{1}{\alpha^2}\left(e^{-\alpha t} - 1\right)\right]. \end{aligned}$$

Proof. By the Lévy-Ito decomposition, we know that $L_t^i = at + \sigma W_t^i + \int_{|x|<1} x\tilde{N}^i(t, dx) + \int_{|x|\geq 1} xN^i(t, dx)$. Thus,

$$\begin{aligned} L_s^{i,j} &= L_s^i - L_s^j \\ &= \sigma\left(W_s^i - W_s^j\right) + \int_{|x|<1} x\left(\tilde{N}^i(t, dx) - \tilde{N}^j(t, dx)\right) \\ &\quad + \int_{|x|\geq 1} x\left(N^i(t, dx) - N^j(t, dx)\right). \end{aligned}$$

Since the first two terms are martingales, we then have

$$\begin{aligned}
\mathbb{E}\left(\hat{v}_t^{i,j}\right) &= e^{-\alpha t}\mathbb{E}\left(\hat{v}_0^{i,j}\right)+\mathbb{E}\left[\int_0^t e^{-\alpha(t-s)}\int_{|x|\geq 1} x\left(N^i(ds,dx)-N^j(ds,dx)\right)\right] \\
&= e^{-\alpha t}\mathbb{E}\left(\hat{v}_0^{i,j}\right)+\int_0^t e^{-\alpha(t-s)}ds\left(\int_{|x|\geq 1} x\rho^i(dx)-\int_{|x|>1} x\rho^j(dx)\right) \\
&= e^{-\alpha t}\mathbb{E}\left(\hat{v}_0^{i,j}\right)+\frac{1}{\alpha}\left(1-e^{-\alpha t}\right)\mu^{i,j}.
\end{aligned}$$

The second estimate directly follows by integrating $e^{-\alpha t}\mathbb{E}\left[\hat{v}_0^{i,j}\right]+\frac{1}{\alpha}\left(1-e^{-\alpha t}\right)\mu^{i,j}$.

□

As we can see from the above lemma, the expectation of the gap process $\hat{v}_t^{i,j}$ depends not only on the initial value, but also on the mean of the jumps $\mu^{i,j}$, which will not converge to zero as time tends to infinity. The term $\mu^{i,j}$ contributed by jumps may cause the flocking to fail. In other words, the dynamics of velocity with distinct jumps will not satisfy definition (2.2) . On the contrary, flocking emerges in the SCS model (2.3) with identical jumps, i.e., $\mu^i=\mu^j$ as shown in the next theorem:

Theorem 2.4. *Let* (x_t^i, v_t^i) *be the solutions to* (2.3) *with* $\mu^i=\mu^j$, $i\neq j\in\{1,...,N\}$, *then we have the time-asymptotic flocking in velocity alignment and group formation given by*

$$\begin{aligned}
\lim_{t\to\infty}\left|\mathbb{E}\left(v_t^i\right)-\mathbb{E}\left(v_t^j\right)\right| &= 0, \\
\sup_{0\leq t<\infty}\left|\mathbb{E}\left(x_t^i\right)-\mathbb{E}\left(x_t^j\right)\right| &< \infty.
\end{aligned}$$

Proof. With the assumptions on jumps $\mu^i = \mu^j$, $\forall i \neq j \in \{1, ..., N\}$, by Lemma 2.1, we have

$$\begin{aligned}
\lim_{t\to\infty} \left|\mathbb{E}\left(v_t^i\right) - \mathbb{E}\left(v_t^j\right)\right| &= \lim_{t\to\infty} \left|\mathbb{E}\left(v_t^i - \bar{v}_t\right) - \mathbb{E}\left(v_t^j - \bar{v}_t\right)\right| \\
&= \lim_{t\to\infty} \left|\mathbb{E}\left(\hat{v}_t^i\right) - \mathbb{E}\left(\hat{v}_t^j\right)\right| \\
&= \lim_{t\to\infty} \left|\mathbb{E}\left(\hat{v}_t^{i,j}\right)\right| \\
&= \lim_{t\to\infty} e^{-\alpha t} \left|\mathbb{E}\left(\hat{v}_0^{i,j}\right)\right| = 0.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\left|\mathbb{E}\left(x_t^i\right) - \mathbb{E}\left(x_t^j\right)\right| &= \left|\mathbb{E}\left(x_t^i - \bar{x}_t\right) - \mathbb{E}\left(x_t^j - \bar{x}_t\right)\right| \\
&= \left|\mathbb{E}\left(\hat{x}_t^{i,j}\right)\right| \\
&\leq \left|\mathbb{E}\left(\hat{x}_0^{i,j}\right)\right| + \left|\mathbb{E}\left(\hat{v}_0^{i,j}\right)\right| \frac{(1 - e^{-\alpha t})}{\alpha},
\end{aligned}$$

which yields the desired result. □

The above theorem indicates that flocking emerges in the SCS model (2.3) as long as we make some restrictions on jumps. In other words, the jumps for each particle must be identical, otherwise some particles may move too far away from the group due to dramatic jumps in velocity, and, therefore, fail to have group formation and velocity alignment asymptotically. As a result, it is worth investigating the situation where each particle has only identical jumps, i.e., $\mu^i = \mu^j$ for all $i \neq j$. The following theorem focuses only on particles with identical jumps. Assume that the noise terms are identical, i.e.,

$$\begin{aligned}\mathbb{E}\left(L_t^i\right) &= \mathbb{E}\left(L_t^j\right) \\ Var\left(L_t^i\right) &= Var\left(L_t^j\right),\end{aligned}$$

for all $i \neq j \in \{1, ..., N\}$.

Theorem 2.5. *(Estimate of fluctuation of all particles) For every $t \geq 0$, define the variance of all particles*

$$S_N(t) = \frac{1}{N-1}\sum_{i=1}^{N}\left(v_t^i - \bar{v}_t\right)^2 \tag{2.6}$$

Then, the expectation of $S_N(t)$ is given by

$$\mathbb{E}\left(S_N(t)\right) = \frac{1}{N-1}\sum_{i=1}^{N}\left\{e^{-2\alpha t}\left(\hat{v}_0^i\right)^2\right\} + \frac{\left(1-e^{-2\alpha t}\right)}{2\alpha}var\left(L_1\right),$$

Furthermore, as $t \to \infty$, we have

$$\lim_{t\to\infty}\mathbb{E}\left(S_N(t)\right) = \frac{var\left(L_1\right)}{2\alpha}.$$

Proof. Recall the solution $\hat{v}_t^i = e^{-\alpha t}\hat{v}_0^i + \int_0^t e^{-(t-s)}dZ_t^i$. It is well known that the characteristic function of $\hat{v}_t^i$, see details in [Sato, 1999], is given by

$$\phi^j(\theta) = E\left(e^{i\theta\hat{v}_t^j}\right) = \exp\left[ie^{-\alpha t}v_0^j\theta + \int_0^t \psi^j\left(e^{-\alpha(t-s)}\theta\right)ds\right],$$

where $\psi^j(\theta) = \log E\left[e^{i\theta Z_1^j}\right]$. The second moment of $\hat{v}_t^i$ is then obtained by taking derivative at $\theta = 0$, so we now have

$$\begin{aligned}E\left[\left(\hat{v}_t^i\right)^2\right] &= \left[e^{-\alpha t}E\left(v_0^i\right) + \frac{\left(1-e^{-\alpha t}\right)}{\alpha}E\left(Z_1^i\right)\right]^2 + \frac{\left(1-e^{-2\alpha t}\right)}{2\alpha}var\left(Z_1^i\right) \\ &= e^{-2\alpha t}\left[E\left(v_0^i\right)\right]^2 + \frac{\left(1-e^{-2\alpha t}\right)}{2\alpha}var\left(Z_1^i\right),\end{aligned}$$

since $E\left(Z_1^i\right) = \left(1 - \frac{1}{N}\right) E\left(L_1^i\right) - \frac{1}{N} \sum_{j\neq i} E\left(L_1^j\right) = E\left(L_1^i\right) \left(\left(1 - \frac{1}{N}\right) - \frac{(N-1)}{N}\right) = 0$.

Additionally,

$$
\begin{aligned}
var\left(Z_1^i\right) &= var\left[\left(1 - \frac{1}{N}\right) L_t^i - \frac{1}{N} \sum_{j\neq i} L_t^j\right] = \left[\left(1 - \frac{1}{N}\right)^2 + \frac{(N-1)}{N^2}\right] var\left(L_1\right) \\
&= \frac{N-1}{N} var\left(L_1\right),
\end{aligned}
$$

the asymptotic estimate of variance is then given by

$$
\begin{aligned}
\lim_{t\to\infty} E\left(S_N\left(t\right)\right) &= \lim_{t\to\infty} \frac{1}{N-1} \sum_{i=1}^{N} E\left[\left(\hat{v}_t^i\right)^2\right] \\
&= \lim_{t\to\infty} \frac{1}{N-1} \sum_{i=1}^{N} \left\{ e^{-2\alpha t} \left[E\left(v_0^i\right)\right]^2 + \frac{\left(1 - e^{-2\alpha t}\right)}{2\alpha} var\left(Z_1^i\right) \right\} \\
&= \frac{var\left(L_1\right)}{2\alpha}. \qquad (2.7)
\end{aligned}
$$

□

Remark 2.1. *The asymptotic estimate of variance does not depend on the number of particles N but on the coupling strength α as $t \to \infty$.*

We provide two concrete examples with numerical results to illustrate the asymptotic estimate that will tend to a function of coupling strength α.

Example 2.2. (*Brownian motions*) *If $L_t^i = \sigma W_t^i$, then the dynamics are given by*

$$
dv_t^i = \frac{\alpha}{N} \sum_{j=1}^{N} \left(v_t^j - v_t^i\right) dt + \sigma dW_t^i, \qquad 1 \leq i \leq N. \qquad (2.8)
$$

Note that these are the dynamics investigated in [Ha et al., 2009]. From (2.7), we have the asymptotic estimate of the variance

$$
\lim_{t\to\infty} \mathbb{E}\left(S_N\left(t\right)\right) = \frac{\sigma^2}{2\alpha},
$$

which is a constant that depends on α. It is worth knowing that although the system (2.8) satisfies the flocking criteria (2.2), the fluctuation of all particles might not converge to zero except in the case when the coupling strength α tends to infinity.

Example 2.3. (*Compound Poisson processes*) *Let*

$$L_t^i = \sum_{j=1}^{N_t} \xi_j,$$

where N_t is a Poisson process with intensity λ and the jump sizes $\{\xi_j, j \geq 1\}$ are i.i.d. random variables with common distribution $F \sim normal(\mu_F, \sigma_F)$, independent of N_t. Thus, the mean and variance of L_t^i are given by $\mathbb{E}\left(L_t^i\right) = \lambda t \mu_F$ and $var\left(L_t^i\right) = \lambda t \mathbb{E}\left(\xi_j^2\right) = \lambda t\left(\sigma_F^2 + \mu_F^2\right)$. The asymptotic estimate of the variance is then given by

$$\lim_{t\to\infty} \mathbb{E}\left(S_N\left(t\right)\right) = \frac{\lambda\left(\sigma_F^2 + \mu_F^2\right)}{2\alpha}.$$

From the above example, we see that the asymptotic estimate of the variance tends to a function of not only in α but also in the parameters λ, μ_F and σ_F. Both examples may have the same asymptotic variance under some parameters settings which we show in the following figures.

We simulate the linear SCS model (2.3) in both examples 2.2 and 2.3 by employing an Euler scheme with time dividend $\Delta t = 0.0001$ and illustrate the effect of coupling strength α. In Figure 2.1 with smaller coupling strength, for instance $\alpha = 1$, we see that the flocking is not obvious while the asymptotic variance in both examples tend to the same theoretical value 0.5 with parameters $T = 10$, $\lambda = 1$, $\mu_F = 0$ and $\sigma_F = \sigma = 1$ as shown in Figure 2.3. In Figure 2.2, both asymptotic variances tend to zero with large coupling strength, $\alpha = 100$, and the flocking emerges in both systems.

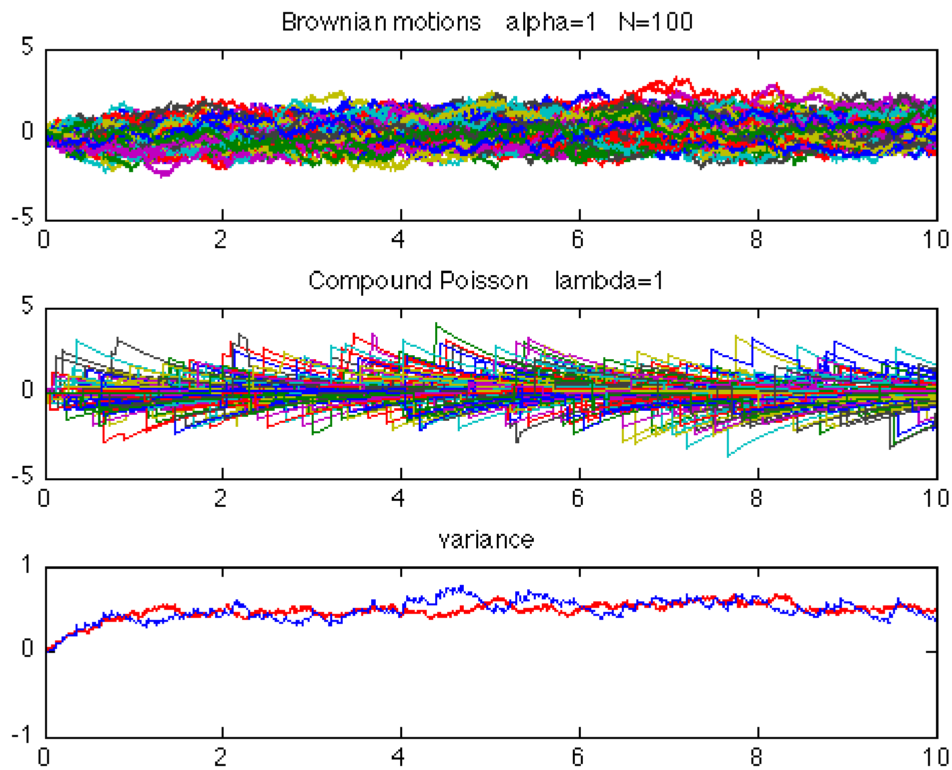

Figure 2.1: The velocity plots of both with Brownian motions, compound Poisson processes and the fluctuation. With smaller coupling strength $\alpha = 1$, flocking is not obvious while the asymptotic variances in both examples tend to the same theoretical value $\frac{\lambda}{2\alpha}(\sigma_F^2 + \mu_F^2) = 0.5$. The parameters used: $N = 100$, $T = 10$, $\lambda = 1$, $\mu_F = 0$, $\sigma_F = \sigma = 1$.

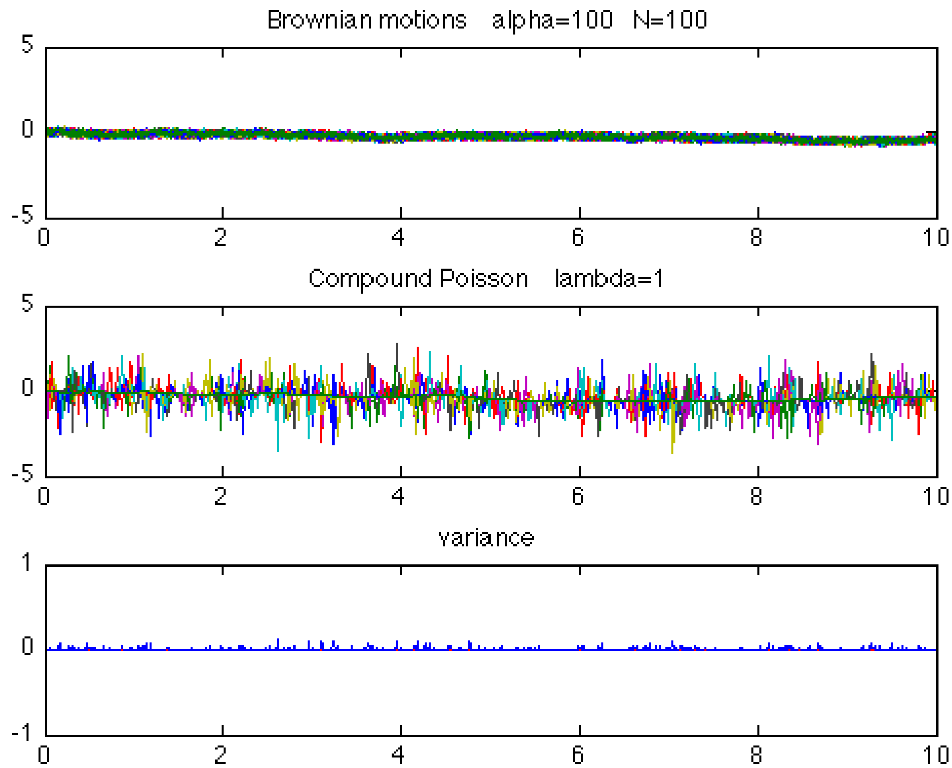

Figure 2.2: The velocity plots of both with Brownian motions, compound Poisson processes and the fluctuation. With larger coupling strength $\alpha = 100$, flocking is obvious while the asymptotic variances in both examples tend to the same theoretical value $\frac{\lambda}{2\alpha}(\sigma_F^2 + \mu_F^2) \approx 0$. The parameters used: $N = 100$, $T = 10$, $\lambda = 1$, $\mu_F = 0$, $\sigma_F = \sigma = 1$.

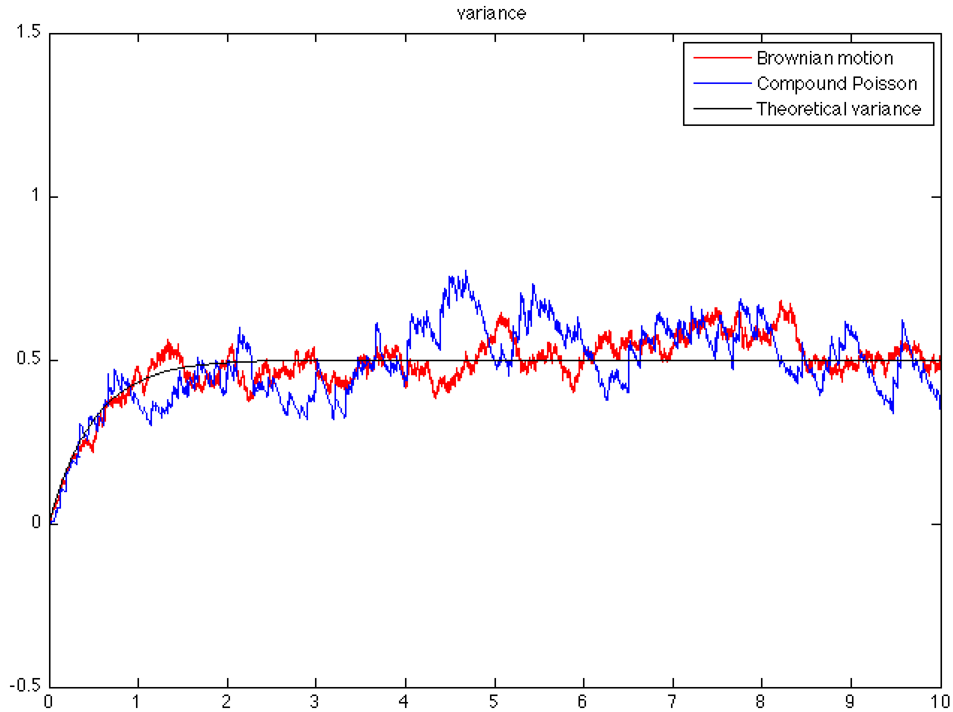

Figure 2.3: The fluctuation plot of both with Brownian motions, compound Poisson processes and the fluctuation. The parameters used: $T = 1$, $\lambda = 1$, $\mu_F = 0$, $\sigma_F = \sigma = 1$.

2.3.2 Radially symmetric communication rate

In this section, we follow the analysis in [Ha et al., 2009] for which the noise terms are now driven by Lévy processes. When the communication rate function satisfies some boundedness conditions, we also give the upper bound for the estimate of the variance of all particles that we define in (2.6) .

Assume that ψ is a non-increasing function in its argument and depends only on the distance between particles, i.e.,

$$\psi = \bar{\psi}\left(|x_j - x_i|^2\right).$$

Then (2.1) can be rewritten as

$$\begin{aligned} dx_t^i &= v_t^i dt \\ dv_t^i &= \frac{\alpha}{N}\sum_{j=1}^{N} \bar{\psi}\left(|x_j - x_i|^2\right)\left(v_t^j - v_t^i\right) dt + dL_t^i, 1 \le i \le N. \end{aligned} \tag{2.9}$$

By using the assumption of symmetric and translation invariance for the communication rate, we may write the dynamics for the micro variables $(\hat{x}_t^i, \hat{v}_t^i)$ as

$$\begin{aligned} dv_t^i &= \frac{\alpha}{N}\sum_{j=1}^{N} \bar{\psi}\left(\left|\hat{x}_t^j - \hat{x}_t^i\right|^2\right)\left(\hat{v}_t^j - \hat{v}_t^i\right) dt + dZ_t^i, \\ d\hat{x}_t^i &= \hat{v}_t^i dt, \end{aligned}$$

where $Z_t^i = \left(1 - \frac{1}{N}\right) L_t^i - \frac{1}{N}\sum_{j\neq i}^{N} L_t^j$.

Recall that the variance is given by

$$S_N(t) = \frac{1}{N-1}\sum_{i=1}^{N}\left(\hat{v}_t^i\right)^2.$$

We need an auxiliary function to obtain an upper bound for $\mathbb{E}\left[S_N(t)\right]$. Define

$$\chi(t) = \sum_{i=1}^{N}\left(\hat{x}_t^i\right)^2.$$

Observe that

$$\left|\hat{x}_t^j - \hat{x}_t^i\right|^2 \le 2\chi(t)$$

and since $\bar{\psi}$ is non-increasing, we have

$$\bar{\psi}\left(\left|\hat{x}_t^j - \hat{x}_t^i\right|^2\right) \geq \bar{\psi}\left(2\chi\left(t\right)\right).$$

We now give the boundedness of $\mathbb{E}\left[S_N\left(t\right)\right]$ in the following theorem:

Theorem 2.6. *Assume that $\bar{\psi}$ satisfies the lower bound condition*

$$\min_{s>0} \bar{\psi}\left(s\right) \geq \psi^* > 0,$$

for some constant ψ^*. Then, the estimate of the variance $S_N\left(t\right)$ is bounded in t and depends on the distribution of jump size of v_t^i. Specifically, we have

$$\mathbb{E}\left[S_N\left(t\right)\right] \leq e^{-2\psi^*\alpha t} S_N\left(0\right) + \frac{\sigma^2}{2\psi^*\alpha}\left(1 - e^{-2\psi^*\alpha t}\right) + \frac{1}{N-1} e^{-2\psi^*\alpha t} \sum_{i=1}^{N} \sum_{0\leq s\leq t} e^{2\psi^*\alpha s}\mathbb{E}\left(\Delta\hat{v}_s^i\right)^2.$$

Proof. Firstly, we derive the dynamics for $S_N\left(t\right)$ by using the Ito formula for Lévy processes and by changing index $i \leftrightarrow j$.

$$\begin{aligned}
dS_N\left(t\right) &= \frac{1}{N-1}\sum_{i=1}^{N} d\left(\hat{v}_t^i\right)^2 \\
&= \frac{1}{N-1}\sum_{i=1}^{N}\left\{2\hat{v}_t^i dv_t^i + d\left[\hat{v}_t^i, \hat{v}_t^i\right]_t^c + \left(\hat{v}_t^i\right)^2 - \left(\hat{y}_{t-}^i\right)^2 - 2\hat{v}_{t-}^i \Delta\hat{v}_t^i\right\} \\
&= \frac{1}{N-1}\sum_{i=1}^{N}\left\{2\hat{v}_t^i dv_t^i + \sigma^2\left(1 - \frac{1}{N}\right)dt + \left(\Delta\hat{v}_t^i\right)^2\right\} \\
&= \frac{1}{N-1}\sum_{i=1}^{N} 2\hat{v}_t^i\left(\frac{\alpha}{N}\sum_{j=1}^{N}\bar{\psi}\left(\left|\hat{x}_t^j - \hat{x}_t^i\right|^2\right)\left(\hat{v}_t^j - \hat{v}_t^i\right)dt + dZ_t^i\right) \\
&\quad + \sigma^2 dt + \frac{1}{N-1}\sum_{i=1}^{N}\left(\Delta\hat{v}_t^i\right)^2
\end{aligned}$$

$$
\begin{aligned}
&= \frac{-2}{N-1}\sum_{i=1}^{N} v_t^j \left(\frac{\alpha}{N}\sum_{j=1}^{N} \bar{\psi}\left(\left| \hat{x}_t^j - \hat{x}_t^i \right|^2 \right) \left(\hat{v}_t^j - \hat{v}_t^i \right) dt \right) + \frac{1}{N-1}\sum_{i=1}^{N} 2\hat{v}_t^i dZ_t^i \\
&\quad +\sigma^2 dt + \frac{1}{N-1}\sum_{i=1}^{N} \left(\Delta \hat{y}_t^{(i)} \right)^2 \\
&\leq \frac{-\bar{\psi}\left(2\chi\left(t\right)\right)}{N-1}\frac{\alpha}{N}\sum_{i=1}^{N}\sum_{j=1}^{N} 2\hat{v}_t^j \left(\hat{v}_t^j - \hat{v}_t^i \right) dt + \frac{1}{N-1}\sum_{i=1}^{N} 2\hat{v}_t^i dZ_t^{(i)} \\
&\quad +\sigma^2 dt + \frac{1}{N-1}\sum_{i=1}^{N} \left(\Delta \hat{v}_t^i \right)^2 \\
&= \frac{-\bar{\psi}\left(2\chi\left(t\right)\right)\alpha}{N-1}\sum_{j=1}^{N} 2\left(\hat{v}_t^j \right)^2 dt + \frac{1}{N-1}\sum_{i=1}^{N} 2\hat{v}_t^i dZ_t^{(i)} \\
&\quad +\sigma^2 dt + \frac{1}{N-1}\sum_{i=1}^{N} \left(\Delta \hat{v}_t^i \right)^2 \\
&\leq -2\psi^* \alpha S_N\left(t\right) dt + \frac{1}{N-1}\sum_{i=1}^{N} 2\hat{v}_t^i dZ_t^{(i)} + \sigma^2 dt + \frac{1}{N-1}\sum_{i=1}^{N} \left(\Delta \hat{v}_t^i \right)^2 .
\end{aligned}
$$

In order to obtain the upper bound, we then apply Ito's formula again to the function $e^{2\psi^*\alpha t} S_N\left(t\right)$. Thus,

$$
\begin{aligned}
de^{2\psi^*\alpha t} S_N\left(t\right) &= 2\psi^*\alpha e^{2\psi^*\alpha t} S_N\left(t\right) dt + e^{2\psi^*\alpha t} dS_N\left(t\right) \\
&\quad + e^{2\psi^*\alpha t} S_N\left(t\right) - e^{2\psi^*\alpha t^-} S_N\left(t^-\right) - e^{2\psi^*\alpha t^-} \Delta S_N\left(t\right) \\
&= 2\psi^*\alpha e^{2\psi^*\alpha t} S_N\left(t\right) dt + e^{2\psi^*\alpha t} dS_N\left(t\right) \\
&\leq 2\psi^*\alpha e^{2\psi^*\alpha t} S_N\left(t\right) dt \\
&\quad + e^{2\psi^*\alpha t} \left\{ \begin{array}{c} -2\psi^*\alpha S_N\left(t\right) dt + \frac{1}{N-1}\sum_{i=1}^{N} 2\hat{y}_t^i dZ_t^i + \sigma^2 dt \\ + \frac{1}{N-1}\sum_{i=1}^{N} \left(\Delta \hat{y}_t^i\right)^2 \end{array} \right\} \\
&= e^{2\psi^*\alpha t} \left\{ \frac{1}{N-1}\sum_{i=1}^{N} 2\hat{y}_t^i dZ_t^i + \sigma^2 dt + \frac{1}{N-1}\sum_{i=1}^{N} \left(\Delta \hat{y}_t^i \right)^2 \right\}
\end{aligned}
$$

Then we integrate with respect to t to get

$$\begin{aligned} S_N(t) \leq\; & e^{-2\psi^*\alpha t} S_N(0) + \frac{2}{N-1} \sum_{i=1}^{N} \int_0^t e^{-2\psi^*\alpha(t-s)} \hat{v}_s^i dZ_s^i + \frac{\sigma^2}{2\psi^*\alpha} \left(1 - e^{-2\psi^*\alpha t}\right) \\ & + \frac{1}{N-1} e^{-2\psi^*\alpha t} \sum_{i=1}^{N} \sum_{0 \leq s \leq t} e^{2\psi^*\alpha s} \left(\Delta \hat{v}_s^i\right)^2 . \end{aligned}$$

Finally, we take the ensemble average to obtain

$$\mathbb{E}\left[S_N(t)\right] \leq e^{-2\psi^*\alpha t} S_N(0) + \frac{\sigma^2}{2\psi^*\alpha} \left(1 - e^{-2\psi^*\alpha t}\right) + \frac{1}{N-1} e^{-2\psi^*\alpha t} \sum_{i=1}^{N} \sum_{0 \leq s \leq t} e^{2\psi^*\alpha s} \mathbb{E}\left(\Delta \hat{v}_s^i\right)^2 ,$$

since $\mathbb{E}\left[\int_0^t e^{-2\psi^*\alpha(t-s)} \hat{v}_s^i dZ_s^i\right] = 0$ by the Lévy-Ito decomposition and the martingale argument. □

It is difficult to prove when the nonlinear system (2.9) satisfies the flocking criteria (2.2). However, the above theorem provides a different perspective on flocking by investigating the fluctuation of all particles. Flocking will emerge in such systems as long as the rate α is large enough under some boundedness of communication rate ψ. Specifically, the fluctuation of all particles will decrease exponentially with large α despite the lack of flocking criteria (2.2).

2.3.3 Numerical results

In this section, we simulate the linear SCS model (2.3) by employing an Euler scheme with time dividend $\Delta t = 0.0001$ and illustrate the effect of coupling strength α as well as jumps. Assuming the noise terms in (2.3) is given by $L_t^i = \sigma W_t^i + \sum_{j=1}^{N_t^i} \xi_j$, where ξ_j has distribution $f(y;\theta) = \frac{\theta}{2} e^{-|y|\theta}, \theta > 0$ and N_t^i is a Poisson process with rate λ. We use the parameters $N = 10, T = 1, \sigma = 1, \theta = 1$ and $\lambda = 3$ to illustrate the model. For the purpose of simplification, we only provide the plots for the position of all particles. Note that Figure 2.4 to Figure 2.7 show the trajectories of the model (2.3) in space (with 3-dimensional noises terms).

In both Figure 2.4 and Figure 2.5, it is obvious that the flocking does not emerge for a small coupling strengh, $\alpha = 1$. Additionally, as shown in blue line in Figure 2.4, a particle only changes its position due to a sudden jump in its velocity part, whereas the blue line shows no obvious change in course in Figure 2.5.

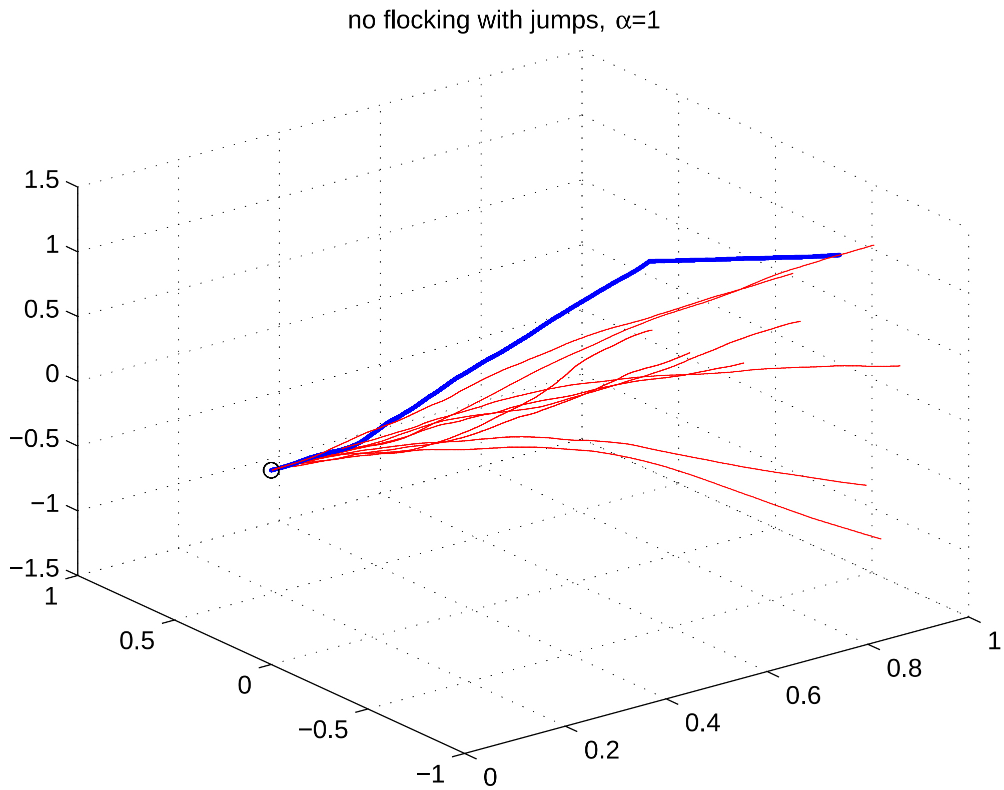

Figure 2.4: Plot of 3-dimensional position for the model (2.3) with one particular jump (blue line). Flocking does not emerge for a small coupling strengh, $\alpha = 1$, since the failure of group formation. The parameters used: $\sigma = 1$, $\theta = 1$ and $\lambda = 3$.

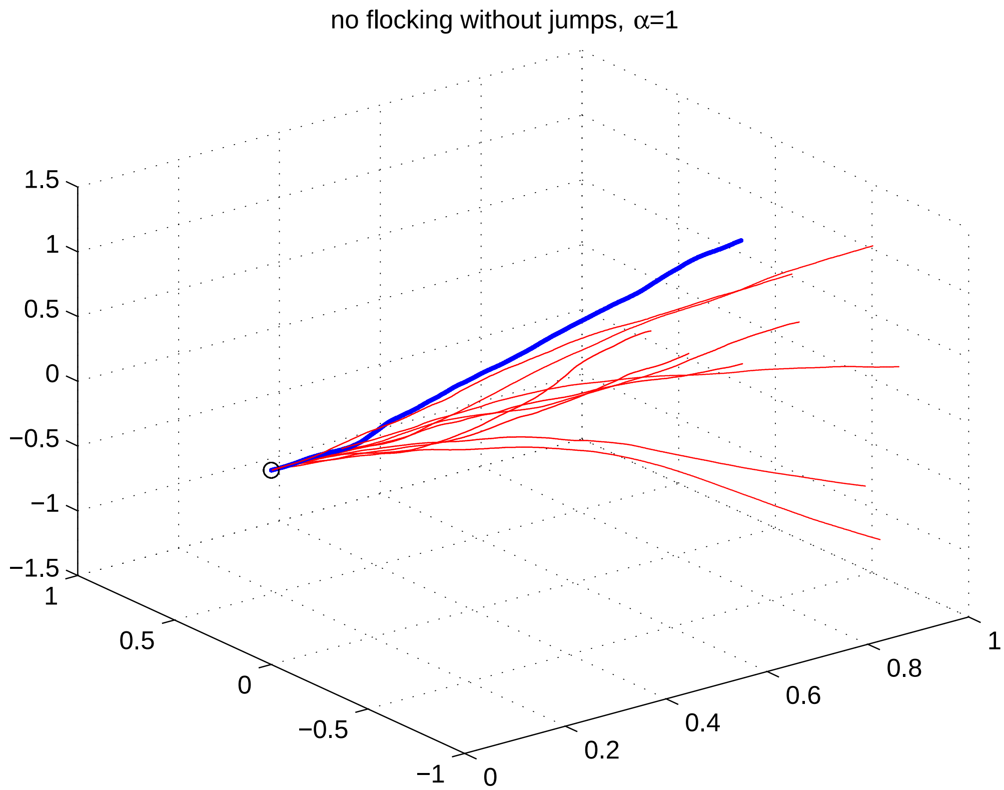

Figure 2.5: Plot of 3-dimensional position for the model (2.3) without any jumps. Compare to Figure 2.4, the blue line shows no obvious change in course.

In Figure 2.6 and Figure 2.7, flocking emerges in the SCS model for larger coupling strength $\alpha = 100$ regardless the presence of jumps. Figure 2.6 shows flocking with jumps where jumps might be considered as a characteristic of a leading particle (i.e. the blue line) who has a faster or lower velocity than other particles in the group. Everyone else follows the trajectory of this leading particle in Figure 2.6. In contrast, every particle is moving around closely to each other without jumps in Figure 2.7.

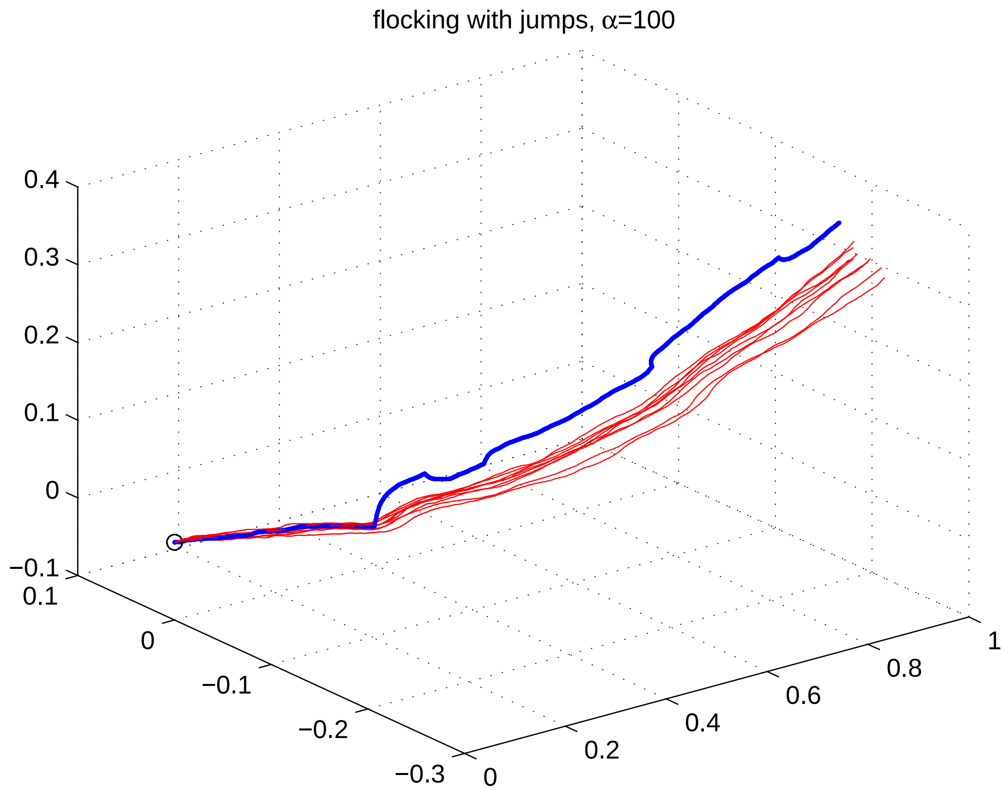

Figure 2.6: Plot of 3-dimensional position for the model (2.3) with one particular jump (blue line). Flocking emerges in the SCS model for larger coupling strength $\alpha = 100$ regardless of the presence of jumps. Everyone else (red lines) follows the trajectory of this particular particle (blue line). The parameters used: $N = 10$, $T = 1$, $\sigma = 1$, $\theta = 1$ and $\lambda = 3$.

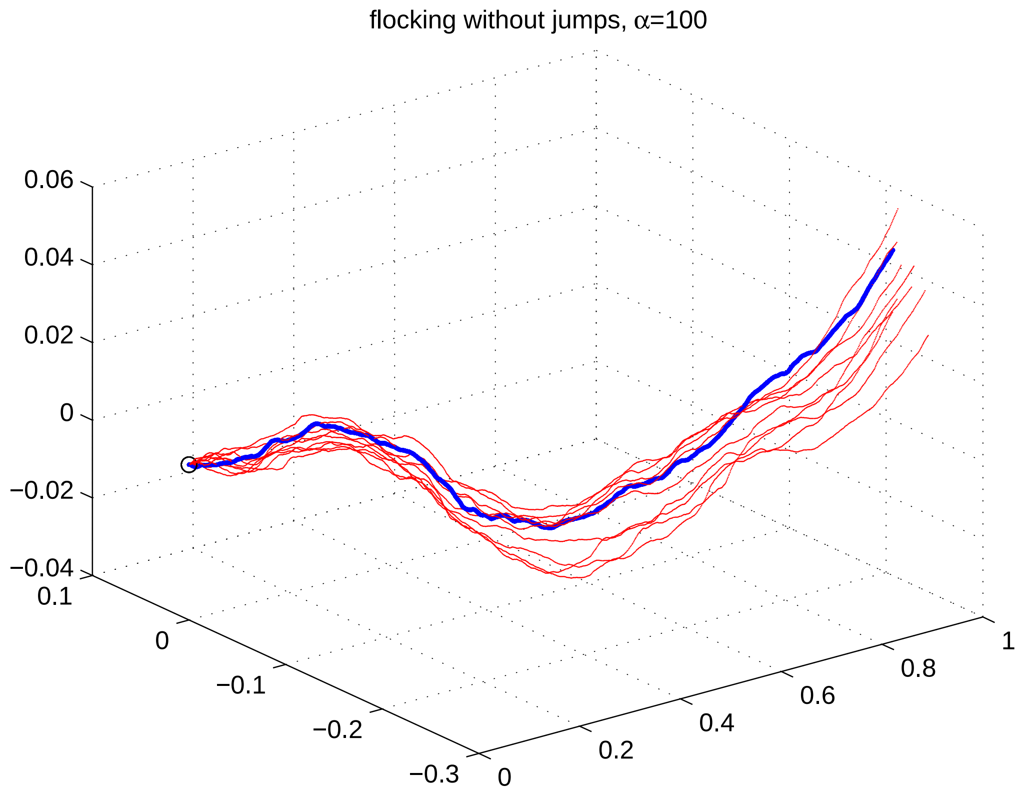

Figure 2.7: Plot of 3-dimensional position for the model (2.3) without any jumps. Compare to Figure 2.6, the blue line shows no dramatic change in course and every particle is moving around closely to each other without jumps.

In the following two figures, we provide a one dimensional plot for both positions and velocities to better describe the jump phenomenon. As we can see in Figure 2.8, there is a negative jump around time 0.6. Compared to the velocity plot in Figure 2.9, it shows that the velocity of a particular particle (blue line) decreases suddenly around time 0.6 and then is attracted to the rest of the group. While the corresponding position of this particle is affected and is moving out of the group slightly, the rest of the group is still attracted to its trajectory. Therefore, we may consider this particle as a leading particle.

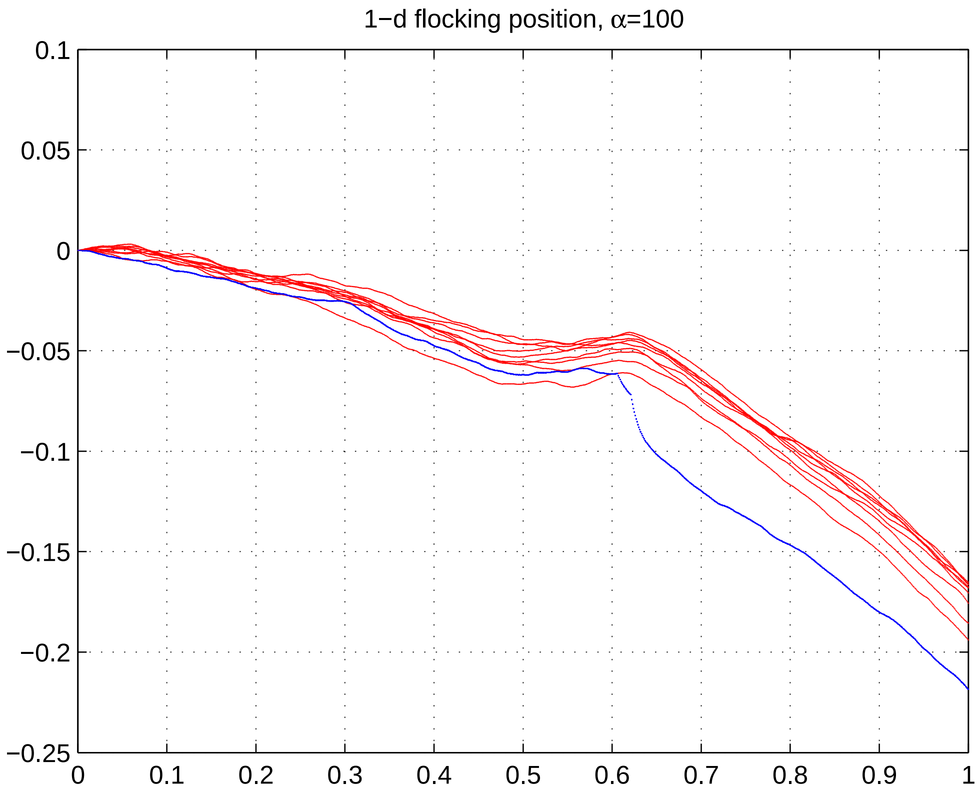

Figure 2.8: Plot of one-dimensional position, a negative jump presents around time 0.6. The parameters used: $N = 10$, $T = 1$, $\sigma = 1$, $\theta = 1$ and $\lambda = 3$.

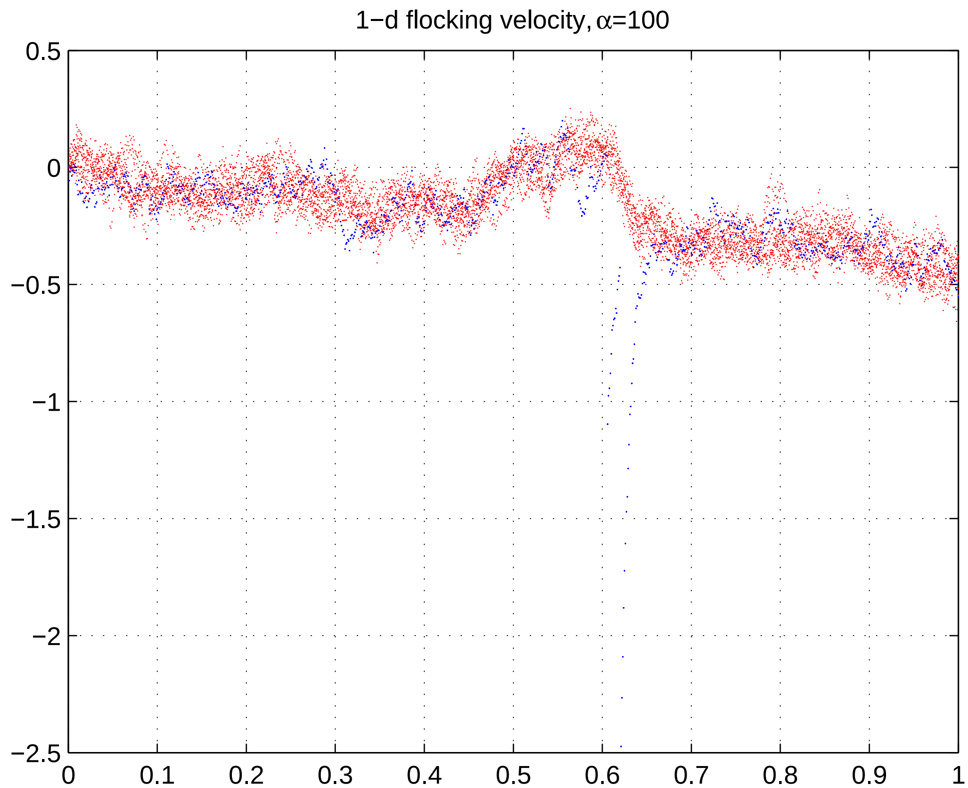

Figure 2.9: The corresponding velocity of a particular particle (blue line) decreases suddenly around time 0.6. The parameters used are the same as Figure 2.8.

2.4 Stochastic flocking model with a central particle

In the previous sections, we did not take into account the interactions between individual particles and one central particle in a stochastic system. Within the new system, all particles now communicate with each other indirectly through a central particle. The central particle is now playing an important role as it acts as an intermediary to bring all particles into communication.

In this section, we propose a new model that would capture the interactions between individual particles and the central particle. For the simplicity of analysis, we consider the noise terms in the dynamics are driven by Brownian motions only but can easily be extended to Lévy processes. We consider a particle (x_t^0, v_t^0) as the central particle with dynamics:

$$\begin{aligned} dv_t^0 &= \frac{\alpha}{N}\sum_{j=1}^{N}\phi\left(x_t^j, x_t^0\right)\left(v_t^j - v_t^0\right)dt + \sigma_0 dW_t^0, \\ dx_t^0 &= v_t^0 dt, \end{aligned}$$

while the other individual particles, called peripheral particles, have dynamics

$$\begin{aligned} dv_t^i &= \beta\phi\left(x_t^0, x_t^i\right)\left(v_t^0 - v_t^i\right)dt + \sigma_i dW_t^i \\ dx_t^i &= v_t^i dt, \qquad i = 1, ..., N, \end{aligned}$$

where $\alpha, \beta \geq 0$ are the coupling strength and $\phi \geq 0$ is the communication rate as we defined earlier in this chapter.

From the above dynamics, we observe that individual peripheral particles communicate with each other only through the communication rate $\phi\left(x_t^0, x_t^i\right)$ with the central particle x_t^0. In other words, the closer the peripheral particle x_t^i is to the central particle x_t^0, the stronger communication rate x_t^i will be. We again assume that the initial condition $(x_0^i, v_0^i) = (x_0, v_0)$, for $i = 0, 1, .., N$. We will also prove that such stochastic systems have the time-asymptotic flocking as in the case of linear communication rate.

We consider a constant communication rate $\phi = 1$. Hence, the dynamics read as

$$
\begin{aligned}
dv_T^0 &= \frac{\alpha}{N}\sum_{j=1}^{N}\left(v_t^j - v_t^0\right)dt + \sigma_0 dW_t^0 \\
dv_t^i &= \beta\left(v_t^0 - v_t^i\right)dt + \sigma_I dW_t^i \\
dx_t^i &= v_t^i dt, \qquad i = 0,1,...,N
\end{aligned}
\tag{2.10}
$$

Define the ensemble average $\bar{v}_t = \frac{1}{N}\sum_{j=1}^{N} v_t^j$ and then we have the dynamics of $\bar{v}_t$ to be

$$
d\bar{v}_t = \beta\left(v_t^0 - \bar{v}_t\right)dt + \frac{1}{N}\sum_{i=1}^{N}\sigma_i dW_t^i.
$$

Our goal is to show that v_t^i will satisfy the time-asymptotic flocking criteria

$$
\lim_{t\to\infty}\left|\mathbb{E}\left(v_t^i\right) - \mathbb{E}\left(v_t^j\right)\right| = 0, \;\; for\ i \neq j \in \{0,...,N\}.
$$

We consider the micro and macro decomposition again and recall that the micro variable is defined as $\hat{v}_t^i = v_t^i - \bar{v}_t$ for $i = 0,1,...,N$. First of all, we investigate the micro variable only for $i = 1,..,N$; the dynamics are given by

$$
\begin{aligned}
d\hat{v}_t^i &= \beta\left(v_t^0 - v_t^i\right)dt + \sigma_i dW_t^i - \beta\left(v_t^0 - \bar{v}_t\right)dt - \frac{1}{N}\sum_{i=1}^{N}\sigma_i dW_t^i \\
&= -\beta\left(v_t^i - \bar{v}_t\right) + \sigma_i\left(1 - \frac{1}{N}\right)dW_t^i + \frac{1}{N}\sum_{j\neq i}^{N}\sigma_j dW_t^j \\
&= -\beta\hat{X}_t^i + dZ_t^i,
\end{aligned}
$$

where $Z_t^i = \sigma_i\left(1 - \frac{1}{N}\right)W_t^i + \frac{1}{N}\sum_{j\neq i}^{N}\sigma_j W_t^j$.

Since $\hat{v}_t^i$ is an OU process with mean reverting rate β, and it is easy to check that the solution $\hat{v}_t^i$ are given by

$$\hat{v}_t^i = e^{-\beta t}\hat{v}_0^i + \int_0^t e^{-\beta(t-s)} dZ_s^i, i = 1, .., N,$$

we have the estimate

$$\mathbb{E}\left(\hat{v}_t^i\right) = e^{-\beta t}\mathbb{E}\left(\hat{v}_0^i\right) = 0.$$

Secondly, we see that the dynamic of the micro variable for $i = 0$ is given by

$$\begin{aligned} d\hat{v}_t^0 &= dv_t^0 - d\bar{v}_t \\ &= \alpha\left(\bar{v}_t - v_t^0\right)dt + \sigma_0 dW_t^0 - \beta\left(v_t^0 - \bar{v}_t\right)dt - \frac{1}{N}\sum_{i=1}^N \sigma_i dW_t^i \\ &= -\left(\alpha+\beta\right)\left(v_t^0 - \bar{v}_t\right)dt + \sigma_0 dW_t^0 - \frac{1}{N}\sum_{i=1}^N \sigma_i dW_t^i \\ &= -\left(\alpha+\beta\right)\hat{v}_t^0 dt + dZ_t^0, \end{aligned}$$

where $Z_t^0 = \sigma_0 W_t^0 - \frac{1}{N}\sum_{i=1}^N \sigma_i W_t^i$. The solution is, again, given by

$$\hat{v}_t^0 = e^{-(\alpha+\beta)t}\hat{v}_0^0 + \int_0^t e^{-(\alpha+\beta)(t-s)} dZ_s^0,$$

and, also, we have the estimate

$$\mathbb{E}\left(\hat{v}_t^0\right) = e^{-(\alpha+\beta)t}\mathbb{E}\left(\hat{v}_0^0\right).$$

We now know the particles (x_t^i, v_t^i) have time-asymptotic flocking by the following theorem:

Theorem 2.7. *Let* (x_t^i, v_t^i) *be the solutions to* (2.10) *for* $i \neq j \in \{0, 1, ..., N\}$, *we then have*

$$\lim_{t\to\infty}\left|\mathbb{E}\left(v_t^i\right) - \mathbb{E}\left(v_t^j\right)\right| = 0.$$

Furthermore,

$$\sup_{0 \le t < \infty} \left| \mathbb{E}\left(x_t^i\right) - \mathbb{E}\left(x_t^j\right) \right| < \infty.$$

Proof. By a similar argument shown in the previous section, we would have for $i \neq j \in \{1, .., N\}$

$$\lim_{t\to\infty} \left| \mathbb{E}\left(v_t^i\right) - \mathbb{E}\left(v_t^j\right) \right| = \lim_{t\to\infty} e^{-at} \left| \mathbb{E}\left(\hat{v}_0^i - \hat{v}_0^j\right) \right| = 0.$$

In addition, for $i = \{1, ..., N\}$, we have

$$\begin{aligned}
\lim_{t\to\infty} \left| \mathbb{E}\left(v_t^0\right) - \mathbb{E}\left(v_t^j\right) \right| &= \lim_{t\to\infty} \left| \mathbb{E}\left(v_t^0 - \bar{v}_t\right) - \mathbb{E}\left(v_t^j - \bar{v}_t\right) \right| \\
&= \lim_{t\to\infty} \left| \mathbb{E}\left(\hat{v}_t^0\right) - \mathbb{E}\left(\hat{v}_t^j\right) \right| \\
&= \lim_{t\to\infty} \left| e^{-(\alpha+\beta)t} \mathbb{E}\left(\hat{v}_0^0\right) - e^{-\beta t} \mathbb{E}\left(\hat{v}_0^i\right) \right| \\
&= \lim_{t\to\infty} \left| e^{-(\alpha+\beta)t} \mathbb{E}\left(\hat{v}_0^0\right) - e^{-\beta t} \mathbb{E}\left(\hat{v}_0^i\right) \right| \\
&\le \lim_{t\to\infty} e^{-(\alpha+\beta)t} \left| \mathbb{E}\left(\hat{v}_0^0\right) \right| + e^{-\beta t} \left| \mathbb{E}\left(\hat{v}_0^i\right) \right| \\
&= 0.
\end{aligned}$$

□

The above theorem provides us the fact that flocking still emerges even in such stochastic particle systems with a central particle. We will apply this result to further characterize systemic risk in Chapter 5. The particle dynamics studied in this chapter will represent the dynamics of log-monetary reserves of banks later in Chapter 3 and will use this time-asymptotic flocking result.

Chapter 3

Systemic risk with jumps

3.1 Systemic risk model

Systemic risk is the risk of financial system instability or failure that may occur under certain circumstances and can sometimes led to catastrophic consequences to the interconnected financial system. Many researchers have investigated systemic risk and its impact on the banking market [Bo and Capponi, 2013], [Fouque and Ichiba, 2013], [Fouque and sun, 2013], [Garnier et al., 2012]. In [Fouque and Ichiba, 2013], the authors analyze systemic risk in interbank lending systems by modeling monetary reserves of banks as a system of interacting Feller diffusions. Their model demonstrates that growth rate and lending preference are key factors in understanding systemic risk in an interbank lending system. In [Fouque and Sun, 2013], the authors propose a simpler system with log-monetary reserves of N banks where the rates of borrowing\lending between individual banks are proportional to the difference between their log-monetary reserves. They further define systemic risk as the risk that all banks will be in default simultaneously for a given horizon time.

In this chapter, we aim to extend the model proposed in [Fouque and Sun, 2013] and establish a mathematical model to characterize the banking system and the effect on the systemic risk when each bank is exposed to the possibility of having instantaneous shock or sudden bankruptcy that we identify as jumps. To get a more in-depth understanding of systemic risk that may help us to navigate and further prevent such event, we propose an interbank borrowing and lending model that allows sudden increase or decrease in individual bank reserve to illustrate a systemic risk by using a stochastic flocking system described in Chapter 2.

We consider a system of N banks with log-monetary reserves interacting with each other through interbank borrowing and lending. We assume that, for $i = 1, .., N$, the log-monetary reserves of the i^{th} bank satisfies the following dynamics

$$dX_t^i = \frac{\alpha}{N} \sum_{j=1}^{N} \left(X_t^j - X_t^i \right) dt + dL_t^i, \tag{3.1}$$

where $\alpha \geq 0$ is defined as the rate of borrowing or lending to each other. For $i = 1, ..., N$, the processes L_t^i are independent and one-dimensional Lévy processes with generating triplet (a, σ, ρ^i) . We assume that the initial log-monetary reserves X_0^i is zero. The overall rate of mean-reversion $\frac{\alpha}{N}$ is normalized by the number of banks. Moreover, the drift term indicates the interaction between the reserves of bank i and bank j. For bank i with reserve X_t^i, it will borrow from bank j

if $X_t^i < X_t^j$ and lend to bank j if $X_t^i > X_t^j$. Note that this is the case of linear communication rate in the SCS model where we regard the velocity as the log-monetary reserve. As a result, we can expect that flocking emerges whenever the rate α is large or in the long run. In the sense of this banking system, for each bank i, the log-monetary reserve X_t^i will be almost the same as long as there are more interbank activities, i.e., the rate of borrowing or lending is large. It is shown in [Fouque and Sun, 2013] that increasing the rate will not only increases the stability of this banking system but also systemic risk.

As stated in [Fouque and Sun, 2013], in order to study the systemic risk, we first define the first default time for bank i as

$$\tau_i = \inf\left\{t \geq 0; X_t^i \leq \eta\right\}, \ \eta < 0.$$

We are interested in the event that all banks will be in default simultaneously for a given horizon time T. Therefore, we investigate the joint probability that

$$P\left(\tau_i \leq T, i = 1, ..., N\right) = P\left(\min_{0\leq s\leq T} X_s^i \leq \eta, i = 1, ..., N\right). \tag{3.2}$$

According to the literature [Di Crescenzo et al., 1995], it is difficult to compute the joint probability explicitly even when $N = 2$. It is even more complicated to find this probability in a high dimensional banking system with interacting drift terms.

However, within the special structured system where noise terms are driven by particular processes, we will be able to compute the joint probability (3.2) approximately by the behavior of flocking, which will be discussed in section 3.3. In the following section, we investigate the mean-field limit in this interbank lending system.

3.2 Mean-field limit

To understand system (3.1) , we rewrite the dynamics as

$$\begin{aligned} dX_t^i &= \frac{\alpha}{N}\sum_{j=1}^{N}\left(X_t^j - X_t^i\right)dt + dL_t^i \\ &= \alpha\left[\frac{1}{N}\sum_{j=1}^{N}X_t^j - X_t^i\right]dt + dL_t^i,\ \ i = 1, ..., N. \end{aligned}$$

Observe that processes X_t^i are now Lévy-type OU processes which mean-revert to the ensemble average $\bar{X}_t = \frac{1}{N}\sum_{j=1}^{N}X_t^j$, which indicates that $\bar{X}_t$ approximately leads the entire banking system. Moreover, by taking the average, the ensemble average $\bar{X}_t$ satisfies

$$d\bar{X}_t = \frac{1}{N}\sum_{i=1}^{N}dL_t^i.$$

Recall that $X_0^i = 0$, for $i = 1, ..., N$; we then have $\bar{X}_t = \frac{1}{N}\sum_{i=1}^{N}L_t^i$, and so

$$dX_t^i = \alpha\left[\frac{1}{N}\sum_{i=1}^{N}L_t^i - X_t^i\right]dt + dL_t^i.$$

The solution X_t^i is given by

$$X_t^i = \frac{1}{N}\sum_{i=1}^{N}L_t^i + e^{-\alpha t}\int_0^t e^{\alpha s}dL_s^i - \frac{1}{N}\sum_{j=1}^{N}e^{-\alpha t}\int_0^t e^{\alpha s}dL_s^j.$$

Note that the ensemble average is still a Lévy process.

As $N \to \infty$, the strong law of large numbers gives

$$\frac{1}{N}\sum_{i=1}^{N} L_t^i \to \gamma \equiv \mathbb{E}\left(L_t^1\right) = t\left(a + \int_{|x|\geq 1} x\rho\,(dx)\right) \qquad \text{a.s.,}$$

and therefore $(X_t^i)^{'}s$ converges to independent Lévy-type OU processes

$$\left(a + \int_{|x|\geq 1} x\rho\,(dx)\right)\left(1 - \frac{1-e^{-\alpha t}}{\alpha}\right) + e^{-\alpha t}\int_0^t e^{\alpha s} dL_s^i$$

with long-run mean γ. In fact, this is a simple example of a mean-field limit and propagation of chaos studied in [Sznitman, 1991]. In conclusion, getting more banks involved in this lending system will make the system more complicated intuitively; however, the banks will eventually act independently.

3.3 Systemic risk illustrated with jump diffusion processes

The stability of this interbank system with coupled diffusions has been illustrated in [Fouque and Sun, 2013], the authors show that more interactivities between individual banks will not only create stability but also systemic risk. In this section, we aim to extend their result by adding jumps which can be regarded as optimistic news or sudden bankruptcy.

We assume that, for $i = 1, .., N$, the log-monetary reserves of the i^{th} bank satisfies the following dynamics:

$$dX_t^i = \frac{\alpha}{N}\sum_{j=1}^{N}\left(X_t^j - X_t^i\right)dt + dL_t^i, \tag{3.3}$$

where α is defined as the rate of borrowing and lending. The processes $L_t^i = \sigma W_t^i + \sum_{j=1}^{N_t^i} \xi_j$ are one-dimensional jump diffusion processes, where W_t^i are independent Brownian motions, $\sigma > 0$, jump sizes ξ_j have distribution $f(y;\theta) = \frac{\theta}{2} e^{-|y|\theta}, \theta > 0$ and N_t^i are Poisson processes independent of W_t^i, with intensity parameter λ. We assume that the initial log-monetary reserves $X_0^i = 0$. The parameter σ is the volatility of Brownian motions, while the parameters θ and λ are described as the strength and the frequency of instantaneous shock, respectively.

To understand the systemic risk, we now investigate the joint default probability (3.2), i.e., how many banks have reached the default level $\eta < 0$ before $t = 1$. In order to illustrate the systemic risk in this setting, we assume the default level $\eta = -0.7$. Define $\{\text{default event}\} = \{\min_{0 \leq t \leq 1} X_t^i \leq \eta, 1 \leq i \leq N\}$ and $K \equiv \{\# \text{ of default}\}$. We are interested in the loss distribution

$$p = \mathbb{P}(K = k), \quad \text{where } k = 0, 1..., N.$$

We aim to know the probability that all banks will be in default simultaneously before $t = 1$, i.e., $K = N$; however, such probability is difficult to obtain explicitly. Alternatively, we compute this probability numerically through simulations to illustrate the systemic risk. We simulate 10^4 trajectories for the dynamics (3.3) by employing an Euler scheme with time dividend $\Delta t = 0.0001$ and illustrate the effect of rate α as well as jumps. Below, we provide figures for the loss distribution in both coupled diffusions and jump diffusions for $\alpha = 1$, indicating weak interbank lending activities, and for $\alpha = 100$, indicating strong interbank lending activities. These figures also show one realization of the trajectories for different values α.

In Figure 3.1 , we assume that the noise terms in (3.3) are driven by Brownian motions and when $\alpha = 1$, the model appears to be driven by independent Brownian motions. Thus, the chance that the log-monetary reserve of each bank i will be in default before $t = 1$, i.e., $\{\min_{0 \leq t \leq 1} X_t^i \leq -0.7\}$, is equally likely.

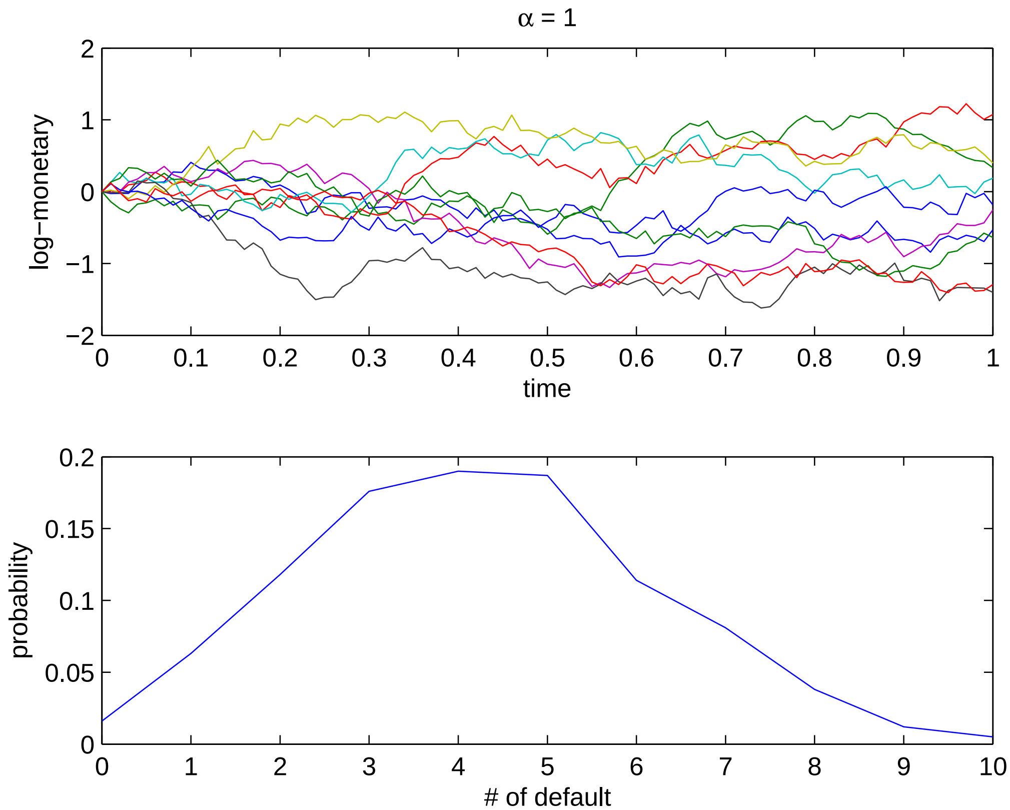

Figure 3.1: Plots of trajectories for the model (3.3) with Brownian motions (upper one) and the corresponding loss distribution (bottom one) in a fixed time $t = 1$. Weak flocking for $\alpha = 1$ in the upper one; the model appears to be driven by independent Brownian motions. The parameters used: $N = 10$, $\sigma = 1$ and $\eta = -0.7$..

The loss distribution is shown as a Binomial distribution (N, p), $N = 10$, $p \approx \mathbb{P}\left\{\min_{0 \le t \le 1} X_t^i \le -0.7\right\}$. In addition, the default probability can be computed in general for $\alpha = 0$ with the initial reserve $X_0^i = 0$ and noise terms $L_t^i = \sigma W_t^i$ in

(3.3). We conclude that

$$\begin{aligned} p &= \mathbb{P}\left\{\min_{0\le t\le T} X_t^i \le \eta\right\} \\ &= \mathbb{P}\left\{\min_{0\le t\le T} L_t^i \le \eta\right\} \\ &= \mathbb{P}\left\{\min_{0\le t\le T} \sigma W_t^i \le \eta\right\} \\ &= 2\Phi\left(\frac{\eta}{\sigma\sqrt{T}}\right), \end{aligned}$$

where Φ denotes the cdf of $N(0,1)$ and we use the distribution of the first passage time for Brownian motion, which will be shown in section 3.3.1.

In Figure 3.2, we see that the loss distribution corresponds to either no default or all defaults. It appears that the fat tail corresponds to the small probability of the ensemble average reaching the default level, and to almost all diffusions following this average due to flocking behavior for large α. The authors in [Fouque and Sun, 2013] identify this small probability as a "systemic risk" probability which can be obtained by the distribution of the first passage time of Brownian motions due to flocking in this stable system.

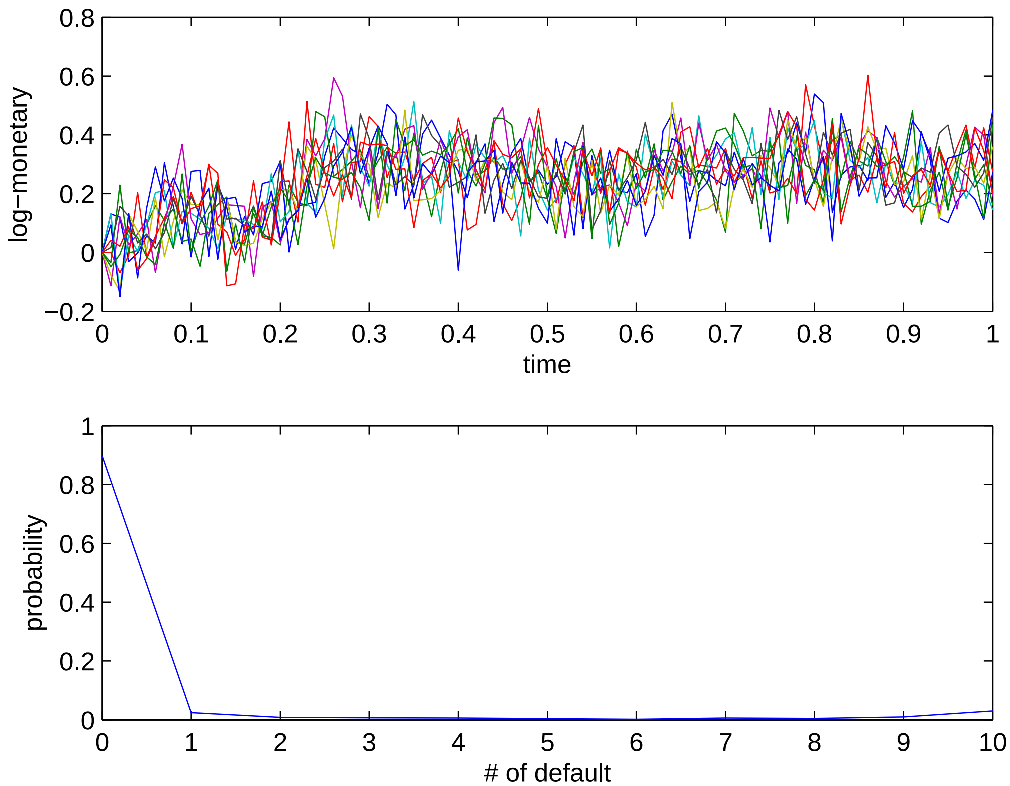

Figure 3.2: Plots of trajectories for the model (3.3) with Brownian motions (upper one) and the corresponding loss distribution (bottom one) in a fixed time $t = 1$. Compare to Figure 3.1, flocking emerges for $\alpha = 100$ in the upper one and the loss distribution corresponding to either no default or all defaults. The parameters used: $N = 10$, $\sigma = 1$ and $\eta = -0.7$.

We next illustrate the effect of jumps on the systemic risk through another simulation using a model with jump diffusions shown in Figure 3.3. We simulate the model in (3.3):

$$dX_t^i = \frac{\alpha}{N} \sum_{j=1}^{N} \left(X_t^j - X_t^i \right) dt + dL_t^i,$$

where α is defined as the rate of borrowing and lending. The processes $L_t^i = \sigma W_t^i + \sum_{j=1}^{N_t^i} \xi_j$ are one-dimensional jump diffusion processes, where W_t^i are independent Brownian motions, $\sigma > 0$, jump sizes ξ_j have distribution $f(y;\theta) = \frac{\theta}{2} e^{-|y|\theta}, \theta > 0$ and N_t^i are Poisson processes independent of W_t^i, with intensity parameter λ.

In Figure 3.3, we obtain a higher probability that all banks will be in default before $t = 1$. Note that $\theta = 1$ means we have a greater chance that one of the banks will reach the default $\eta = -0.7$. As a result, in the jump diffusion model with double exponential jump size, we obtain a similar result as the Brownian motion case but higher probability on the fat tail. Observe that the small probability, i.e., systemic risk increases and so does the risk of individual banks, that is, the probability that one bank will be in default.

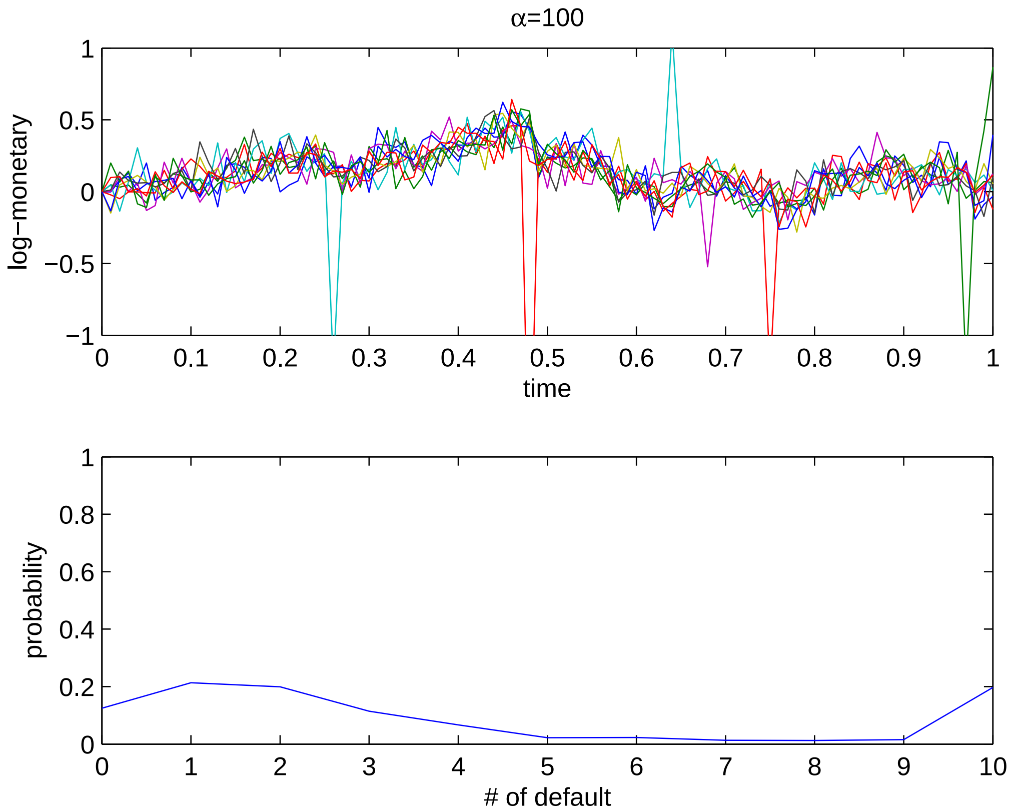

Figure 3.3: Plot of trajectories for the model (3.3) with jump-diffusion noises as well as the corresponding loss distribution in a fixed time $t = 1$. Note that the upper one is just one realization of the model (3.3), flocking still emerges for $\alpha = 100$ according to the analysis in section 2.3. The parameters used: $N = 10$, $\sigma = 1, \theta = 1$ and $\lambda = 1$.

In [Fouque and Sun, 2013], the authors identify $\left\{\min_{0\le t\le T} \bar{X}_t \le \eta\right\}$ as a systemic event in a coupled diffusion model. We now focus on the event where the ensemble average $\bar{X}_t = \frac{1}{N}\sum_{i=1}^{N} L_t^i$ reaches the default level $\eta < 0$. The probability that this event, which we call systemic risk, can be written as follows:

$$P\left(\min_{0\le t\le T}\bar{X}_t\le\eta\right)=P\left(\min_{0\le t\le T}\frac{1}{N}\sum_{i=1}^{N}L_t^i\le\eta\right). \tag{3.4}$$

Note that this probability might only be computed explicitly under certain situations (see [Kou and Wang, 2003] and [Novikov et al, 2003]) such as when L_t^i is a Brownian motion, only possesses one-sided jumps or there is some special distribution for the jump size.

3.3.1 First passage time for Brownian motions

For searching the distribution of the first passage, we define $\tau_b=\inf\{t\ge 0; L_t\ge b\}$, $b>0$. It is well known that, if $L_t=W_t$, the Laplace transform of τ_b is then given by, for $s\in(0,\infty)$,

$$\mathbb{E}\left[e^{-s\tau_b}\right]=e^{-b\sqrt{2s}}.$$

Moreover, the density function of τ_b is given by

$$f_{\tau_b}(t)=\frac{b}{\sqrt{2\pi}}t^{-\frac{3}{2}}e^{-\frac{b^2}{2t}},\quad t\ge 0.$$

In terms of (3.4), let $L_t^i=\sigma W_t^i$. Then the probability can be computed explicitly as

$$\begin{aligned}P\left(\min_{0\le t\le T}\frac{1}{N}\sum_{i=1}^{N}L_t^i\le\eta\right)&=P\left(\min_{0\le t\le T}\frac{\sigma}{N}\sum_{i=1}^{N}W_t^i\le\eta\right)\\&=P\left(\min_{0\le t\le T}\tilde{W}_t\le\frac{\eta\sqrt{N}}{\sigma}\right)\\&=2\Phi\left(\frac{\eta\sqrt{N}}{\sigma T}\right),\end{aligned} \tag{3.5}$$

by a change of measure, by the reflection principle or by computing the Laplace transform through some appropriate martingale and the optional stopping theorem. Here, $\tilde{W}_t$ is a standard Brownian motion.

3.3.2 First passage time for jump-diffusion processes

If L_t is a jump-diffusion process, there are some difficulties for finding the distribution of the first passage time depending on the structure of jumps. The intuition is given as follows:

- Without a jump part, the distribution of the first passage time can be obtained by the reflection principle or by calculating the Laplace transform via some appropriate martingale and optional sampling theorem.

- With a jump part, it is difficult to find the distribution of the first passage time since the process may hit the boundary exactly or it incurs an "overshoot", $L_{\tau_b} - b$, over the boundary.

- The overshoot presents many problems.

  1. We have to know the exact distribution of the overshoot, i.e, $P\left(L_{\tau_b} - b = 0\right)$ and $P\left(L_{\tau_b} - b > x\right)$ for $x > 0$.

  2. We need to know the dependent structure between the overshoot $L_{\tau_b} - b$ and the first passage time τ_b.

3. If we want to use the reflection principle, the dependent structure between the overshoot and the terminal value of L_t is needed.

However, the special case that the size of jump is double-exponentially distributed has been solved in [Kou and Wang, 2003]. We state the important results needed for systemic risk in what follows.

Let L_t^i be a jump-diffusion process with double-exponentially distributed jump sizes, i.e.,

$$L_t^i = \sigma W_t^i + \sum_{j=1}^{N_t^i} \xi_j, \tag{3.6}$$

where N_t^i is a Poisson process with intensity rate $\lambda > 0$ and $\{\xi_j, j \geq 1\}$ are i.i.d. random variables with common distribution

$$f\left(y;\theta\right) = \frac{\theta}{2} e^{-|y|\theta}, \theta > 0.$$

For all $u \in C^2$, the infinitesimal generator of L_t^i is given by

$$\mathcal{L}u\left(x\right) = \frac{1}{2}\sigma^2 x^2 u''\left(x\right) + \lambda \int_R \left[u\left(x+y\right) - u\left(x\right)\right] f_\xi\left(y\right) dy.$$

Moreover, assume that $z \in \left(-\theta, \theta\right)$. The moment generating function of L_t is given by

$$\mathbb{E}\left[e^{zL_t}\right] = \exp\left\{G\left(z\right)t\right\},$$

where $G\left(x\right) \equiv \frac{1}{2}\sigma^2 x^2 + \lambda\left(\frac{\theta^2}{\theta^2 - x^2} - 1\right)$. Then the Laplace transform of $P\left(\tau_b \leq T\right) = P\left(\max_{0 \leq t \leq T} L_t \geq b\right)$ is given by the following theorem:

Theorem [Kou and Wang, 2003]. For any $s \in (0, \infty)$, let β_1 and β_2 be the only two positive roots of the equation

$$s = G(\beta),$$

where $0 < \beta_1 < \theta < \beta_2 < \infty$. Then the Laplace transform of τ_b is given by

$$\mathbb{E}\left[e^{-s\tau_b}\right] = \frac{\theta - \beta_1}{\theta} \frac{\beta_2}{\beta_2 - \beta_1} e^{-b\beta_1} + \frac{\beta_2 - \theta}{\theta} \frac{\beta_1}{\beta_2 - \beta_1} e^{-b\beta_2}.$$

Remark 3.1. *See Theorem 3.1 in [Kou and Wang, 2003] for the general case if interested, i.e.,* $f(y;\theta) = p\theta_1 e^{-y\theta_1} 1\{y \geq 0\} + q\theta_2 e^{y\theta_2} 1\{y < 0\}, \theta_1, \theta_2 > 0$ *and* $p + q = 1$.

Back to the systemic risk (3.4), the probability is given by

$$\begin{aligned} P\left(\min_{0 \leq t \leq T} \frac{1}{N} \sum_{i=1}^{N} L_t^i \leq \eta\right) &= P\left(\max_{0 \leq t \leq T} \frac{1}{N} \sum_{i=1}^{N} L_t^i \geq b\right), \text{ where } b = -\eta. \\ &= P\left(\max_{0 \leq t \leq T} \sum_{i=1}^{N} L_t^i \geq Nb\right) \\ &= P\left(\max_{0 \leq t \leq T} \tilde{L}_t \geq Nb\right) \\ &= P\left(\tau_{Nb} \leq T\right), \end{aligned} \tag{3.7}$$

where $\tilde{L}_t = \tilde{\sigma} W_t + \sum_{j=1}^{\tilde{N}_t} \xi_j$, $\tilde{\sigma} = \sqrt{N}\sigma$ and $\tilde{N}_t$ is a Poisson process with rate $\tilde{\lambda} = N\lambda$.

By using the above theorem and the inverse Laplace transform, we can therefore obtain the numerical result of the probability (3.7). As stated in [Kou and Wang, 2003], the Gaver-Stehfest algorithm by [Gaver, 1966] and [Stehfest, 1970] is a crucial method since it is the only algorithm that can deal with the inversion on the real line. Alternatively, one may see [Abate and Whitt, 1992] for more details about Laplace inversion algorithms. The algorithm is described as follows:

For any bounded real-valued function f on $[0, \infty)$ and continuous at t we have that

$$f(t) = \lim_{n \to \infty} \tilde{f}_n(t),$$

where $f_n(t) = \frac{\ln(2)}{t} \frac{(2n)!}{n!(n-1)!} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \hat{f}\left((n+k)\frac{\ln(2)}{t}\right)$ and $\hat{f}$ is the Laplace transform of f. Since the explicit form of $\hat{f}$ is given in the previous theorem, we will now be able to compute the distributions of the first passage times for the double exponential jump diffusion process and thus the systemic risk probability (3.4) with (3.6) can be computed numerically.

It is worth discussing (3.4) for a Lévy process L_t; however, finding an explicit formula for the distribution of the first hitting time for a Lévy process is quite difficult. According to [Kou and Wang, 2003], the only case of an explicit formula, so far, is the jump diffusion with double exponential jump size. Problem (3.4) can be solved either through a martingale approach or an integral equation approach, see [Hadjiev, 1985], [Novikov, 1981]. In addition, see [Novikov et al., 2003] for a survey of the first hitting time problems of compound Poisson processes. Moreover, in [Braverman, 2009] the author provides an asymptotic probability for (3.4) as $N \to \infty$ that may help us to see the systemic risk in the general case.

At the end of this section, we provide the plot for the probability that the ensemble average reaches the default barrier on both models for different number of banks N given in Figure 3.4.

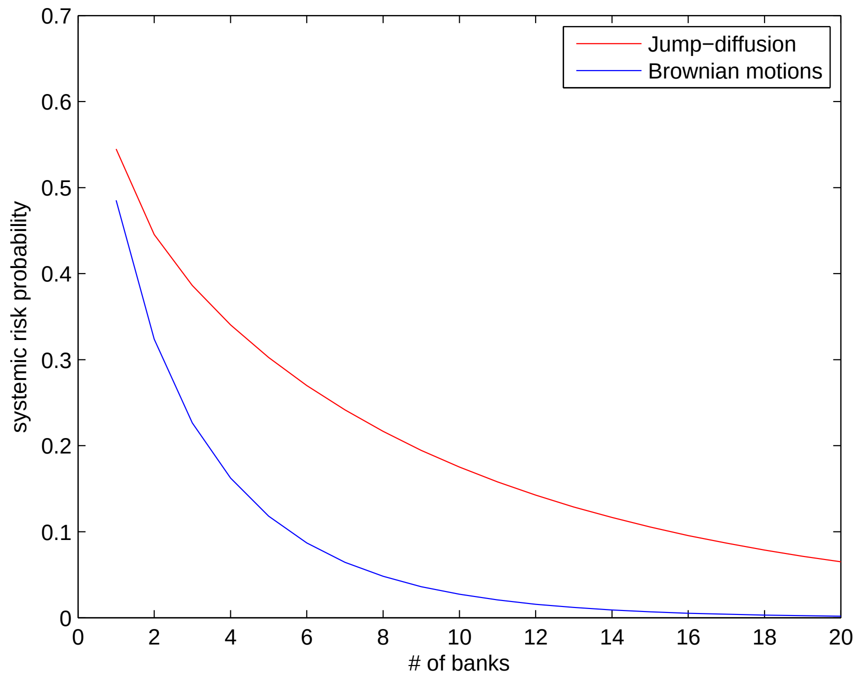

Figure 3.4: Plot of systemic risk probability for different number of banks with both Brownian motions and jump diffusions. The probability that the ensemble average reaches the default level $\eta = -0.7$ for both Brownian motions and jump diffusions decreases as N increases.

One can observe in Figure 3.4 that, compared to systemic risk with Brownian motions, the systemic risk probability is higher with jump diffusion processes and decays to 0 slower as N increases. Note that in [Fouque and Sun, 2013], the authors show the probability that the ensemble average reaches the default barrier is of order $\exp\left(-\frac{\eta^2 N}{2\sigma^2 T}\right)$ by the theory of large deviations when L_t is a Brownian motion.

To conclude this chapter, we see that including jumps in the dynamic of reserve processes increases not only the systemic risk but also the risk of individual banks. However, the model with jumps provides more flexibility by including more parameters such as intensity rates λ and jump sizes θ, for the purpose of calibration. In April 2015, China's central bank made a huge reduction in the amount of cash that banks must hold in reserves in order to add liquidity to help stimulate bank lending and fight slowing growth. This kind of dramatic change can be regarded as a sudden jump in the reserve processes that will expose the entire banking system to a higher risk environment. Making some slight changes in the reserve is reasonable and acceptable in a stable banking system such as studied in [Fouque and Sun, 2013] but might not be a good idea with dramatic changes (jumps) since it may result in a higher systemic risk.

Chapter 4

Stochastic differential games with jumps and systemic risk

In this chapter, we integrate a game feature with jumps, where each bank controls its rate of borrowing/lending with a central bank. We will use game theory and stochastic optimal control with jump-diffusion processes to analyze the impact of jumps to our interbank borrowing and lending model. We will solve a feedback equilibrium with jumps in game theory—Nash equilibria—with finitely many banks using both an FBSDE approach and an HJB approach in section 4.2.

4.1 Stochastic optimal control with jump-diffusion processes

In this section, we give some background and concepts related to stochastic optimal control with jump-diffusion processes. There are typically two ways to solve an optimal control problem: one way is through dynamic programming with corresponding HJB equations [Framstad et al, 1998] and the other approach is by using maximum principle [Framstad and Øksendal, 2004]. The deterministic case of maximum principle was first introduced in [Pontryagin et al., 1962]. The author in [Kushner, 1972] and [Bismut, 1973] further investigate a corresponding maximum principle for Ito diffusions. Subsequently, the maximum principle for Ito diffusions was developed in [Bensoussan, 1983], [Bensoussan, 1991] and [Haussmann, 1986]. Recently, a sufficient maximum principle for jump diffusions was formulated in [Framstad et al, 2004] and is further summarized in the textbook [Øksendal and Sulem, 2007] from which we state some related results in the following.

Maximum principle for jump diffusion processes

Assuming that the state process $X_t = X_t^{(\alpha)}$ of a controlled jump diffusion in R satisfies the dynamics

$$\begin{aligned} dX_t &= b\left(t, X_t, \alpha_t\right) dt + \sigma\left(t, X_t, \alpha_t\right) dW_t \\ &\quad + \int_R \gamma\left(t, X_{t^-}, \alpha_{t^-}, z\right) \tilde{N}\left(dt, dz\right), \end{aligned} \tag{4.1}$$

$X_0 = x \in R$, where

$$\begin{aligned} b &: [0,T] \times R \times U \to R, \\ \sigma &: [0,T] \times R \times U \to R \text{ and} \\ \gamma &: [0,T] \times R \times U \times R \to R \end{aligned}$$

are given functions, and $U \subset R$ is a given set. Let $\tilde{N}(dt, dz) = N(dt, dz) - \rho(dz)\, dt$ be the compensated Poisson random measure with Lévy measure ρ. The process $\alpha_t = \alpha(t, \omega) : [0,T] \times \Omega \to R$ is the control process and assumed to be càdlàg and adapted. We say the control process α_t is admissible if there exists a unique and strong solution to (4.1), and we denote $\mathcal{A}$ the set of all admissible controls.

The performance criterion (objective function) is given by

$$J(\alpha) = \mathbb{E}\left[\int_0^T f(t, X_t, \alpha_t)\, dt + g(X_T)\right],$$

where $T < \infty$ is deterministic, f is continuous and g is concave. We consider an optimal control problem that finds an admissible $\alpha^* \in \mathcal{A}$ such that

$$J(\alpha^*) = \sup_{\alpha \in A} J(\alpha).$$

The approach is similar to the case of Ito diffusion except we now have to deal with the jump part. Define the Hamiltonian $H : [0,T] \times R \times U \times R \times R \times R \to R$ by

$$\begin{aligned} H(t, x, \alpha, p, q, r) &= f(t, x, \alpha) + b(t, x, \alpha)\, p + \sigma(t, x, \alpha)\, q \\ &\quad + \int_R \gamma(t, x, \alpha, z)\, r(t, z)\, \nu(dz), \end{aligned} \tag{4.2}$$

where R is the set of functions $r : [0,T] \times R \to R$ such that the integrals in (4.2) converge. Functions p_t, q_t and r satisfy the forward and backward stochastic differential equation (FBSDE)

$$\begin{aligned} dp_t &= -H_x(t,x,\alpha,p,q,r)\,dt + q_t dW_t + \int_R r\left(t^-, z\right)\tilde{N}(dt,dz), \\ p_T &= g'(X_T) \end{aligned}$$

A sufficient maximum principle in [Framstad and Øksendal, 2004] is stated in the following:

Theorem 4.1. *Let $\alpha \in \mathcal{A}$ with corresponding solution $X^* = X^{(\alpha^*)}$ and suppose there exists a solution $(p_t^*, q_t^*, r^*(t,z))$ of the corresponding adjoint equation. Moreover, suppose that*

$$H(t, X_t^*, \alpha_t^*, p_t^*, q_t^*, r^*(t,\cdot)) = \sup_{u \in U} H(t, X_t^*, \alpha, p_t^*, q_t^*, r^*(t,\cdot))$$

and

$$H(x) := \max_{\alpha \in U} H(t, x, \alpha, p_t^*, q_t^*, r^*(t,\cdot)) \tag{4.3}$$

exists and is a concave function of x, for all $t \in [0,T]$. Then α^* is an optimal control.

Next, we provide a concrete example as provided by [Øksendal and Sulem, 2006] to see how to implement the above theorem.

Example : The stochastic linear regulator problem

Assuming that the state process $X_t = X_t^{(\alpha)}$ is given by

$$dX_t = \alpha_t dt + \sigma dW_t + \int_R z\tilde{N}(dt,dz), X_0 = x,$$

and $T > 0$ is a constant, we aim to solve a stochastic control problem that minimizes the objective function

$$J(x) = \mathbb{E}^x \left[\int_0^T \left(X_t^2 + \theta \alpha_t^2 \right) dt + \lambda X_T^2 \right].$$

Our goal is to find an admissible $\alpha^* \in \mathcal{A}$ such that

$$J(\alpha^*) = \inf_{\alpha \in \mathcal{A}} J(\alpha).$$

We can solve this problem by using the stochastic maximum principle. Define the Hamiltonian

$$H(t, x, \alpha, p, q, r) = x^2 + \theta \alpha^2 + \alpha p + \sigma q + \int_R z r\left(t^-, z\right) \nu(dz)$$

Then the corresponding adjoint equation is given by

$$\begin{aligned} p_t &= -2X_t dt + q_t dW_t + \int_R r\left(t^-, z\right) \tilde{N}(dt, dz); \quad t < T \\ p_T &= 2\lambda X_T. \end{aligned}$$

By the stochastic maximum principle, we minimize H with respect to α to obtain that $H(t, x, \alpha, p, q, r)$ is minimal for

$$\alpha_t = \hat{\alpha}_t = -\frac{1}{2\theta} p_t.$$

To find a solution of the adjoint equation, we consider an ansatz

$$p_t = \xi_t X_t,$$

where $\xi_t : R \to R$ is a deterministic function such that $\xi_T = 2\lambda$.

Note that $\alpha_t = -\frac{\xi_t X_t}{2\theta}$ and

$$dX_t = -\frac{\xi_t X_t}{2\theta}dt + \sigma dW_t + \int_R z\tilde{N}(dt, dz); \quad X_0 = x.$$

Moreover, differentiating the ansatz, we obtain

$$\begin{aligned} dp_t &= \xi_t dX_t + \acute{\xi}_t X_t dt \\ &= X_t\left[-\frac{\xi_t^2}{2\theta} + \acute{\xi}_t\right]dt + \xi_t \sigma dW_t + \xi_t \int_R z\tilde{N}(dt, dz). \end{aligned}$$

Hence, ξ_t is the solution to

$$\begin{aligned} \acute{\xi}_t &= \frac{\xi_t^2}{2\theta} - 2; \quad t < T \\ \xi_T &= 2\lambda. \end{aligned}$$

The solution is then given by

$$\xi_t = 2\sqrt{\theta}\frac{1 + \beta e^{\frac{2t}{\sqrt{\theta}}}}{1 - \beta e^{\frac{2T}{\sqrt{\theta}}}},$$

where $\beta = \frac{\lambda - \sqrt{\theta}}{\lambda + \sqrt{\theta}} e^{-\frac{2T}{\sqrt{\theta}}}$. By using the stochastic maximum principle, we can conclude that

$$\alpha_t^* = -\frac{\xi_t X_t}{2\theta}$$

is the optimal control, $p_t = \xi_t X_t$ and $q_t = \sigma \xi_t$, $r(t^-, z) = \xi_t z$.

4.2 Stochastic differential games with jumps

The noncooperative games analysis without jumps has been investigated in [Carmona et al., 2014]. Within our proposed interbank lending system, individual banks can control their rates of borrowing from and lending to the central bank. In addition, individual banks are allowed to have jumps as we mentioned in Chapter 3. The lending/borrowing interaction is determined by the amount in their reserves. If the reserve of a bank is below the average reserve of all banks, it will borrow money from the central bank. Conversely, if the reserve of a bank is above the average reserve, it will be lending money to the central bank. Each transaction comes with certain cost and fees. To minimize the cost, banks will seek the optimal strategies considering the distance between the average reserve and their own reserves. Here we want to see how this game feature with jumps may affect the systemic risk.

In this chapter, we look at N player games, where N is finite as studied in [Carmona et al., 2014]. Considering N controls for N players, each optimal strategy depends on all the other optimal strategies. We are looking for the equilibria in this game where the state processes are allowed to have jumps. We first construct closed-loop (feedback) equilibria using an FBSDE approach and then follow it up with an HJB approach.

Recall in Chapter 3 that the log monetary reserve X_t^i satisfies the dynamics

$$dX_t^i = \left[a\left(\bar{X}_t - X_t^i\right) + \alpha_t^i\right]dt + \sigma^i dW_t^i + \int_R \gamma^i\left(t^-, z\right) N^i\left(dt, dz\right), \tag{4.4}$$

where $W_t^i, i = 1, ..., N$ are independent Brownian motions, $a \geq 0$ and $\sigma^i > 0$. Here, $\int_R \gamma^i(t, z) N^i(dt, dz)$ is an independent jump process for $i = 1, ..., N$, where R is the set such that the integral converges with Poisson random measure $N^i(dt, dz)$ and of jump size $\gamma^i(t, z)$.

Control problem

As stated in [Carmona et al., 2013], bank $i \in \{1, ..., N\}$ controls its rate of lending and borrowing (to a central bank) at time t by choosing the control α_t^i in order to minimize

$$J^i\left(\alpha^i, ..., \alpha^N\right) = \mathbb{E}\left\{\int_0^T f_i\left(X_t, \alpha_t^i\right) dt + g_i\left(X_T^i\right)\right\} \tag{4.5}$$

with

$$\begin{aligned} f_i\left(x, \alpha^i\right) &= \frac{1}{2}\left(\alpha^i\right)^2 - q\alpha^i\left(\bar{x} - x^i\right) + \frac{\epsilon}{2}\left(\bar{x} - x^i\right)^2 \text{ and} & (4.6)\\ g_i\left(x\right) &= \frac{c}{2}\left(\bar{x} - x^i\right)^2, & (4.7)\end{aligned}$$

where the running cost function $f_i(x, \alpha)$ is convex in (x, α) under the assumption $q^2 \leq \epsilon$. Notice that the running quadratic cost $\frac{1}{2}\left(\alpha^i\right)^2$ has been normalized and that the effect of the parameter $q > 0$ is to control the incentive to borrowing or lending: bank i will want to borrow ($\alpha_t^i > 0$) if X_t^i is smaller than the empirical mean $\bar{X}_t$ and lend ($\alpha_t^i < 0$) if X_t^i is larger than $\bar{X}_t$. Equivalently, after dividing by $q > 0$, this parameter can be thought as a control by the regulator of the cost of borrowing or lending (with q large meaning low fees). The quadratic terms in $\left(\bar{x} - x^i\right)^2$ in the running cost ($\epsilon > 0$) and in the terminal cost ($c > 0$) penalize departure from the average.

Recall the state processes X_t in the stochastic maximum principle; we need the jump part of the state process to be a martingale in order to apply the principle to solve the control problem. As a result, we assume that $\int_R \gamma^i(t, z)\, \rho^i(dz) < \infty$, and then rewrite (4.4) as

$$\begin{aligned} dX_t^i &= \left[a\left(\bar{X}_t - X_t^i\right) + \alpha_t^i + \int_R \gamma^i(t,z)\,\rho^i(dz)\right] dt + dL_t^i \\ &= \left[a\left(\bar{X}_t - X_t^i\right) + \alpha_t^i + \upsilon_t^i\right] dt + dL_t^i, \end{aligned} \tag{4.8}$$

where now $dL_t^i = \sigma^i dW_t^i + \int_R \gamma^i(t^-,z)\,\tilde{N}^i(dt,dz)\,, i = 1,...,N$ are independent martingales with compensated Poisson random measures $\tilde{N}^i(dt,dz) = N^i(dt,dz) - \rho^i(dz)\,dt$ and $\upsilon_t^i = \int_R \gamma^i(t,z)\,\rho^i(dz)\,.$ Processes (4.8) are now the state processes for solving control problem (4.5).

4.2.1 Closed-loop equilibria : FBSDE approach

With the state processes (4.8) and the objective function J^i, we solve for an exact closed-loop Nash equilibrium when banks at time t use Markovian strategies and have the complete information of states of all other banks. When all other banks $k \neq i$ have chosen their strategies $\alpha^k(t,x)\,,$ bank i has to solve a Markovian control problem to search for its best strategy among these choices.

Using the Pontryagin approach, the Hamiltonian for bank i is given by

$$\begin{aligned} & H^i\left(x, y^{i,1},...,y^{i,N}, \alpha^1(t,x)\,,...,\alpha_t^i,...,\alpha^N(t,x)\right) \\ = & \sum_{k\neq i}^{N} \left[a\left(\bar{x} - x^k\right) + \alpha^k(t,x) + \upsilon_t^k\right] y^{i,k} + \left[a\left(\bar{x} - x^i\right) + \alpha^i + \upsilon_t^i\right] y^{i,i} \\ & + \frac{1}{2}\left(\alpha^i\right)^2 - q\alpha^i\left(\bar{x} - x^i\right) + \frac{\epsilon}{2}\left(\bar{x} - x^i\right)^2. \end{aligned}$$

The state processes X_t^i are given by (4.8) with initial condition $X_0^i = x^i$. Based on the Pontryagin principle, the adjoint processes $Y_t^i = \left(Y_t^{i,j} : j = 1,...,N\right)\,, Z_t^i =$

$\left(Z_t^{i,j,k} : j = 1, ..., N, k = 1, ..., N\right)$ and $\left(r^{i,j,k}\left(t^-, z\right) : j = 1, ..., N, k = 1, ..., N\right)$ for $i = 1, ..., N$ are defined as the solutions to the BSDEs

$$dY_t^{i,j} = -\partial_{x^j} H^i \left(X_t, Y_t^i, \alpha_t\right) dt + \sum_{k=1}^{N} Z_t^{i,j,k} dW_t^k + \sum_{k=1}^{N} \int_R r^{i,j,k}\left(t^-, z\right) \tilde{N}^k\left(dt, dz\right) \tag{4.9}$$

with terminal conditions $Y_T^{i,j} = \partial_{x^j} g_i\left(X_T\right)$.

Without any information about strategy $\alpha^i, i = 1, ..., N$, the partial derivative with respect to x^j of the Hamiltonian H^i is given by

$$\begin{aligned} \partial_{x^j} H^i &= a \sum_{k=1}^{N} \left(\frac{1}{N} - \delta_{k,j}\right) y^{i,k} + \sum_{k \neq i} \left(\partial_{x^j} \alpha^k\left(t, x\right)\right) y^{i,k} \\ &\quad - q\alpha^i \left(\frac{1}{N} - \delta_{i,j}\right) + \epsilon\left(\bar{x} - x^i\right)\left(\frac{1}{N} - \delta_{i,j}\right). \end{aligned} \tag{4.10}$$

The stochastic maximum principle (4.3) suggests that one minimizes H^i with respect to α^i yields choices

$$\hat{\alpha}^i = -y^{i,i} + q\left(\bar{x} - x^i\right), \qquad i = 1, ..., N. \tag{4.11}$$

We assume that all banks are making that choice so as to prove that this choice is a Nash equilibrium. Our goal is to find a solution to (4.9) and identify each bank's own adjoint equation so that bank i obtains its best response. However, the solution is difficult to find in general but can be obtained in the linear case.

We make the ansatz

$$Y_t^{i,j} = \left(\frac{1}{N} - \delta_{i,j}\right)\left[\eta_t\left(\bar{X}_t - X_t^i\right) + \varphi_t^i\right], \tag{4.12}$$

where η_t and φ_t^i are deterministic functions satisfying the terminal conditions $\eta_T = c$ and $\varphi_T^i = 0$.

With ansatz (4.12) , we have the choices of control (4.11)

$$\begin{aligned} \alpha^k &= \left[q + \eta_t \left(1 - \frac{1}{N}\right)\right] \left(\bar{x} - x^k\right) + \left(1 - \frac{1}{N}\right) \varphi_t^k, \\ \partial_{x^j} \alpha^k &= \left[q + \eta_t \left(1 - \frac{1}{N}\right)\right] \left(\frac{1}{N} - \delta_{k,j}\right). \end{aligned} \tag{4.13}$$

Plugging these choices into (4.10) , the derivation of $\partial_{x^j} H^i$ is then given by

$$\begin{aligned} \partial_{x^j} H^i &= a \sum_{k=1}^{N} \left(\frac{1}{N} - \delta_{k,j}\right) \left(\frac{1}{N} - \delta_{i,k}\right) \left[\eta_t \left(\bar{x} - x^i\right) + \varphi_t^i\right] \\ &\quad + \sum_{k \neq i} \left[q + \eta_t \left(1 - \frac{1}{N}\right)\right] \left(\frac{1}{N} - \delta_{k,j}\right) \left(\frac{1}{N} - \delta_{i,k}\right) \left[\eta_t \left(\bar{x} - x^i\right) + \varphi_t^i\right] \\ &\quad - q \left(\frac{1}{N} - \delta_{i,j}\right) \left\{\left[q + \eta_t \left(1 - \frac{1}{N}\right)\right] \left(\bar{x} - x^i\right) + \left(1 - \frac{1}{N}\right) \varphi_t^i\right\} \\ &\quad + \epsilon \left(\bar{x} - x^i\right) \left(\frac{1}{N} - \delta_{i,j}\right) \\ &= a \left[\eta_t \left(\bar{x} - x^i\right) + \varphi_t^i\right] \sum_{k=1}^{N} \left(\frac{1}{N} - \delta_{k,j}\right) \left(\frac{1}{N} - \delta_{i,k}\right) \\ &\quad + \left[q + \eta_t \left(1 - \frac{1}{N}\right)\right] \left[\eta_t \left(\bar{x} - x^i\right) + \varphi_t^i\right] \frac{1}{N} \sum_{k \neq i} \left(\frac{1}{N} - \delta_{k,j}\right) \\ &\quad + \left(\frac{1}{N} - \delta_{i,j}\right) \left(\bar{x} - x^i\right) \left[\epsilon - q^2 - q\eta_t \left(1 - \frac{1}{N}\right)\right] \\ &\quad - \left(\frac{1}{N} - \delta_{i,j}\right) \left(1 - \frac{1}{N}\right) q\varphi_t^i \\ &= - \left(\frac{1}{N} - \delta_{i,j}\right) \left(\bar{x} - x^i\right) \left[(a + q)\, \eta_t + \frac{1}{N} \left(1 - \frac{1}{N}\right) \eta_t^2 - \epsilon + q^2\right] \\ &\quad - \left(\frac{1}{N} - \delta_{i,j}\right) \left[(a + q)\, \varphi_t^i + \frac{1}{N} \left(1 - \frac{1}{N}\right) \eta_t \varphi_t^i\right], \end{aligned}$$

where we used the fact that $\sum_{k=1}^{N} \left(\frac{1}{N} - \delta_{k,j}\right) \left(\frac{1}{N} - \delta_{i,k}\right) = -\left(\frac{1}{N} - \delta_{i,j}\right)$ and $\sum_{k \neq i} \left(\frac{1}{N} - \delta_{k,j}\right) = -\left(\frac{1}{N} - \delta_{i,j}\right)$.

We now plug $\partial_{x^j} H^i$ into BSDE (4.9) and so the backward equation is given by

$$\begin{aligned} dY_t^{i,j} &= -\partial_{x^j} H^i dt + \sum_{k=1}^{N} Z_t^{i,j,k} dW_t^k + \sum_{k=1}^{N} \int_R r^{i,j,k}\left(t^-, z\right) \tilde{N}^k\left(dt, dz\right) \\ &= \left(\frac{1}{N} - \delta_{i,j}\right)\left(\bar{X}_t - X_t^i\right)\left[\left(a+q\right)\eta_t + \frac{1}{N}\left(1-\frac{1}{N}\right)\eta_t^2 - \epsilon + q^2\right] dt \\ &\quad + \left(\frac{1}{N} - \delta_{i,j}\right)\left[\left(a+q\right)\varphi_t^i + \frac{1}{N}\left(1-\frac{1}{N}\right)\eta_t\varphi_t^i\right] dt \\ &\quad + \sum_{k=1}^{N} Z_t^{i,j,k} dW_t^k + \sum_{k=1}^{N} \int_R r^{i,j,k}\left(t^-, z\right) \tilde{N}^k\left(dt, dz\right), \end{aligned} \tag{4.14}$$

with the terminal conditions $Y_T^{i,j} = \left(\frac{1}{N} - \delta_{i,j}\right)\left[c\left(\bar{X}_T - X_T^i\right)\right]$.

With choice controls (4.13), the forward dynamics are given by

$$\begin{aligned} dX_t^i &= \partial_{y^{i,i}} H^i\left(X_t, Y_t^i, \alpha_t\right) dt + dL_t^i \\ &= \left\{\left[a + q + \left(1 - \frac{1}{N}\right)\eta_t\right]\left(\bar{X}_t - X_t^i\right) + \left(1-\frac{1}{N}\right)\varphi_t^i + \upsilon_t^i\right\} dt \\ &\quad + d\tilde{L}_t^i, \end{aligned} \tag{4.15}$$

with initial conditions $X_0^i = x^i$ for $i = 1, ..., N$. In addition, the dynamics of the ensemble average $\bar{X}_t$ are given by

$$d\bar{X}_t = \left[\left(1 - \frac{1}{N}\right)\bar{\varphi}_t + \bar{\upsilon}_t\right] dt + \frac{1}{N}\sum_{k=1}^{N} d\tilde{L}_t^k \tag{4.16}$$

,where $\bar{\varphi}_t = \frac{1}{N}\sum_{k=1}^{N}\varphi_t^i$ and $\bar{\upsilon}_t = \frac{1}{N}\sum_{k=1}^{N}\upsilon_t^k$.

On the other hand, we differentiate the ansatz to obtain

$$dY_t^{i,j} = \left(\frac{1}{N} - \delta_{i,j}\right)\left[\dot{\eta}_t\left(\bar{X}_t - X_t^i\right) + \dot{\varphi}_t^i\right] dt + \left(\frac{1}{N} - \delta_{i,j}\right)\eta_t d\left(\bar{X}_t - X_t^i\right),$$

where $\dot{\eta}_t$ and $\dot{\varphi}_t^i$ denote the time-derivative of η_t and φ_t^i, respectively. Using equations (4.15) and (4.16), we further have

$$
\begin{aligned}
dY_t^{i,j} = & \left(\frac{1}{N}-\delta_{i,j}\right)\left(\bar{X}_t - X_t^i\right)\left[\dot{\eta}_t - \eta_t\left(a+q+\left(1-\frac{1}{N}\right)\eta_t\right)\right]dt \\
& + \left(\frac{1}{N}-\delta_{i,j}\right)\left[\dot{\varphi}_t^i + \eta_t\left(1-\frac{1}{N}\right)\left(\bar{\varphi}_t - \varphi_t^i\right) + \eta_t\left(\bar{\upsilon}_t - \upsilon_t^i\right)\right]dt \\
& + \eta_t\left(\frac{1}{N}-\delta_{i,j}\right)\left(\frac{1}{N}\sum_{k=1}^{N} d\tilde{L}_t^k - d\tilde{L}_t^i\right) \\
= & \left(\frac{1}{N}-\delta_{i,j}\right)\left(\bar{X}_t - X_t^i\right)\left[\dot{\eta}_t - \eta_t\left(a+q+\left(1-\frac{1}{N}\right)\eta_t\right)\right]dt \\
& + \left(\frac{1}{N}-\delta_{i,j}\right)\left[\dot{\varphi}_t^i + \eta_t\left(1-\frac{1}{N}\right)\left(\bar{\varphi}_t - \varphi_t^i\right) + \eta_t\left(\bar{\upsilon}_t - \upsilon_t^i\right)\right]dt \\
& + \eta_t\left(\frac{1}{N}-\delta_{i,j}\right)\left[\begin{array}{c} \frac{1}{N}\sum_{k=1}^{N}\left(\sigma^k dW_t^k + \int_R \gamma^k\left(t^-,z\right)\tilde{N}^k\left(dt,dz\right)\right) \\ -\sigma^i dW_t^i - \int_R \gamma^i\left(t^-,z\right)\tilde{N}^i\left(dt,dz\right) \end{array}\right]
\end{aligned}
\tag{4.17}
$$

Comparing term by term between the two decompositions (4.14) and (4.17), we obtain

$$
\begin{aligned}
Z_t^{i,j,k} &= \eta_t\left(\frac{1}{N}-\delta_{i,j}\right)\left(\frac{1}{N}-\delta_{i,k}\right)\sigma^k, \\
r^{i,j,k}\left(t,z\right) &= \eta_t\left(\frac{1}{N}-\delta_{i,j}\right)\left(\frac{1}{N}-\delta_{i,k}\right)\gamma^k\left(t,z\right), \text{ for } k=1,...,N.
\end{aligned}
$$

From the drift term:

$$
\begin{aligned}
\dot{\eta}_t - \eta_t\left(a+q+\left(1-\frac{1}{N}\right)\eta_t\right) &= \left(a+q\right)\eta_t + \frac{1}{N}\left(1-\frac{1}{N}\right)\eta_t^2 - \epsilon + q^2, \\
\dot{\varphi}_t^i + \eta_t\left(1-\frac{1}{N}\right)\left(\bar{\varphi}_t - \varphi_t^i\right) + \eta_t\left(\bar{\upsilon}_t - \upsilon_t^i\right) &= \left(a+q\right)\varphi_t^i + \frac{1}{N}\left(1-\frac{1}{N}\right)\eta_t\varphi_t^i.
\end{aligned}
$$

Therefore, η_t must satisfy the scalar Riccati equation

$$
\dot{\eta}_t = 2\left(a+q\right)\eta_t + \left(1-\frac{1}{N^2}\right)\eta_t^2 - \left(\epsilon - q^2\right) \tag{4.18}
$$

with the terminal condition $\eta_T = c$, and φ_t^i must satisfy the equation

$$\dot{\varphi}_t^i = (a+q)\,\varphi_t^i - \eta_t\left(1-\frac{1}{N}\right)\left[\bar{\varphi}_t - \left(1+\frac{1}{N}\right)\varphi_t^i\right] - \eta_t\left(\bar{\upsilon}_t - \upsilon_t^i\right), i = 1, ..., N, \tag{4.19}$$

with the terminal condition $\varphi_T^i = 0$.

By taking the average in (4.19), $\bar{\varphi}_t$ will satisfy the equation

$$\dot{\bar{\varphi}}_t = \left[a + q + \frac{1}{N}\left(1-\frac{1}{N}\right)\eta_t\right]\bar{\varphi}_t \tag{4.20}$$

with terminal condition $\bar{\varphi}_T = 0$. Therefore, the solution should be $\bar{\varphi}_t = 0$.

As a result, (4.19) becomes

$$\begin{aligned}\dot{\varphi}_t^i &= (a+q)\,\varphi_t^i + \left(1-\frac{1}{N}\right)\eta_t\varphi_t^i - \eta_t\left(\bar{\upsilon}_t - \upsilon_t^i\right)\\ &= \left[a+q+\left(1-\frac{1}{N}\right)\eta_t\right]\varphi_t^i - \eta_t\left(\bar{\upsilon}_t - \upsilon_t^i\right)\\ &= \kappa_t\varphi_t^i - \lambda_t^i,\end{aligned} \tag{4.21}$$

with terminal condition $\varphi_T^i = 0$, where $\kappa_t = a + q + \left(1-\frac{1}{N}\right)\eta_t$ and $\lambda_t^i = \eta_t\left(\bar{\upsilon}_t - \upsilon_t^i\right), i = 1, ..., N$.

According to the exact solutions in [Polyanin and Zaitsev, 2002], the solutions of (4.21) for $i = 1, ..., N$ are then given by

$$\varphi_t^i = e^{-\int_t^T \kappa_s ds} \cdot \int_t^T \left(e^{\int_s^T k_u du}\lambda_s^i\right) ds.$$

With the optimal control $\hat{\alpha}^i$, the forward dynamics become

$$\begin{aligned}dX_t^i &= \left\{\left[a+q+\left(1-\frac{1}{N}\right)\eta_t\right]\left(\bar{X}_t - X_t^i\right) + \left(1-\frac{1}{N}\right)\varphi_t^i + \upsilon_t^i\right\}dt + d\tilde{L}_t^i\\ &= \left\{\kappa_t\left(\bar{X}_t - X_t^i\right) + e^{-\int_t^T \kappa_s ds} \cdot \int_t^T \left(e^{\int_s^T k_u du}\left(1-\frac{1}{N}\right)\lambda_s^i\right)ds\right\}dt + dL_t^i\end{aligned}$$

with initial conditions $X_0^i = x^i$.

Observe the drift term in the forward dynamics; each bank finds its best response with additional liquidity $\kappa_t \left(\bar{X}_t - X_t^i\right)$ as well as a growth contributed by compensators υ_t^i. Next, we provide a concrete example for the impact of jumps.

The first example is when the noise terms are driven by a jump-diffusion process with common intensity rate λ while the second one is with different intensity rate λ^i.

Example 4.1. *Let* $L_t^i = \sigma W_t^i + \sum_{j=1}^{N_t^i} \xi_j$, ξ_j *has distribution* $f(y;\theta) = \frac{\theta}{2} e^{-|y-\mu|\theta}, \theta > 0$ *and* N_t^i *is a Poisson process with rate* λ. *Then, for i=1,...,N,* $\upsilon_t^i = \int_R \gamma^i(t,z)\rho^i(dz) = \lambda t \mathbb{E}(\xi_j) = \lambda t \mu$. *Furthermore, equation* (4.21) *becomes*

$$\dot{\varphi}_t^i = \kappa_t \varphi_t^i - \lambda_t^i = \kappa_t \varphi_t^i$$

with terminal condition $\varphi_T^i = 0$. *Hence,*

$$\varphi_t^i = 0, i = 1, ..., N.$$

As a result, the forward dynamics of X_t^i*'s become*

$$\begin{aligned} dX_t^i &= \left[\kappa_t \left(\bar{X}_t - X_t^i\right) + \lambda t \mu\right] dt + d\tilde{L}_t^i \\ &= \kappa_t \left(\bar{X}_t - X_t^i\right) dt + dL_t^i, \end{aligned} \tag{4.22}$$

where $\kappa_t = a + q + \left(1 - \frac{1}{N}\right)\eta_t$.

Example 4.2. *(Compound Poisson with different jump rates) Let* $L_t^i = \sigma W_t^i + \sum_{j=1}^{N_t^i} \xi_j$, ξ_j *has distribution* $f(y;\theta) = \frac{\theta}{2} e^{-|y-\mu|\theta}, \theta > 0$ *and* N_t^i *is a Poisson process* *with rate* λ^i. *Then, for i=1,...,N,* $\upsilon_t^i = \int_R \gamma^i(t,z)\,\rho^i(dz) = \lambda^i t \mathbb{E}(\xi_j) = \lambda^i t\mu$. *So, equation* (4.21) *becomes*

$$\begin{aligned}\dot{\varphi}_t^i &= \kappa_t \varphi_t^i - \lambda_t^i \\ &= \kappa_t \varphi_t^i - \eta_t\left(\bar{\upsilon}_t - \upsilon_t^i\right) \\ &= \kappa_t \varphi_t^i - \eta_t t \mu \tilde{\lambda},\end{aligned}$$

where $\tilde{\lambda} = \left(\bar{\lambda} - \lambda^i\right)$, *with terminal condition* $\varphi_T^i = 0$. *The solutions are then given by*

$$\varphi_t^i = e^{-\int_t^T \kappa_s ds} \cdot \int_t^T \left(e^{\int_s^T k_u du} \eta_s \tilde{\lambda} \mu s\right) ds.$$

As a result, the forward dynamics with the best response α^i *become*

$$\begin{aligned}dX_t^i &= \left[\kappa_t\left(\bar{X}_t - X_t^i\right) + \left(1 - \frac{1}{N}\right)\varphi_t^i + \upsilon_t^i\right] dt + d\tilde{L}_t^i \\ &= \left[\kappa_t\left(\bar{X}_t - X_t^i\right) + e^{-\int_t^T \kappa_s ds} \cdot \int_t^T \left(e^{\int_s^T k_u du}\left(1 - \frac{1}{N}\right)\eta_s \tilde{\lambda}\mu s\right) ds\right] dt \qquad (4.23) \\ &\quad + dL_t^i.\end{aligned}$$

In the first example, although jumps are present in the interbank lending system, the impact of jumps disappear after we obtain the Nash equilibria as shown in (4.22) since each bank now has a common intensity rate λ. In fact, the effect contributed by compensators is absorbed into the Nash equilibrium. However, if the intensity rate λ^i is distinct for all i, after obtaining the Nash equilibria we have an additional drift term $(1 - \frac{1}{N})\varphi_t^i$ from the compensator υ_t^i. As we can see from the drift term in (4.23), $(1 - \frac{1}{N})\varphi_t^i$ is a function of κ_t and the intensity rate λ^i. In the next section, we provide an HJB approach in searching for the feedback Nash equilibria with jumps.

4.2.2 Closed-loop equilibria : HJB approach

In order to search for a closed-loop equilibrium, we start with some settings as stated in the section 3 in [Matatamvura and Øksendal, 2008]. We assume that the set of all admissible controls $\mathcal{A}$ contains the set of controls such that (4.4) has a unique strong solution and such that

$$\mathbb{E}_{t,x}\left\{\int_0^{\tau_S}\left|f_i\left(X_t,\alpha_t^i\right)\right|dt+\left|g_i\left(X_T^i\right)\right|\right\}<\infty \tag{4.24}$$

for all $x\in S$, where $S\subset R$ is a given open set (called the solvency region),

$$\tau_S=\inf\left\{t>0;X_t\neq S\right\}.$$

Recall that the value function of bank i in our problem is given by

$$V^i\left(t,x\right)=\inf_{\alpha}\mathbb{E}_{t,x}\left\{\int_t^T f_i\left(X_t,\alpha_t^i\right)dt+g_i\left(X_T^i\right)\right\} \tag{4.25}$$

with the cost function f_i and g_i given in (4.6) and (4.7), respectively, where the log-monetary reserve X_t^i satisfies the dynamics

$$dX_t^i=\left[a\left(\bar{X}_t-X_t^i\right)+\alpha_t^i+\upsilon_t^i\right]dt+dL_t^i,\qquad i=1,...,N,$$

with $dL_t^i=\sigma^i dW_t^i+\int_R\gamma^i\left(t^-,z\right)\tilde{N}^i\left(dt,dz\right), i=1,...,N$ being independent martingales with compensated Poisson random measures $\tilde{N}^i\left(dt,dz\right)=N^i\left(dt,dz\right)-\rho^i\left(dz\right)dt$ and $\upsilon_t^i=\int_R\gamma^i\left(t,z\right)\rho^i\left(dz\right).$

Analogous to the classical HJB for optimal control of jump diffusions by using the dynamic programming principle (see theorem 3.1 in [Øksendal and Sulem, 2007]), one can formulate a verification theorem (see theorem 5.2 in [Matatamvura and Øksendal, 2008]) in searching for a closed-loop equilibrium in a differential game. The corresponding HJB equations read as

$$0=\inf_{\alpha^i}\left\{A^{\left(\alpha^1(t,x),\dots\alpha^i_t,\dots,\alpha^N(t,x)\right)}V\left(t,x\right)+f_i\left(x,\alpha^i_t\right)\right\},\tag{4.26}$$

where $A^{\left(\alpha^1(t,x),\dots\alpha^i_t,\dots,\alpha^N(t,x)\right)}$ is defined as

$$\begin{aligned}A^{\left(\alpha^1(t,x),\dots\alpha^i_t,\dots,\alpha^N(t,x)\right)}\left(V^i\right) &= \sum_{j\neq i}^{N}\left[a\left(\bar{x}-x^j\right)+\alpha^j\left(t,x\right)+\upsilon^j_t\right]\partial_{x^j}V^i\\ &\quad+\left[a\left(\bar{x}-x^i\right)+\alpha^i+\upsilon^i_t\right]\partial_{x^j}V^i\\ &\quad+\frac{\sigma^2}{2}\sum_{j=1}^{N}\sum_{k=1}^{N}\delta_{j,k}\partial_{x^jx^k}V^i\\ &\quad+\sum_{j=1}^{N}\int_R\left\{\begin{array}{c}V^i\left(t,x+\gamma^j\left(t,z\right)\right)-V^i\left(t,x\right)\\ -\partial_{x^j}V^i\left(t,x\right)\gamma^j\left(t,z\right)\end{array}\right\}\rho^j\left(dz\right)\end{aligned}$$

and

$$f_i\left(x,\alpha^i_t\right)=\frac{\left(\alpha^i\right)^2}{2}-q\alpha^i\left(\bar{x}-x^i\right)+\frac{\epsilon}{2}\left(\bar{x}-x^i\right)^2$$

with the terminal conditions $V^i\left(T,x\right)=g_i\left(X^i_T\right)=\frac{c}{2}\left(\bar{x}-x^i\right)^2$.

Assume that all players chose $\alpha^j\left(t,x\right)$ for $j\neq i$, minimizing the above equation with respect to α^i to obtain the control for bank i

$$\hat{\alpha}^i=q\left(\bar{x}-x^i\right)-\partial_{x^i}V^i,$$

where V^i is unknown. Put it back to the (4.26), the HJB equations become

$$0=\left\{A^{\left(\hat{\alpha}^1_t,\dots\hat{\alpha}^i_t,\dots,\hat{\alpha}^N_t\right)}V\left(t,x\right)+f_i\left(x,\alpha^i_t\right)\right\}\tag{4.27}$$

and specifically

$$0 = \partial_t V^i + \sum_{j=1}^{N} \left[(a+q)\left(\bar{x} - x^j\right) - \partial_{x^j} V^j + \upsilon_t^j \right] \partial_{x^j} V^i$$
$$+ \frac{\sigma^2}{2} \sum_{j=1}^{N} \sum_{k=1}^{N} \delta_{j,k} \partial_{x^j x^k} V^i$$
$$+ \frac{1}{2}\left(\epsilon - q^2\right)\left(\bar{x} - x^i\right)^2 + \frac{1}{2}\left(\partial_{x^i} V^i\right)^2$$
$$+ \sum_{j=1}^{N} \int_R \left\{ \begin{array}{c} V^i\left(t, x + \gamma^j(t,z)\right) - V^i(t,x) \\ -\partial_{x^j} V^i(t,x)\, \gamma^j(t,z) \end{array} \right\} \rho^j(dz). \tag{4.28}$$

We need to find the function V that satisfies (4.28) and then make the ansatz

$$V^i(t,x) = \frac{\tilde{\eta}_t}{2}\left(\bar{x} - x^i\right)^2 + \tilde{\varphi}_t^i\left(\bar{x} - x^i\right) + \mu_t, \tag{4.29}$$

where $\tilde{\eta}_t$, $\tilde{\varphi}_t^i$ and μ_t are deterministic functions satisfying $\tilde{\eta}_T = c$, $\tilde{\varphi}_T^i = 0$ and $\mu_T = 0$ in order to match the terminal conditions for V^i.

The optimal strategies are then given by

$$\begin{aligned} \hat{\alpha}^i &= q\left(\bar{x} - x^i\right) - \partial_{x^i} V^i \\ &= \left(q + \left(1 - \frac{1}{N}\right)\tilde{\eta}_t\right)\left(\bar{X}_t - X_t^i\right) + \left(1 - \frac{1}{N}\right)\tilde{\varphi}_t^i, \end{aligned} \tag{4.30}$$

and the controlled dynamics become

$$dX_t^i = \left(a + q + \left(1 - \frac{1}{N}\right)\tilde{\eta}_t\right)\left(\bar{X}_t - X_t^i\right) dt + \left[\left(1 - \frac{1}{N}\right)\tilde{\varphi}_t^i + \upsilon_t^i\right] dt + dL_t^i.$$

Next, we compute some useful terms

$$
\begin{aligned}
\partial_{x^j} V^i &= \tilde{\eta}_t \left(\frac{1}{N} - \delta_{i,j} \right) \left(\bar{x} - x^i \right) + \tilde{\varphi}_t^i \left(\frac{1}{N} - \delta_{i,j} \right), \\
\partial_{x^j x^k} V^i &= \tilde{\eta}_t \left(\frac{1}{N} - \delta_{i,j} \right) \left(\frac{1}{N} - \delta_{i,k} \right), \\
V^i \left(t, x + \gamma^j (t,z) \right) - V^i (t,x) &= \frac{\tilde{\eta}_t}{2} \left[\begin{array}{c} 2 \left(\bar{x} - x^i \right) \left(\frac{1}{N} - 1 \right) \gamma^j (t,z) \\ + \left(\frac{1}{N} - 1 \right)^2 \gamma^j (t,z)^2 \end{array} \right] \\
&\quad + \tilde{\varphi}_t^i \left(\frac{1}{N} - 1 \right) \gamma^j (t,z) \text{ and} \\
\partial_{x^j} V^i \gamma^j (t,z) &= \left[\begin{array}{c} \tilde{\eta}_t \left(\frac{1}{N} - \delta_{i,j} \right) \left(\bar{x} - x^i \right) \\ + \tilde{\varphi}_t^i \left(\frac{1}{N} - \delta_{i,j} \right) \end{array} \right] \gamma^j (t,z).
\end{aligned}
$$

Note that the term $\partial_{x^j} V^i$ corresponds to the adjoint variables $y^{i,j}$ introduced in the FBSDE approach, and the jump term $V^i (t, x + \gamma^j (t,z)) - V^i (t,x) - \partial_{x^j} V^i \gamma^j (t,z)$ correspond to the variables $r^{i,j,k} (t,z)$ for $k = 1, ..., N$. Plugging these all into (4.28), and matching the terms in $(\bar{x} - x^i)$, as well as $(\bar{x} - x^i)^2$ and the state-independent terms on both sides, we obtain

$$
\begin{aligned}
\dot{\tilde{\eta}}_t &= 2 (a+q) \tilde{\eta}_t + \left(1 - \frac{1}{N^2} \right) \tilde{\eta}_t^2 - \left(\epsilon - q^2 \right), \\
\dot{\tilde{\varphi}}_t^i &= (a+q) \tilde{\varphi}_t^i - \tilde{\eta}_t \left(1 - \frac{1}{N} \right) \left[\overline{\tilde{\varphi}}_t - \left(1 + \frac{1}{N} \right) \tilde{\varphi}_t^i \right] - \tilde{\eta}_t \left(\bar{\upsilon}_t - \upsilon_t^i \right), i = 1, ..., N, \\
\dot{\mu}_t &= -\frac{1}{2} \sigma^2 \left(1 - \frac{1}{N} \right) \tilde{\eta}_t,
\end{aligned}
$$

with terminal conditions $\tilde{\eta}_T = c$, $\tilde{\varphi}_T^i = 0$ and $\mu_T = 0$. Observe that $\tilde{\eta}_t$ and $\tilde{\varphi}_t^i$ satisfy the same equations given by (4.18) and (4.19), and we conclude that $\tilde{\eta}_t = \eta_t$ and $\tilde{\varphi}_t^i = \varphi_t^i$ for all $t < T$. As a result, we verify that the closed-loop equilibrium obtained through the FBSDE approach is indeed a feedback Nash equilibrium. The value functions V^i using this exact Nash equilibrium are then given by (4.29).

In fact, the theorem 5.2 in [Matatamvura and Øksendal, 2008] for a two players game establishes a relationship between the HJB equations and the Nash equilibrium which is a fundamental theorem for our differential game with N players. We give the extended version of theorem 5.2 in [Matatamvura and Øksendal, 2008] in the following and show that the value functions V^i and controls $\hat{\alpha}^i$ will satisfy the assumptions.

Theorem 4.2. *Suppose there exists functions $V^i \in C^2, i = 1, ..., N,$ and a Markov control $\left(\hat{\alpha}^1, ..., \hat{\alpha}^N\right) \in \mathcal{A}$ such that*

(1) $A^{\left(\alpha^1(t,x), ...\hat{\alpha}^i_t, ..., \alpha^N(t,x)\right)} V^i\left(t, x\right) + f_i\left(x, \alpha^i_t\right) \geq 0$, for all $i = 1, ..., N$.

(2) $A^{\left(\alpha^1(t,x), ..., \alpha^N(t,x)\right)}\left(\tau_S\right) \in \partial S$ $a.s.$ on $\{\tau_S < \infty\}$ and
$\lim_{t \to \tau_S^-} V^i\left(t, X_t^{\left(\alpha^1(t,x), ..., \alpha^N(t,x)\right)}\right) = g_i\left(X_{\tau_S}^{\left(\alpha^1(t,x), ..., \alpha^N(t,x)\right)}\right) \chi_{\{\tau_S < \infty\}}$ $a.s.$
for all $i = 1, ..., N$, where $\tau_S = \inf\left\{t > 0; X_t \neq S\right\}$.

(3) The families $\left\{V^i\left(t, X_\tau^{\left(\alpha^1(t,x), ..., \alpha^N(t,x)\right)}\right)\right\}_{\tau \in \mathcal{T}}$ are uniformly integrable for all $i = 1, ..., N$, where $\mathcal{T}$ is the set of all stopping times $\tau \leq \tau_S$.

Then $\left(\hat{\alpha}^1, ..., \hat{\alpha}^N\right)$ is a Nash equilibrium.

Proof. The proof of this theorem resembles that of theorem 5.2 in [Matatamvura and Øksendal, 2008], except that the partial differential operator is replaced by $A^{\left(\alpha^1(t,x), ...\alpha^i_t, ..., \alpha^N(t,x)\right)}$. □

Indeed, we can check that the (1)-(3) will be satisfied and therefore $\left(\hat{\alpha}^1, ..., \hat{\alpha}^N\right)$ is a Nash equilibrium. First of all, recall that our control problem (4.25) is a finite horizon one and $\tau_S = T$. As a result, (2) is obvious since

$$\lim_{t \to T} V^i \left(t, X_t^{\left(\alpha^1(t,x),...,\alpha^N(t,x)\right)}\right) = g_i \left(X_T^{\left(\alpha^1(t,x),...,\alpha^N(t,x)\right)}\right)$$

according to the terminal condition. Moreover, (3) follows by the assumption (4.24) since the families

$$\left\{V^i \left(X_\tau^{\left(\alpha^1(t,x),...,\alpha^N(t,x)\right)}\right)\right\}_{\tau \in \mathcal{T}} = \left\{V^i \left(X_s^{\left(\alpha^1(t,x),...,\alpha^N(t,x)\right)}\right)\right\}_{s \in [t,T]}$$

is bounded by (4.24) and is therefore uniformly integrable.

Secondly, (1) is satisfied since from (4.27) and the optimal control $\hat{\alpha}_t^i$ is a minimizer of the HJB equation (4.26) we have

$$\begin{aligned} 0 &= \left\{A^{\left(\hat{\alpha}_t^1,...\hat{\alpha}_t^i,...,\hat{\alpha}_t^N\right)} V(t,x) + f_i\left(x, \hat{\alpha}_t^i\right)\right\} \\ &\leq \left\{A^{\left(\alpha^1(t,x),...\hat{\alpha}_t^i,...,\alpha^N(t,x)\right)} V(t,x) + f_i\left(x, \hat{\alpha}_t^i\right)\right\} \end{aligned}$$

for all $\alpha_t^i \in \mathcal{A}$, $i = 1, ..., N$. As a result, we conclude that $\left(\hat{\alpha}^1, ..., \hat{\alpha}^N\right)$ is a Nash equilibrium from the above theorem.

4.3 Numerical results and conclusion

In this section, we provide simulation results of example 4.2 to illustrate the system with or without the game feature. In section 4.2, with the feedback equilibria, we have the controlled dynamic that

$$\begin{aligned} dX_t^i &= \left[\left(a + q + \left(1 - \frac{1}{N} \right) \eta_t \right) \left(\bar{X}_t - X_t^i \right) + \left(1 - \frac{1}{N} \right) \varphi_t^i + \upsilon_t^i \right] dt \\ &\quad + d\tilde{L}_t^i \\ &= \left[A_t \left(\bar{X}_t - X_t^i \right) + B_t^i \right] dt + dL_t^i, \end{aligned} \tag{4.31}$$

where A_t is the effective rate as stated in [Carmona et al., 2014], and B_t^i is the growth contributed by compensators. Here, $\tilde{L}_t^i$ are the compensated independent martingales, and υ_t^i is a deterministic function which is the mean of the jumps. Recall that in example 4.2 where $L_t^i = \sigma W_t^i + \sum_{j=1}^{N_t^i} \xi_j$, ξ_j has distribution $f(y; \theta) = \frac{\theta}{2} e^{-|y-\mu|\theta}, \theta > 0$ and N_t^i is a Poisson process with rate λ^i. Also, $\upsilon_t^i = \int_R \gamma^i(t,z)\, \rho^i(dz) = \lambda^i t \mathbb{E}(\xi_j) = \lambda^i t \mu$.

Assume that $\upsilon_t^i = 1$ which suggests the jump of one particular bank i is positive in average. In order to obtain the Nash equilibrium, bank i has to lend more to other banks via the central bank. This situation can be seen in Figure 4.1 where the growth B_t is negative at about -0.4. Figure 4.2 indicates that this growth remains a constant as time increases. The parameter used in both Figure 4.1 and Figure 4.2 are $N = 10$, $a = 0$, $q = 1$, $\epsilon = 10$, $c = 0$, $\lambda = \theta = 1$ and $\sigma = 1$.

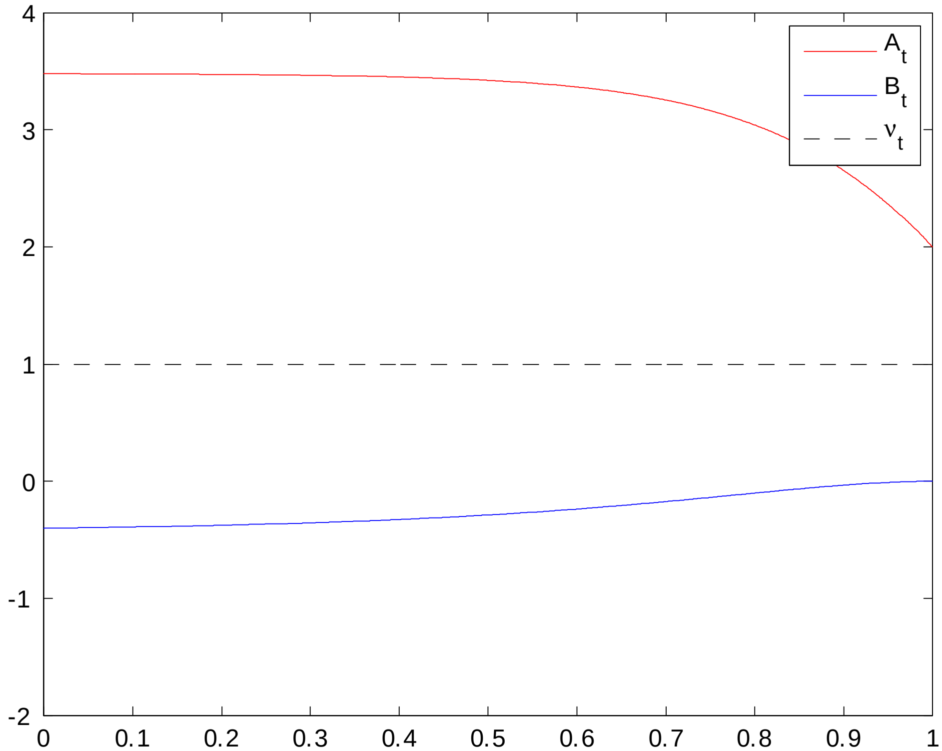

Figure 4.1: Plot of the effective rate A_t (top red line) and the growth B_t^i (bottom blue line) for the closed-loop equilibrium in a fixed time $t = 1$. $v_t^i = 1$ (dash line) suggests the jump of one particular bank i is positive in average.

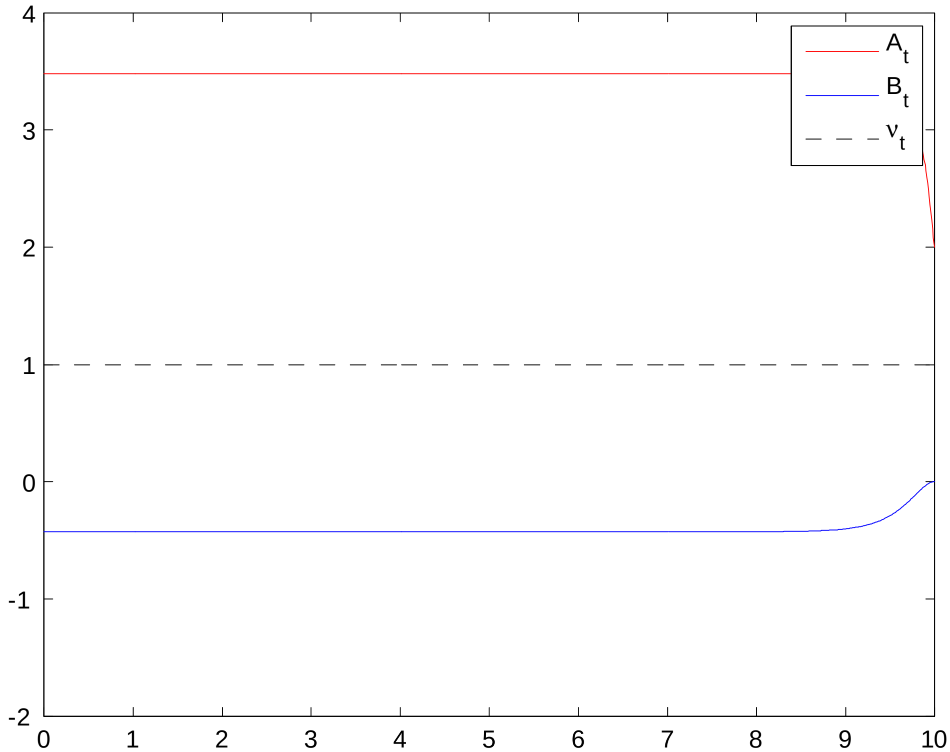

Figure 4.2: Plot of the effective rate A_t (top red line) and the growth B_t^i (bottom blue line) in a fixed time $t = 10$. Both A_t and B_t^i remain a constant most of the time as t increases.

In both Figure 4.3 and Figure 4.4, the upper plots provide the trajectories without the game feature while the bottom ones give the trajectory with the game feature. In the upper plot in Figure 4.3, without a game feature, each bank acts as an independent jump-diffusion process. In the bottom plot in Figure 4.3, after obtaining an equilibrium, we see that the reserve of a particular bank will be forced to reach the ensemble average like the lowest trajectory (light blue line) shown in Figure 4.3. In other words, the lowest trajectory in Figure 4.3 (identified as bank j) has a negative jump at time 0.05, when the level of its reserve is much lower than others, which means it has to borrow more from the central bank in order to obtain an equilibrium. As a result, the level of bank $j's$ reserve rises and gets closer to others in the long run. Figure 4.4 shows that this situation becomes more obvious as time increases since the effective rate and the growth remain a constant as shown in Figure 4.2.

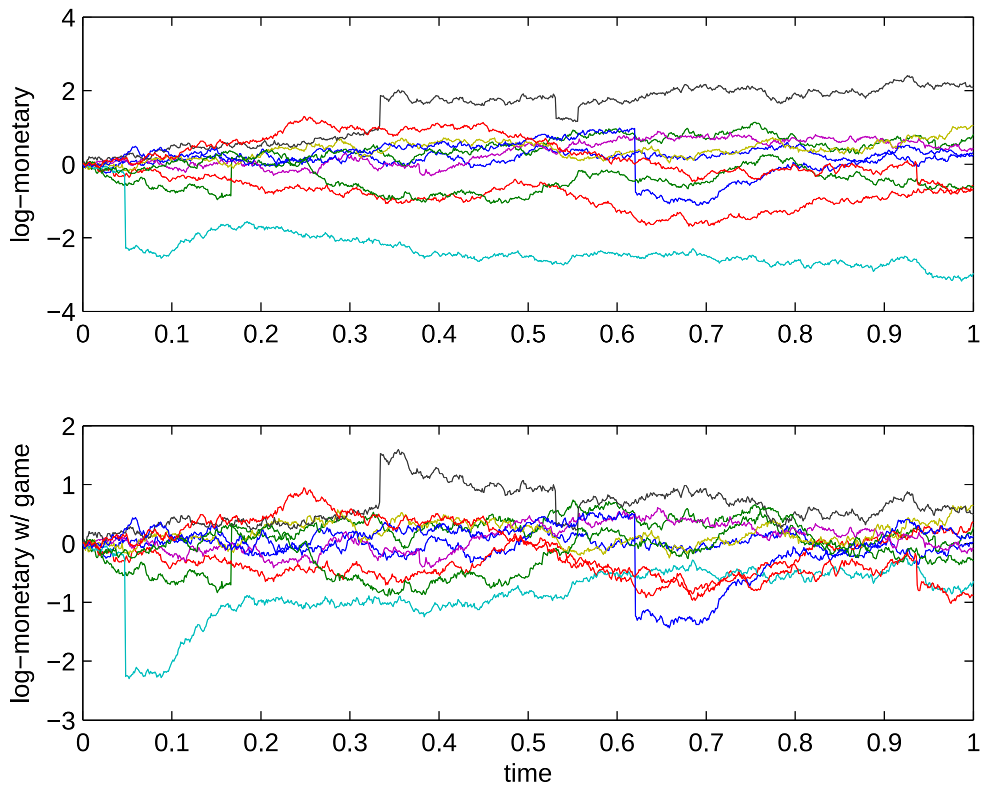

Figure 4.3: Plots of trajectories for the dynamics (4.4) without controls (upper one) and for the controlled dynamics (4.31) (bottom one) in a fixed time $t = 1$. Note that the dynamics in both plots have the same jumps and Brownian increments. The bottom plot shows that, after obtaining an equilibrium, the reserve of a particular bank will be forced to reach the ensemble average like the lowest trajectory (light blue line). The parameters used: $N = 10$, $\sigma = 1$, $\mu = 1, \theta = 1$ and $\lambda = 1$.

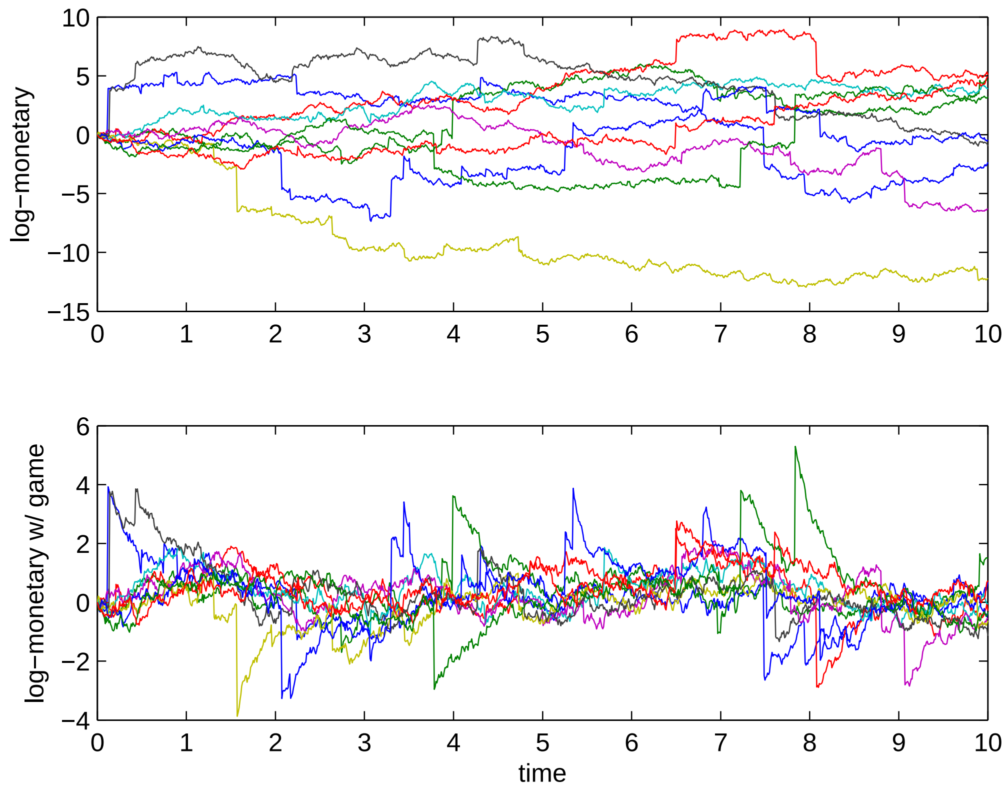

Figure 4.4: Plots of trajectories for the dynamics (4.4) without controls (upper one) and for the controlled dynamics (4.31) (bottom one) in a fixed time $t = 10$. Compare to Figure 4.3, the level of each bank reserve rises or decreases rapidly and gets closer to others in the long run.

We also want to know the effect of the differential game on systemic risk. Considering the systemic event $\left\{\min_{0\leq t\leq T} \bar{X}_t \leq \eta\right\}$, we take the average in (4.31) to obtain

$$\begin{aligned} d\bar{X}_t &= \left(1 - \frac{1}{N}\right) \bar{\varphi}_t dt + \frac{1}{N} \sum_{i=1}^{N} dL_t^i \\ &= \frac{1}{N} \sum_{i=1}^{N} dL_t^i, \end{aligned}$$

since $\bar{\varphi}_t = 0$ is the solution to (4.20). The ensemble average is again the average of the noise terms as studied in Chapter 3. The result indicates that the systemic risk remains even in a differential game with jumps. Hence, allowing a model with jumps will not affect the systemic risk since the average growth contributed by the jumps is zero.

Chapter 5

Systemic risk with a central bank

In this chapter, for the purpose of simplicity, we only consider the diffusion case. Recall that the dynamics of the log-monetary reserve of N banks in Chapter 3 are given by

$$dX_t^i = \frac{\beta}{N} \sum_{j=1}^{N} \left(X_t^j - X_t^i\right) dt + \sigma_1 dW_t^i, \;\; i = 1, .., N. \tag{5.1}$$

Instead, we consider the log-monetary reserves of N banks X_t^i, which we call peripheral banks, satisfy dynamics

$$dX_t^i = \beta \left(X_t^0 - X_t^i\right) dt + \sigma dW_t^i, \;\; i = 1, .., N, \tag{5.2}$$

while the central bank X_t^0 satisfies the following dynamics:

$$\begin{aligned} dX_t^0 &= \frac{\alpha}{N} \sum_{j=1}^{N} \left(X_t^j - X_t^0\right) dt + \sigma_0 dW_t^0 \qquad (5.3) \\ &= \alpha \left(\bar{X}_t - X_t^0\right) dt + \sigma_0 dW_t^0, \end{aligned}$$

where σ_1 and σ_0 are the volatilities of the noise terms for each peripheral bank and for the central bank, respectively. Let W_t^i, $i = 0, ..., N$ be independent 1-dimensional Brownian motions. Also, we define $\bar{X}_t = \sum_{j=1}^{N} X_t^j$ as the ensemble average of reserve of all peripheral banks. Here, α, $\beta > 0$ are the rates of the interactions (borrowing or lending money) between bank i and the central bank. Note that each peripheral bank is now interacting with the central bank only. Within the new system, all banks have interactions with each other indirectly through the central bank. The central bank plays an important role as it acts as an intermediary to bring all banks into communication. In the following, we analyze different values for the rates α and β along with numerical results to indicate how the parameters α and β affect the systemic risk probability. We use the settings that $\sigma_1 = \sigma_0 = 1$, $T = 1$ and a default level $\eta = -0.7$.

Case 1: When $\alpha = 0$ and β is large, for instance $\beta = 100$, the entire dynamics can be read as

$$\begin{aligned} dX_t^i &= \beta\left(X_t^0 - X_t^i\right)dt + \sigma_1 dW_t^i, \ \ i = 1, ..., N, \\ dX_t^0 &= \sigma_0 dW_t^0, \end{aligned} \tag{5.4}$$

which indicate that each peripheral bank is attracted to the central bank X_t^0 where X_t^0 is driven by a Brownian motion with volatility σ_0.

In order to obtain the systemic risk probability, we again investigate the systemic risk event $\left\{\min_{0\leq t\leq T} \bar{X}_t \leq \eta\right\}$. By taking the average in (5.2), we have the dynamic of the ensemble average

$$d\bar{X}_t = \beta \left(X_t^0 - \bar{X}_t \right) dt + \frac{\sigma_1}{N} \sum_{j=1}^{N} dW_t^j. \tag{5.5}$$

As long as the rate of interaction β is large enough, flocking will emerge in this system. The ensemble average of all peripheral banks will be in default if the reserve of the central bank arrives at default level $\eta < 0$. In other words, the dynamics of the central bank is a representative of all other peripheral banks and, hence, can be used to compute the systemic risk. Therefore, we have

$$\begin{aligned} P\left(\min_{0\le t\le T} \bar{X}_t \le \eta\right) &\approx P\left(\min_{0\le t\le T} X_t^0 \le \eta\right) & (5.6)\\ &= P\left(\min_{0\le t\le T} \sigma_0 W_t^0 \le \eta\right) \\ &= 2\Phi\left(\frac{\eta}{\sigma_0 T}\right) & (5.7)\\ &\ge 2\Phi\left(\frac{\eta\sqrt{N}}{\sigma T}\right), \end{aligned}$$

where the last term is the systemic risk probability given in the interbank lending system (5.1) without the dynamic of a central bank (5.3). In Figure 5.1, we observe that the systemic risk with a central bank, which depends on the number of banks N, is indeed higher than the one without a central bank which can be obtained by the formula (5.6).

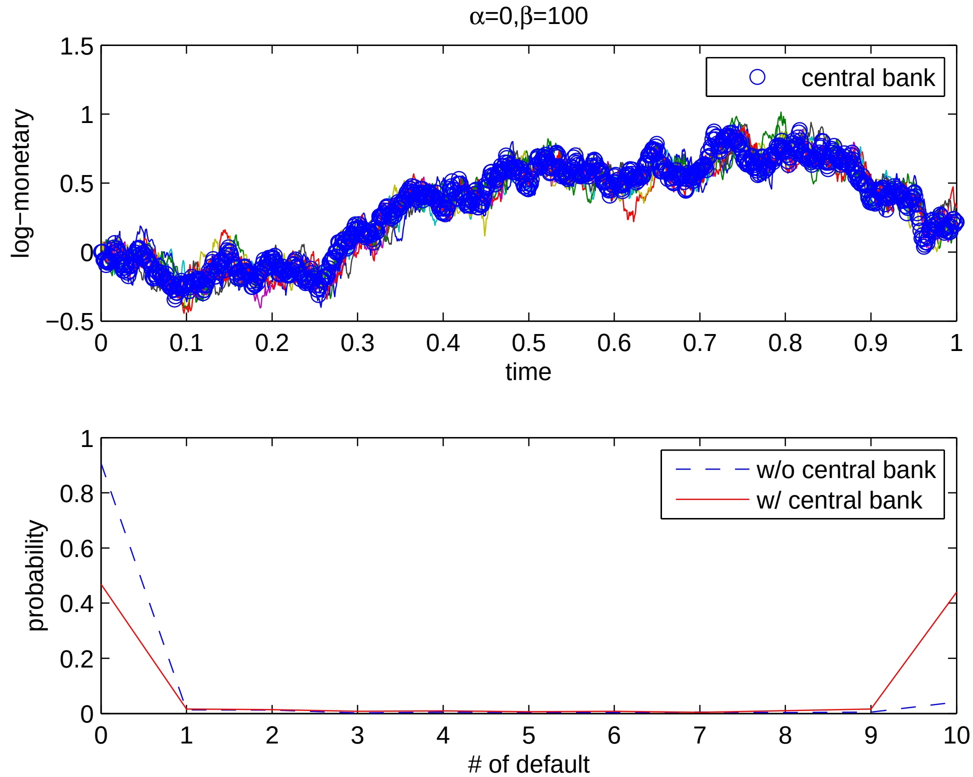

Figure 5.1: The upper plot shows trajectories for dynamics with a central bank and all peripheral banks in case 1. The bottom plot shows the corresponding loss distribution (red line) as well as the loss distribution (dash line) for the dynamics (5.3) without a central bank. The parameters used: $\alpha = 0$, $\beta = 100$, $N = 10$, $t = 1$, $\eta = -0.7$ and $\sigma_1 = \sigma_0 = 1$.

Case 2: When $\beta = 0$, the entire dynamics can be read as

$$
\begin{aligned}
dX_t^i &= \sigma dW_t^i, \;\; i = 1, .., N, \\
dX_t^0 &= \alpha \left(\bar{X}_t - X_t^0\right) dt + \sigma_0 dW_t^0 \\
&= \alpha \left(\frac{\sigma}{N} \sum_{i=1}^{N} W_t^i - X_t^0 \right) dt + \sigma_0 dW_t^0,
\end{aligned}
$$

which indicate that no interaction drift terms in the system of peripheral banks. There is only independent Brownian motions while the central bank is still attracted to the ensemble average of reserves of peripheral banks.

Since the dynamics of peripheral banks are merely Brownian motions, we can expect that the loss distribution will be shown as a Binomial distribution (N, p), $N = 10$, wtih $p \approx \mathbb{P}\{\min_{0 \le t \le 1} X_t^i \le -0.7\}$. This Binomial distribution is shown in both bottom plots of Figure 5.2 and Figure 5.3. Increasing the rate of interaction α will only increase the stability of the system but will not affect the systemic risk. In Figure 5.2, for large α, we can observe that the central bank is attracted to the ensemble average reserve of all peripheral banks. On the other hand, the trajectory of the central bank is independent of other peripheral banks for small α as shown in Figure 5.3.

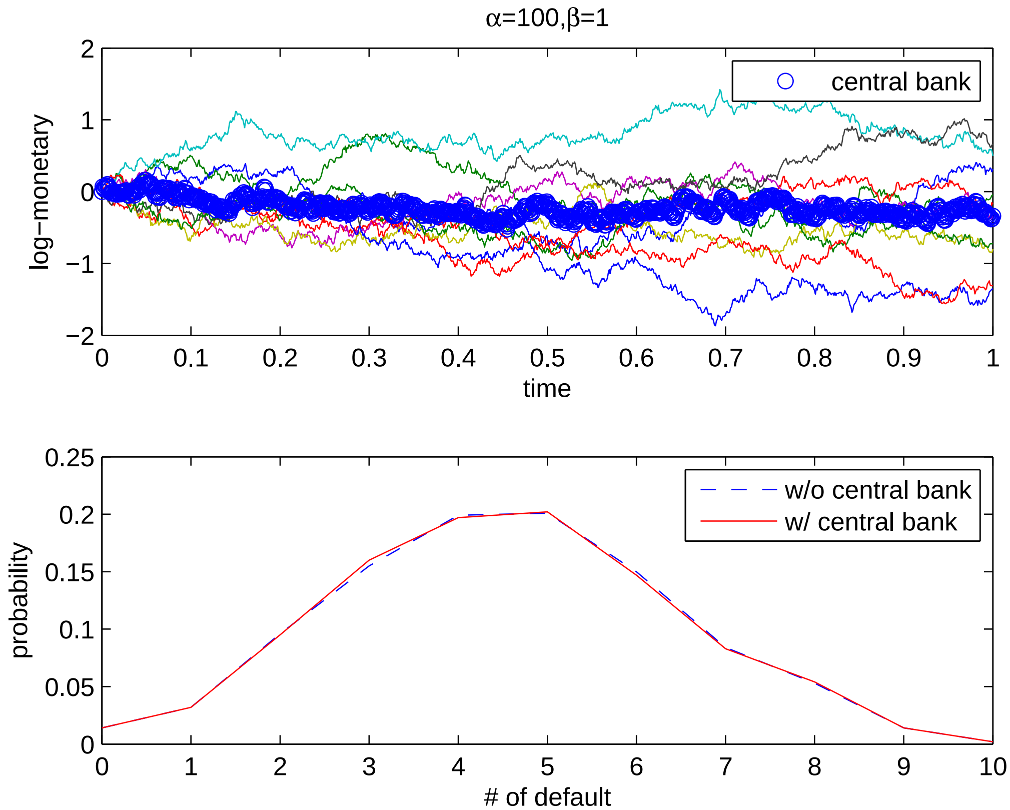

Figure 5.2: The upper plot shows trajectories for dynamics with a central bank and all peripheral banks in case 2. One can observe that the central bank (bold line) is attracted to the ensemble average reserve of all peripheral banks for $\alpha = 100$. The bottom plot shows the corresponding loss distribution (red line) as well as the loss distribution (dash line) for the dynamics (5.3) without a central bank. The parameters used: $N = 10$, $t = 1$, $\eta = -0.7$ and $\sigma_1 = \sigma_0 = 1$.

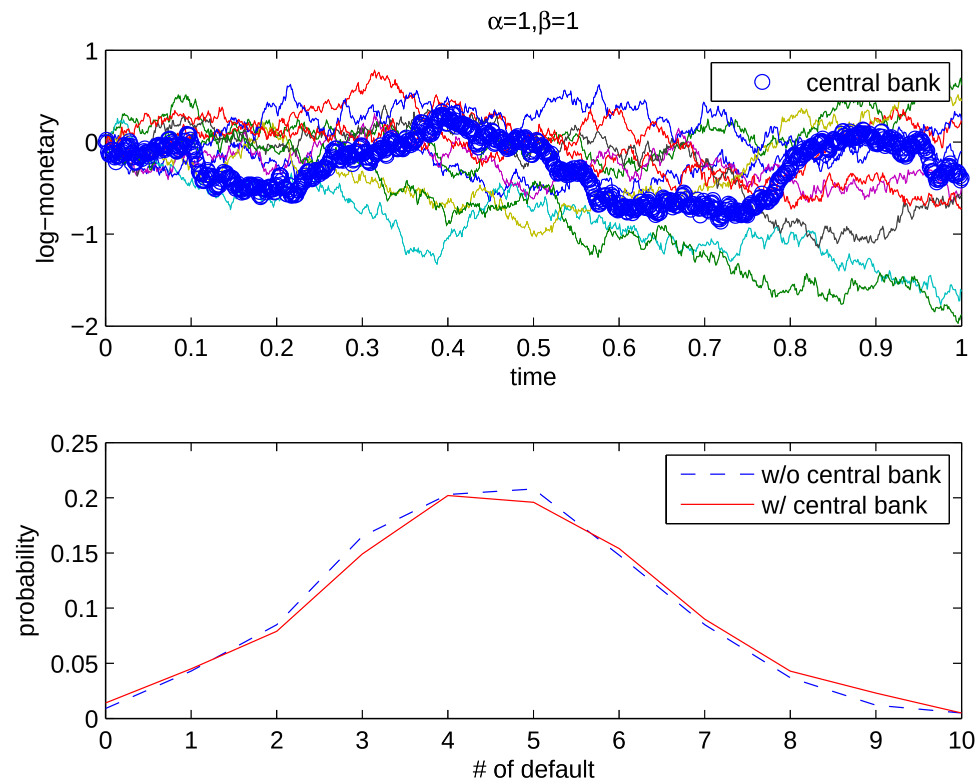

Figure 5.3: The upper plot shows trajectories for dynamics with a central bank and all peripheral banks in case 2. The bottom plot shows the corresponding loss distribution (red line) as well as the loss distribution (dash line) for the dynamics (5.3) without a central bank. The parameters used are the same as Figure 5.2.

Case 3: When α and β are both large, for instance $\alpha = \beta = 100$, the entire system can be read as

$$\begin{aligned} dX_t^i &= \beta\left(X_t^0 - X_t^i\right)dt + \sigma_1 dW_t^i, \;\; i = 1, .., N, \\ dX_t^0 &= \alpha\left(\bar{X}_t - X_t^0\right)dt + \sigma_0 dW_t^0, \\ d\bar{X}_t &= \beta\left(X_t^0 - \bar{X}_t\right)dt + \frac{\sigma_1}{N}\sum_{j=1}^{N} dW_t^j. \end{aligned} \tag{5.8}$$

Now we have two OU processes X_t^0 and $\bar{X}_t$ that are attracted to each other. The reserve of the central bank is mean-reverting to the ensemble average of the peripheral banks, and vice versa.

The systemic risk can be computed by the same flocking argument, that the ensemble average of all peripheral banks will be in default if the reserve of the central bank arrives at default level η. We rewrite (5.8) as

$$\begin{aligned} d\bar{X}_t &= \beta\left(X_t^0 - \bar{X}_t\right) dt + \frac{\sigma_1}{N}\sum_{j=1}^{N} dW_t^j \\ &= 2\beta\left(\frac{\bar{X}_t + X_t^0}{2} - \bar{X}_t\right) dt + \frac{\sigma_1}{N}\sum_{j=1}^{N} dW_t^j, \end{aligned}$$

where the drift term in the second equation indicates that $\bar{X}_t$ is attracted to the average between $\bar{X}_t$ and X_t^0. In addition, the dynamics of the average between $\bar{X}_t$ and X_t^0 are given by

$$d\left(\frac{\bar{X}_t + X_t^0}{2}\right) = \frac{\alpha - \beta}{2}\left(\bar{X}_t - X_t^0\right) dt + \frac{1}{2}\left(\sigma_0 dW_t^0 + \frac{\sigma_1}{N}\sum_{j=1}^{N} dW_t^j\right).$$

As a result, for $\alpha = \beta$, we will be able to use $\frac{\bar{X}_t + X_t^0}{2}$ to approximate the systemic risk probability due to the diffusions flocking towards the average. Therefore, we have

$$
\begin{aligned}
P\left(\min_{0\le t\le T}\bar{X}_t \le \eta\right) &\approx P\left(\min_{0\le t\le T}\frac{\bar{X}_t + X_t^0}{2} \le \eta\right)\\
&= P\left(\min_{0\le t\le T}\frac{1}{2}\left(\sigma_0 dW_t^0 + \frac{\sigma_1}{N}\sum_{j=1}^{N} dW_t^j\right) \le \eta\right)\\
&= P\left(\min_{0\le t\le T}\left(\sqrt{\sigma_0^2 + \frac{\sigma_1^2}{N^2}}\tilde{W}_t\right) \le 2\eta\right)\\
&= P\left(\min_{0\le t\le T}\tilde{W}_t \le \frac{2\eta}{\sqrt{\sigma_0^2 + \frac{\sigma_1^2}{N^2}}}\right)\\
&= 2\Phi\left(\frac{2\eta}{\sqrt{\sigma_0^2 + \frac{\sigma_1^2}{N^2}T}}\right), \qquad (5.9)
\end{aligned}
$$

where $\tilde{W}_t$ is a standard Brownian motion. In Figure 5.4, we see that the systemic risk on the right tail (red line) of the bottom plot is about 18%, which is comparatively higher than the systemic risk when there is no central bank (3%). Contrary to the case with small rate α as shown in Figure 5.1, increasing the rate α when the central bank is attracted more to the average reserve of the peripheral banks will reduce the systemic risk. It suggests that with reasonable monitoring of liquidity, the interbank lending system becomes more stable when the central bank coordinates with peripheral banks closely.

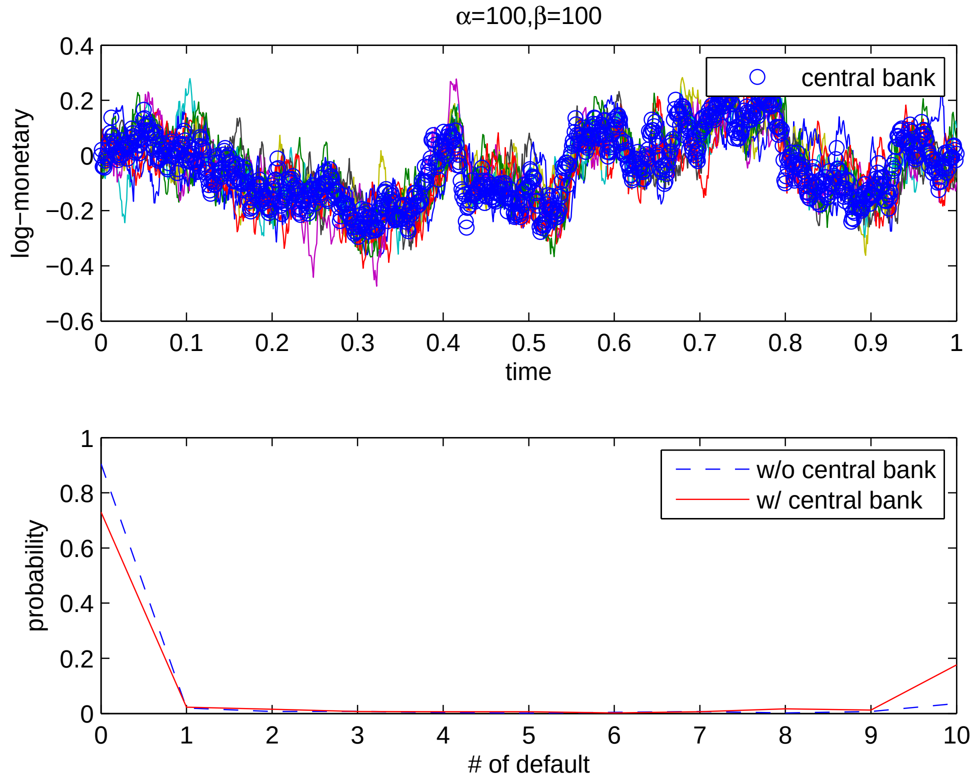

Figure 5.4: The upper plot shows trajectories for dynamics with a central bank and all peripheral banks in case 3. The bottom plot shows the corresponding loss distribution (red line) as well as the loss distribution (dash line) for the dynamics (5.3) without a central bank. The parameter used: $\alpha = 100$, $\beta = 100$, $N = 10$, $t = 1$, $\eta = -0.7$ and $\sigma_1 = \sigma_0 = 1$.

In fact, we investigate two systemic risks: (5.7) with a small rate α and (5.9) with a large rate α. Assume that

$$2\Phi\left(\frac{\eta}{\sigma_0 T}\right) > 2\Phi\left(\frac{2\eta}{\sqrt{\sigma_0^2 + \frac{\sigma_1^2}{N^2}T}}\right), \tag{5.10}$$

where $\eta < 0$, σ_1 is the volatility of peripheral banks and σ_0 is the volatility of the central bank. Then from (5.10), we derive that

$$\frac{\sigma_1^2}{N^2} < 3\sigma_0^2. \tag{5.11}$$

Clearly, our setting $\sigma_1 = \sigma_0 = 1$ and $N = 10$ satisfies (5.11) which indicates that the systemic risk with a large rate α is lower than the systemic risk with a small one. As we mentioned earlier, with reasonable monitoring of liquidity, i.e., in a system with a large α, the system becomes more stable and can therefore reduce the systemic risk with appropriate assumptions about the volatilities of both the central bank and the peripheral banks. In the next section, with the aim to increase system stability with a central bank, we will solve an optimal control problem related to the central bank to further reduce the systemic risk.

5.1 Stochastic optimal control with a central bank

We consider the log-monetary reserves of N banks which we call peripheral banks that satisfy the following dynamics:

$$dX_t^i = \beta\left(X_t^0 - X_t^i\right)dt + \sigma_1 dW_t^i, \;\; i = 1, .., N,$$

whereas the central bank satisfies the following dynamics:

$$dX_t^0 = \alpha_t^0 dt + \sigma_0 dW_t^0, \tag{5.12}$$

where σ_1 and σ_0 are the volatilities of the noise terms for each peripheral bank and for the central bank, respectively. Let $\alpha_t^0 \in R$ be our control process as well as W_t^i, $i = 0, ..., N$, independent 1-dimensional Brownian motions. Recall that $\bar{X}_t = \sum_{j=1}^N X_t^j$ is defined as the ensemble average of reserves of all peripheral banks and the dynamic is given by (5.5). With the control variable α_t^0, the central bank aims to minimize the objective function

$$J^0 = \mathbb{E}\left\{ \begin{array}{c} \int_0^T \left(\frac{1}{2} \left(\alpha_t^0\right)^2 - q_0 \alpha_t^0 \left(\bar{X}_t - X_t^0\right) + \frac{\epsilon_0}{2} \left(\bar{X}_t - X_t^0\right)^2 \right) dt \\ + \frac{c_0}{2} \left(\bar{X}_T - X_T^0\right)^2 \end{array} \right\},$$

where the parameters q_0, ϵ_0 and c_0 are constants that satisfy the same conditions given in Chapter 4.

We solve the control problem by using a similar FBSDE approach that we used in section 4.2. The Hamiltonian is given by

$$H\left(x^0, x^1, ..., x^N, y^0, \alpha^0\right) = y^0 \alpha^0 + \frac{1}{2}\left(\alpha^0\right)^2 - q_0 \alpha^0 \left(\bar{x} - x^0\right) + \frac{\epsilon_0}{2}\left(\bar{x} - x^0\right)^2.$$

Minimizing H with respect to α^0 yields

$$\begin{aligned} \frac{\partial H}{\partial \alpha^0} &= y^0 + \alpha^0 - q_0 \left(\bar{x} - x^0\right) = 0 \text{ and so} \\ \hat{\alpha}^0 &= -y^0 + q_0 \left(\bar{x} - x^0\right). \end{aligned}$$

The partial derivative of $\hat{\alpha}^0$with respect to x^0 gives

$$\frac{\partial \alpha^0}{\partial x^0} = -q_0. \tag{5.13}$$

Thus, with (5.13) , we have

$$\begin{aligned}\frac{\partial H}{\partial x^0} &= y^0\frac{\partial \alpha^0}{\partial x^0} + \alpha^0\frac{\partial \alpha^0}{\partial x^0} - q_0\frac{\partial \alpha^0}{\partial x^0}\left(\bar{x} - x^0\right) + q_0\alpha^0 - \epsilon_0\left(\bar{x} - x^0\right) \\ &= \left(\bar{x} - x^0\right)\left[q_0^2 - \epsilon_0\right] - y^0 q_0.\end{aligned}$$

As a result, the adjoint equation is given by

$$dY_t^0 = \left[\left(\bar{X}_t - X_t^0\right)\left(-q_0^2 + \epsilon_0\right) + Y_t^0 q_0\right]dt + Z_t^0 dW_t^0 + \sum_{i=1}^{N} Z_t^i dW_t^i,$$

where $W_t^i, i = 0, 1, ..., N$ are independent Brownian motions. We then make the ansatz

$$Y_t^0 = -\eta_t\left(\bar{X}_t - X_t^0\right),$$

where η_t is a deterministic function satisfying the terminal condition $\eta_T = c_0$. Given the optimal control $\hat{\alpha}$ with the ansatz, the forward dynamics become

$$dX_t^0 = (\eta_t + q_0)\left(\bar{X}_t - X_t^0\right)dt + \sigma_0 dW_t^0. \tag{5.14}$$

The adjoint equation is then given by

$$dY_t^0 = \left(\bar{X}_t - X_t^0\right)\left(-q_0^2 + \epsilon_0 - \eta_t q_0\right)dt + Z_t^0 dW_t^0 + \sum_{i=1}^{N} Z_t^i dW_t^i. \tag{5.15}$$

Differentiating the ansatz results in

$$\begin{aligned} dY_t^0 &= -\dot{\eta}_t\left(\bar{X}_t - X_t^0\right)dt - \eta_t d\left(\bar{X}_t - X_t^0\right) \\ &= -\dot{\eta}_t\left(\bar{X}_t - X_t^0\right)dt - \eta_t\begin{bmatrix}\beta\left(X_t^0 - \bar{X}_t\right)dt + \frac{\sigma_1}{N}\sum_{j=1}^{N} dW_t^j \\ -(\eta_t + q_0)\left(\bar{X}_t - X_t^0\right)dt - \sigma_0 dW_t^0\end{bmatrix} \\ &= \left(\bar{X}_t - X_t^0\right)\left[-\dot{\eta}_t + \eta_t\beta + \eta_t(\eta_t + q_0)\right]dt \\ &\quad - \eta_t\left[\frac{\sigma_1}{N}\sum_{j=1}^{N} dW_t^j - \sigma_0 dW_t^0\right].\end{aligned} \tag{5.16}$$

Identifying the drift term in (5.15) and (5.16) we obtain

$$\dot{\eta}_t = \eta_t^2 + \eta_t\,(2q_0 + \beta) + q_0^2 - \epsilon_0,$$

with terminal condition $\eta_T = c_0$, which is a Riccati equation with an exact solution. In the controlled dynamic (5.14), the effect of the central bank using its optimal strategy corresponds to the interactions between all peripheral banks and the central bank at the effective rate $\eta_t + q_0$. According to the analysis in [Carmona et al, 2014], η_t will tend to a constant as time goes to infinity. With the effective rate $\eta_t + q_0$, we provide the corresponding loss distributions in Figure 5.5.

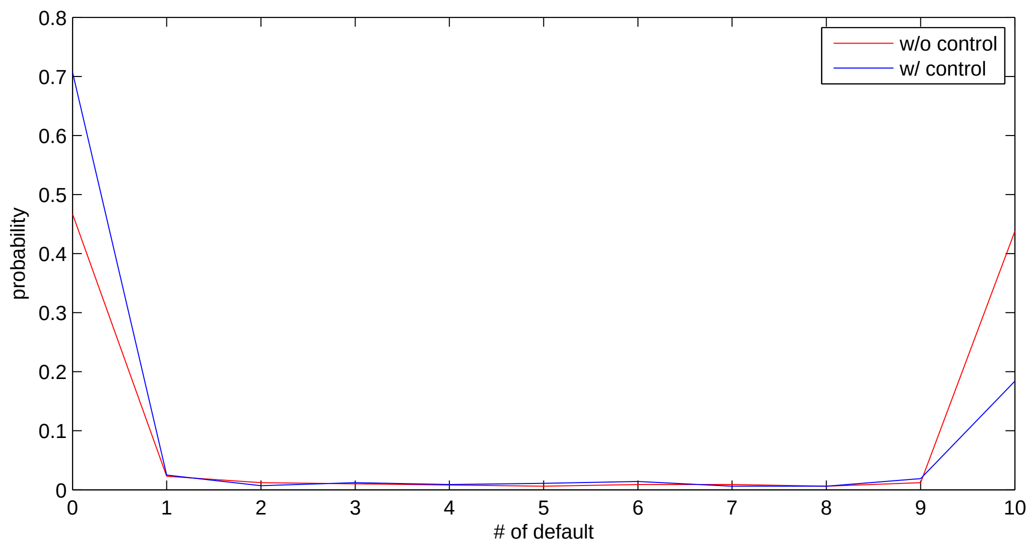

Figure 5.5: Plot of the corresponding loss distribution (blue line) for the controlled dynamic (5.14) as well as the loss distribution (red line) for the dynamics (5.4) without control. The parameter used: $\beta = 100$, $N = 10$, $t = 1$, $\eta = -0.7$, $\sigma_1 = \sigma_0 = 1$, $q_0 = 1$, $\epsilon_0 = 10$ and $c_0 = 0$.

From the right tail of the loss distributions in Figure 5.5, it is interesting to see that the systemic risk (red line) is higher (about 42%) without the control α_t^0 in (5.4). Conversely, if the central bank can now monitor the liquidity between all peripheral banks and itself (i.e., controls difference of the reserve flow $\bar{X}_t - X_t^0$), it will reduce the systemic risk. One of the purposes of having a central bank is to stabilize the entire banking system. However, introducing a central bank in our system without any controls as stated in (5.4) (i.e. all peripheral banks are attracted to a Brownian motion) will result in a higher systemic risk. In other words, in a system where the central bank does not monitor the liquidity, the probability that all peripheral banks will be in default simultaneously will rise. In the next section, we further assume that each peripheral bank can now control its rate of interaction between each other and we aim to solve a stochastic differential game as studied in Chapter 4. However, the equilibrium for the finite players' game has not been solved in our case. In the next section, we will use another approach called a mean-field game inspired by [Carmona and Zhu, 2014] within which the authors provide an approach to mean-field games with major and minor players.

5.2 A mean-field game approach

We consider X_t^0 and X_t^i as the log-monetary reserve of a central bank and peripheral banks, respectively, which satisfy the following dynamics with control variables α_t^0 and α_t^i

$$
\begin{aligned}
dX_t^i &= \alpha_t^i dt + \sigma_1 dW_t^i, i = 1, ..., N, \\
dX_t^0 &= \alpha_t^0 dt + \sigma_0 dW_t^0.
\end{aligned}
$$

The central bank and all peripheral banks aim to minimize the objective functions

$$
\begin{aligned}
J^i(\alpha^0, \alpha^1, ..., \alpha^N) &= \mathbb{E}\left\{ \begin{array}{c} \int_0^T \left(\frac{1}{2} \left(\alpha_t^i\right)^2 - q\alpha_t^i \left(X_t^0 - X_t^i\right) + \frac{\epsilon}{2} \left(X_t^0 - X_t^i\right)^2 \right) dt \\ + \frac{c}{2} \left(X_T^0 - X_T^i\right)^2 \end{array} \right\}, \\
J^0(\alpha^0, \alpha^1, ..., \alpha^N) &= \mathbb{E}\left\{ \begin{array}{c} \int_0^T \left(\frac{1}{2} \left(\alpha_t^0\right)^2 - q_0\alpha_t^0 \left(\bar{X}_t - X_t^0\right) + \frac{\epsilon_0}{2} \left(\bar{X}_t - X_t^0\right)^2 \right) dt \\ + \frac{c_0}{2} \left(\bar{X}_T - X_T^0\right)^2 \end{array} \right\},
\end{aligned}
$$

where $\bar{X}_t = \frac{1}{N} \sum_{i=1}^N X_t^i$ and the parameters q, q_0, ϵ, ϵ_0, c and c_0 satisfy the same conditions as mentioned earlier. We now use a mean-field game approach to obtain an approximate Nash equilibrium. Consider the limit of $\bar{X}_t$ as $N \to \infty$:

$$
m_t = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^N X_t^i.
$$

A key idea in mean-field games states that the control problem for finite players can be regarded as solving the two-player control problem

$$
\inf_{\alpha^1} \mathbb{E}\left\{ \begin{array}{c} \int_0^T \left(\frac{1}{2} \left(\alpha_t^1\right)^2 - q\alpha_t^1 \left(X_t^0 - X_t^1\right) + \frac{\epsilon}{2} \left(X_t^0 - X_t^1\right)^2 \right) dt \\ + \frac{c}{2} \left(X_T^0 - X_T^1\right)^2 \end{array} \right\}, \quad (5.17)
$$

$$
\inf_{\alpha^0} \mathbb{E}\left\{ \begin{array}{c} \int_0^T \left(\frac{1}{2} \left(\alpha_t^0\right)^2 - q_0\alpha_t^0 \left(m_t - X_t^0\right) + \frac{\epsilon_0}{2} \left(m_t - X_t^0\right)^2 \right) dt \\ + \frac{c_0}{2} \left(m_T - X_T^0\right)^2 \end{array} \right\}, \quad (5.18)
$$

subject to the dynamics

$$
\begin{aligned}
dX_t^0 &= \alpha_t^0 dt + \sigma_0 dW_t^0, \\
dX_t^1 &= \alpha_t^1 dt + \sigma_1 dW_t^1,
\end{aligned} \tag{5.19}
$$

where W_t^0 and W_t^1 are independent Brownian motions. Assuming that $X_0^0 = x_0^0$ and $X_0^1 = x_0^1$. We solve the above control problem sequentially by the following procedure.

1. Fix $m_t = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} X_t^i$, where $X_t^i, i = 1, .., N$, are defined as the peripheral banks. In the mean-field game setting, one may consider m_t as a function of X_t^0. As a result, we may assume that $m_t = X_t^0$ since $\bar{X}_t$ might be close to X_t^0 in the mean-field games. Recall that the initial value $X_0^0 = x_0^0 = m_0$.

2. Since m_t is given, we can solve the one player control problem (5.18) using only the dynamic of the central bank.

3. Given α^0 is known, we then solve the control problem (5.17) with dynamics (5.19).

4. Finally, we solve the fixed point problem : find $m_t = \mathbb{E}\left(X_t^1 | \mathcal{F}_t^0\right)$ for all t with $X_0^1 = x_0^1$, where $(\mathcal{F}_t^0)_{t\geq 0}$ denotes the filtration generated by the Brownian motion W_t^0.

With $m_t = X_t^0$ is known.

We solve the control problem (5.18). With $m_t = X_t^0$, the objective function becomes

$$\inf_{\alpha^0} \mathbb{E}\left\{ \int_0^T \frac{1}{2} \left(\alpha_t^0\right)^2 dt \right\},$$

and so the optimal control α^0 is zero by minimizing the above objective function.

The forward dynamics are then given by

$$dX_t^0 = \sigma_0 dW_t^0.$$

With α_t^0 is known.

With $\alpha_t^0 = 0$, the dynamics become

$$\begin{aligned} dX_t^0 &= \sigma_0 dW_t^0 \\ dX_t^1 &= \alpha_t^1 dt + \sigma_1 dW_t^1, \end{aligned}$$

in order to minimize the cost functional

$$\inf_{\alpha^1} \mathbb{E}\left\{ \begin{array}{c} \int_0^T \left(\frac{1}{2} \left(\alpha_t^1\right)^2 - q\alpha_t^1 \left(X_t^0 - X_t^1\right) + \frac{\epsilon}{2} \left(X_t^0 - X_t^1\right)^2 \right) dt \\ + \frac{c}{2} \left(X_T^0 - X_T^1\right)^2 \end{array} \right\}.$$

The Hamiltonian is given by

$$H^1 = y^1 \alpha^1 + \frac{1}{2} \left(\alpha^1\right)^2 - q\alpha^1 \left(x^0 - x^1\right) + \frac{\epsilon}{2} \left(x^0 - x^1\right)^2.$$

Minimizing with respect to α^1, we obtain

$$\begin{aligned} \frac{\partial H^1}{\partial \alpha^1} &= y^1 + \alpha^1 - q\left(x^0 - x^1\right) = 0 \\ \hat{\alpha}^1 &= -y^1 + q\left(x^0 - x^1\right). \end{aligned}$$

Also, we have

$$\frac{\partial H^1}{\partial x^1} = q\alpha^1 - \epsilon\left(x^0 - x^1\right).$$

Plugging in the optimal control $\hat{\alpha}^1$, we have

$$\begin{aligned}\frac{\partial H^1}{\partial x^1} &= q\left(-y^1 + q\left(x^0 - x^1\right)\right) - \epsilon\left(x^0 - x^1\right) \\ &= -y^1 q + \left(x^0 - x^1\right)\left(q^2 - \epsilon\right).\end{aligned}$$

Therefore, the corresponding FBSDEs are given by

$$\begin{aligned}dX_t^1 &= \left[q\left(X_t^0 - X_t^1\right) - Y_t^1\right] dt + \sigma_1 dW_t^1 \text{ and} \\ dY_t^1 &= \left[-\left(X_t^0 - X_t^1\right)\left(q^2 - \epsilon\right) + Y_t^1 q\right] dt + Z_t^1 dW_t^1 + Z_t^0 dW_t^0.\end{aligned}$$

We then make the ansatz

$$Y_t^1 = -\phi_t\left(X_t^0 - X_t^1\right),$$

where ϕ_t is a deterministic function with terminal condition $\phi_T = c$. The FBSDEs become

$$\begin{aligned}dX_t^1 &= \left(\phi_t + q\right)\left(X_t^0 - X_t^1\right) dt + \sigma_1 dW_t^1 \text{ and} \\ dY_t^1 &= \left(X_t^0 - X_t^1\right)\left[\left(\epsilon - q^2\right) - \phi_t q\right] dt + Z_t^1 dW_t^1 + Z_t^0 dW_t^0. \end{aligned} \tag{5.20}$$

Differentiating the ansatz gives

$$\begin{aligned}dY_t^1 &= -\dot{\phi}_t\left(X_t^0 - X_t^1\right) dt - \phi_t d\left(X_t^0 - X_t^1\right) \\ &= -\dot{\phi}_t\left(X_t^0 - X_t^1\right) dt - \phi_t \left[\begin{array}{c}\sigma_0 dW_t^0 \\ -\left(\phi_t + q\right)\left(X_t^0 - X_t^1\right) dt - \sigma_1 dW_t^1\end{array}\right] \\ &= \left(X_t^0 - X_t^1\right)\left[-\dot{\phi}_t + \phi_t\left(\phi_t + q\right)\right] dt - \phi_t\left[\sigma_0 dW_t^0 - \sigma_1 dW_t^1\right].\end{aligned} \tag{5.21}$$

Identifying the drift terms in (5.20) and (5.21) yields

$$\dot{\phi}_t = \phi_t^2 + 2\phi_t q - \epsilon + q^2,$$

with terminal condition $\phi_T = c$, which is again a Riccati equation with an exact solution. Moreover, identifying the noise terms in the two decompositions gives

$$\begin{aligned} Z_t^1 &= \phi_t \sigma_1, \\ Z_t^0 &= -\phi_t \sigma_0. \end{aligned}$$

Given the optimal control $\hat{\alpha}_t^1$, the forward dynamics become

$$\begin{aligned} dX_t^0 &= \sigma_0 dW_t^0 \\ dX_t^1 &= (\phi_t + q)\left(X_t^0 - X_t^1\right) dt + \sigma_1 dW_t^1. \end{aligned}$$

In addition, with the initial condition x_0, we have

$$dX_t^1 = (\phi_t + q)\left[\left(x_0 + \sigma_0 W_t^0\right) - X_t^1\right] dt + \sigma_1 dW_t^1. \tag{5.22}$$

By conditioning with respect to $\mathcal{F}_t^0$, we have

$$d\mathbb{E}\left(X_t^1 | \mathcal{F}_t^0\right) = (\phi_t + q)\left[\left(x_0^0 + \sigma_0 W_t^0\right) - \mathbb{E}\left(X_t^1 | \mathcal{F}_t^0\right)\right] dt.$$

To complete the last step of the procedure, one has to solve the fixed point problem : find $m_t = \mathbb{E}\left(X_t^1 | \mathcal{F}_t^0\right)$ for all t, where $\mathbb{E}\left(X_t^1 | \mathcal{F}_t^0\right)$ is given by the above equation with initial value $\mathbb{E}\left(X_0^1 | \mathcal{F}_t^0\right) = x_0$. One can observe that in a mean-field game setting, the dynamic of peripheral bank X_t^1 is now attracted to the central bank, which is in fact a Brownian motion. As a result of the mean-field game, the optimal control $\hat{\alpha}_t^1$ might be the approximate Nash equilibrium if there are infinitely many players.

Chapter 6

Conclusion

In Chapter 2, we investigated a stochastic Cucker-Smale model when jumps were present in the noise term. We showed that the time-asymptotic flocking defined in [Ha et. al., 2009] was still valid even with jumps. However, there was one situation that might lead to flocking failure—when the distribution of each jump were not identical. In order to obtain flocking behavior, we had to make the assumption that each jump was identically distributed, otherwise any strong positive or negative jump may break the symmetry of the stochastic system and further cause flocking failure. In some cases, a jump may also be considered as the characteristic of a leading particle.

In Chapter 3, we successfully extended the result in [Fouque and Sun, 2013] to a jump-diffusion case, that is, the systemic risk in an interbank lending system would increase when the noise terms were driven by jump-diffusion processes.

Moreover, both the systemic risk and the risk of individual banks would increase. This makes sense since if there was a dramatic increase or decrease in the reserves of one bank, this bank will more likely be in default. We gave an approach to compute the systemic risk in the jump-diffusion case and investigated the order of convergence as the number of banks N increased.

In Chapter 4, each bank controlled its rate of borrowing/lending money through a central bank in order to minimize a cost function where the dynamics of the reserve processes were driven by jump diffusion processes. We utilized the method of stochastic optimal control with jump-diffusions to solve a Nash equilibria in a differential game. The closed-loop equilibria can be obtained by both an FBSDE approach through the maximum principle and an HJB approach through dynamic programming. We obtained a similar result as stated in [Carmona et al, 2014] that the system created additional liquidity and the central bank acted as a clearing house. Additionally, linear growth contributed by jumps were also presented after obtaining the equilibrium. Based on the assumption that each bank had chosen their best strategy to obtain an equilibrium, one had to pay more if it possessed more reserve at any time and vice versa. In other words, if one of the banks possessed more reserves than the average reserve, it had to contribute more (lending money) in the system. As for the systemic risk, the effect of jumps in this differential game was zero since the average growth contributed by jumps was zero. The systemic risk would not be reduced in a game setting with jumps.

In Chapter 5, we obtained a higher systemic risk with a specified dynamics for the central bank. Each peripheral bank was interacting with one another indirectly through the central bank. The activity of borrowing and lending had to go through the central bank. We showed that the systemic risk depended on the relation of the volatility between a central bank and the peripheral banks. The central bank played an important role in stabilizing the banking system by controlling the liquidity between all peripheral banks and itself, otherwise the system would be vulnerable and result in a higher systemic risk. However, if the central bank monitored the liquidity closely and minimized the difference of reserves between all peripheral banks and itself, it would reduce the systemic risk. By solving an optimal control problem, we could reduce the systemic risk. At the end of Chapter 5, we also gave a mean-field game approach for approximately solving a differential game with finite players. Another extended model with major banks and small banks is given and solved in a sequential way in the appendix.

The mathematical model we developed showed that the systemic risk would rise when instantaneous shocks occur. In other words, the probability that all banks will reach the critical level at the same time would be higher when jumps are taken into consideration. Moreover, under the circumstance that each bank was acting toward their best self-interest, the model showed that systemic risk would possibly increase when one individual bank experienced slightly more shocks than the rest. By establishing this mathematical model, we have characterized the banking system and understood the effect of jumps on systemic risk. Through the incorporation of a game feature with jumps, the result of this study also aimed to shed light on the regulation of banking system and contribute to the understanding of the relationship between individual banks, as well as the role that central banks should play to provide more liquidity.

Chapter 7

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Appendix A

Stochastic sequential differential games

We consider $X_t^{0,i}$ and $X_t^{1,i}$ as the log-monetary reserve of major banks and small banks, respectively. We consider the following dynamics with control variables $\alpha_t^{0,i}$ and $\alpha_t^{1,i}$

$$
\begin{aligned}
dX_t^{0,i} &= \alpha_t^{0,i} dt + \sigma_0 dW_t^{0,i}, i = 1, ..., N \\
dX_t^{1,i} &= \alpha_t^{1,i} dt + \sigma_1 dW_t^{1,i}.
\end{aligned}
$$

Major banks $X_t^{0,i}$ control their rate $\alpha_t^{0,i}$ of borrowing/lending from/to a central bank while small banks control their rate $\alpha_t^{1,i}$ of borrowing/lending from/to the major banks $X_t^{0,i}$, $i = 1, ..., N$, in order to minimize their respective objective functions:

$$J^{0,i}(\alpha) = \mathbb{E}\left\{ \begin{array}{c} \int_0^T \left(\frac{1}{2}\left(\alpha_t^{0,i}\right)^2 - q_0\alpha_t^{0,i}\left(\bar{X}_t^0 - X_t^{0,i}\right) + \frac{\epsilon_0}{2}\left(\bar{X}_t^0 - X_t^{0,i}\right)^2\right) dt \\ +\frac{c_0}{2}\left(\bar{X}_T^0 - X_T^{0,i}\right)^2 \end{array} \right\},$$

$$J^{1,i} = \mathbb{E}\left\{ \begin{array}{c} \int_0^T \left(\begin{array}{c} \frac{1}{2}\left(\alpha_t^{1,i}\right)^2 - q_1\alpha_t^{1,i}\left(X_t^{0,i} - X_t^{1,i}\right) + \frac{\epsilon_1}{2}\left(X_t^{0,i} - X_t^{1,i}\right)^2 \\ -\alpha_t^{0,i}\alpha_t^{1,i} \end{array} \right) dt \\ +\frac{c_1}{2}\left(X_T^{0,i} - X_T^{1,i}\right)^2 \end{array} \right\}.$$

where $\alpha = (\alpha^{0,1}, ..., \alpha^{0,N}, \alpha^{1,1}, ..., \alpha^{1,N})$. We now solve the open-loop equilibria by using the Pontryagin approach. The Hamiltonian for major banks $X^{0,i}$ and small banks $X^{1,i}$ are given by

$$\begin{aligned} & H^{0,i}\left(x, y^{0,i,1}, ..., y^{0,i,N}, \alpha^{0,i}, ..., \alpha^{0,N}\right) \\ = & \sum_{k=1}^N \alpha^{0,k} y^{0,i,k} + \frac{1}{2}\left(\alpha^{0,i}\right)^2 - q_0\alpha^{0,i}\left(\bar{x}^0 - x^{0,i}\right) + \frac{\epsilon_0}{2}\left(\bar{x}^0 - x^{0,i}\right)^2, \end{aligned}$$

and

$$\begin{aligned} & H^{1,i}\left(x, y^{1,i,1}, ..., y^{1,i,N}, \alpha^{1,i}, ..., \alpha^{1,N}\right) \\ = & \sum_{k=1}^N \alpha^{1,k} y^{1,i,k} + \frac{1}{2}\left(\alpha^{1,i}\right)^2 - q_1\alpha^{1,i}\left(x^{0,i} - x^{1,i}\right) + \frac{\epsilon_1}{2}\left(x^{0,i} - x^{1,i}\right)^2 \\ & -\alpha_t^{0,i}\alpha_t^{1,i}. \end{aligned}$$

The major banks choose to minimize the cost function $J^{0,i}$, so we minimize $H^{0,i}$ over $\alpha^{0,i}$ to obtain choices

$$\begin{aligned} y^{0,i,i} + \alpha^{0,i} - q_0\left(\bar{x}^0 - x^{0,i}\right) &= 0, \qquad i = 1, ..., N \\ \hat{\alpha}^{0,i} &= -y^{0,i,i} + q_0\left(\bar{x}^0 - x^{0,i}\right). \end{aligned}$$

The ansatz for $y^{0,i,j}$:

$$Y_t^{0,i,j} = \phi_t^0 \left(\frac{1}{N} - \delta_{i,j} \right) \left(\bar{x}^0 - x^{0,i} \right),$$

where ϕ_t^0 is a deterministic function satisfying the terminal condition $\phi_T^0 = c_0$.

So, the optimal controls for the major banks are given by

$$\hat{\alpha}^{0,i} = \left(\phi_t^0 \left(1 - \frac{1}{N} \right) + q_0 \right) \left(\bar{x}^0 - x^{0,i} \right).$$

The adjoint equation for $Y_t^{0,i,j}$ is given by

$$dY_t^{0,i,j} = -\frac{\partial H^{0,i}}{\partial x^{0,j}} dt + \sum_{k=1}^{N} Z_t^{0,i,j,k} dW_t^{0,k}.$$

The small banks anticipate this response and insert major banks' reaction into their own cost function $J^{1,i}$.

$$J^{1,i} = \mathbb{E} \left\{ \begin{array}{c} \int_0^T \left(\begin{array}{c} \frac{1}{2} \left(\alpha_t^{1,i} \right)^2 - q_1 \alpha_t^{1,i} \left(X_t^{0,i} - X_t^{1,i} \right) + \frac{\epsilon_1}{2} \left(X_t^{0,i} - X_t^{1,i} \right)^2 \\ -\alpha_t^{1,i} \left(\phi_t^0 \left(1 - \frac{1}{N} \right) + q_0 \right) \left(\bar{x}^0 - x^{0,i} \right) \end{array} \right) dt \\ + \frac{c_1}{2} \left(X_T^{0,i} - X_T^{1,i} \right)^2 \end{array} \right\}.$$

Using the Pontryagin approach, the Hamiltonian for small banks $X^{1,i}$ is given by

$$\begin{aligned} & H^{1,i} \left(x, y^{1,i,1}, ..., y^{1,i,N}, \alpha^{1,i}, ..., \alpha^{1,N} \right) \\ = & \sum_{k=1}^{N} \alpha^{1,k} y^{1,i,k} + \frac{1}{2} \left(\alpha^{1,i} \right)^2 - q_1 \alpha^{1,i} \left(x^{0,i} - x^{1,i} \right) + \frac{\epsilon_1}{2} \left(x^{0,i} - x^{1,i} \right)^2 \\ & -\alpha_t^{1,i} \left(\phi_t^0 \left(1 - \frac{1}{N} \right) + q_0 \right) \left(\bar{x}^0 - x^{0,i} \right), \end{aligned}$$

The small banks choose to minimize the cost function $J^{1,i}$, so we minimize $H^{1,i}$ over $\alpha^{1,i}$ to obtain choices

$$y^{1,i,i} + \alpha^{1,i} - q_1\left(x^{0,i} - x^{1,i}\right) - \left(\phi_t^0\left(1 - \frac{1}{N}\right) + q_0\right)\left(\bar{x}^0 - x^{0,i}\right) = 0$$
$$\hat{\alpha}^{1,i} = -y^{1,i,i} + q_1\left(x^{0,i} - x^{1,i}\right) + \left(\phi_t^0\left(1 - \frac{1}{N}\right) + q_0\right)\left(\bar{x}^0 - x^{0,i}\right).$$

The ansatz for $y^{1,i,j}$:

$$Y_t^{1,i,j} = \phi_t^1 \delta_{i,j}\left(\bar{X}_t^0 - X_t^{0,i}\right) - \eta_t \delta_{i,j}\left(X_t^{0,i} - X_t^{1,i}\right),$$

where ϕ_t^1 and η_t are deterministic functions satisfying the terminal conditions $\phi_T^1 = 0$ and $\eta_T = c_1$, respectively.

So, the optimal control $\hat{\alpha}^{1,i}$ is given by

$$\hat{\alpha}^{1,i} = \left(\bar{X}_t^0 - X_t^{0,i}\right)\left[-\phi_t^1 + \phi_t^0\left(1 - \frac{1}{N}\right) + q_0\right] + \left(X_t^{0,i} - X_t^{1,i}\right)(\eta_t + q_1)$$

The adjoint equations are then given by

$$\begin{aligned} dY_t^{1,i,j} &= -\frac{\partial H^{1,i}}{\partial x^{1,j}}dt + \sum_{k=1}^{N} Z_t^{1,i,j,k} dW_t^{1,k} \\ &= -\delta_{i,j}\left(\bar{X}_t^0 - X_t^{0,i}\right)\left[(\eta_t + q_1)\left(-\phi_t^1 + \phi_t^0\left(1 - \frac{1}{N}\right) + q_0\right)\right]dt \\ &\quad -\delta_{i,j}\left(X_t^{0,i} - X_t^{1,i}\right)\left[(\eta_t + q_1)^2 - \epsilon_1\right]dt + \sum_{k=1}^{N} Z_t^{1,i,j,k} dW_t^{1,k} \end{aligned}$$

The dynamics for the difference $X_t^{0,i} - X_t^{1,i}$ are given by

$$\begin{aligned} d\left(X_t^{0,i} - X_t^{1,i}\right) &= \left[\phi_t^1\left(\bar{X}_t^0 - X_t^{0,i}\right) - (\eta_t + q_1)\left(X_t^{0,i} - X_t^{1,i}\right)\right]dt \\ &\quad +\sigma_0 dW_t^{0,i} - \sigma_1 dW_t^{1,i}. \end{aligned}$$

Differentiating the ansatz to obtain

$$
\begin{aligned}
dY_t^{1,i,j} &= \left[\dot{\phi}_t^1 \delta_{i,j}\left(\bar{X}_t^0 - X_t^{0,i}\right) - \dot{\eta}_t \delta_{i,j}\left(X_t^{0,i} - X_t^{1,i}\right)\right] dt \\
&\quad + \phi_t^1 \delta_{i,j} d\left(\bar{X}_t^0 - X_t^{0,i}\right) - \eta_t \delta_{i,j} d\left(X_t^{0,i} - X_t^{1,i}\right) \\
&= \left[\dot{\phi}_t^1 \delta_{i,j}\left(\bar{X}_t^0 - X_t^{0,i}\right) - \dot{\eta}_t \delta_{i,j}\left(X_t^{0,i} - X_t^{1,i}\right)\right] dt \\
&\quad + \phi_t^1 \delta_{i,j}\left[-\left(\phi_t^0\left(1 - \frac{1}{N}\right) + q_0\right)\left(\bar{X}_t^0 - X_t^{0,i}\right) dt + \frac{\sigma_0}{N}\sum_{j=1}^N dW_t^{0,j} - \sigma_0 dW_t^{0,i}\right] \\
&\quad - \eta_t \delta_{i,j}\left\{\begin{array}{c} \left[\left(\phi_t^1\right)\left(\bar{X}_t^0 - X_t^{0,i}\right) - \left(\eta_t + q_1\right)\left(X_t^{0,i} - X_t^{1,i}\right)\right] dt \\ + \sigma_0 dW_t^{0,i} - \sigma_1 dW_t^{1,i} \end{array}\right\} \\
&= \left\{\begin{array}{c} -\delta_{i,j}\left(\bar{X}_t^0 - X_t^{0,i}\right)\left[-\dot{\phi}_t^1 + \phi_t^1\left(\phi_t^0\left(1 - \frac{1}{N}\right) + q_0\right) + \eta_t\left(\phi_t^1\right)\right] \\ -\delta_{i,j}\left(X_t^{0,i} - X_t^{1,i}\right)\left[\dot{\eta}_t - \eta_t\left(\eta_t + q_1\right)\right] \end{array}\right\} dt \\
&\quad + \phi_t^1 \delta_{i,j}\left[\frac{\sigma_0}{N}\sum_{j=1}^N dW_t^{0,j} - \sigma_0 dW_t^{0,i}\right] - \eta_t \delta_{i,j}\left(\sigma_0 dW_t^{0,i} - \sigma_1 dW_t^{1,i}\right).
\end{aligned}
$$

Identifying the term $\left(\bar{X}_t^0 - X_t^{0,i}\right)$ and $\left(X_t^{0,i} - X_t^{1,i}\right)$ to obtain the following equations:

$$
\begin{aligned}
\dot{\phi}_t^1 &= \phi_t^1\left[\phi_t^0\left(1 - \frac{1}{N}\right) + q_0 + 2\eta_t + q_1\right] - \left(\eta_t + q_1\right)\left[\phi_t^0\left(1 - \frac{1}{N}\right) + q_0\right] \\
\dot{\eta}_t &= 2\eta_t^2 + 3\eta_t q_1 + q_1^2 - \epsilon_1,
\end{aligned}
$$

where $\phi_T^1 = 0$ and $\eta_T = c_1$.

Thus, for $i = 1, ..., N$, the optimal control for the major players are give by

$$
\hat{\alpha}^{1,i} = \left(\bar{X}_t^0 - X_t^{0,i}\right)\left[-\phi_t^1 + \phi_t^0\left(1 - \frac{1}{N}\right) + q_0\right] + \left(X_t^{0,i} - X_t^{1,i}\right)\left(\eta_t + q_1\right)
$$

As a result, with the optimal control, the dynamics will become

$$
\begin{aligned}
dX_t^{0,i} &= \left[\left(\phi_t^0\left(1-\frac{1}{N}\right)+q_0\right)\left(\bar{X}_t^0-X_t^{0,i}\right)\right]dt+\sigma_0 dW_t^{0,i} \\
dX_t^{1,i} &= \left[\left(\phi_t^0\left(1-\frac{1}{N}\right)+q_0-\phi_t^1\right)\left(\bar{X}_t^0-X_t^{0,i}\right)+(\eta_t+q_1)\left(X_t^{0,i}-X_t^{1,i}\right)\right]dt \\
&\quad +\sigma_1 dW_t^{1,i}.
\end{aligned}
$$