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# Longtime behavior of small solutions to viscous perturbations of nonlinear hyperbolic systems in 3D

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Boyan Yavorov Jonov

Committee in Charge:

Professor Thomas C. Sideris, Chair

Professor Gustavo Ponce

Professor Hector Ceniceros

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The Dissertation of
Boyan Yavorov Jonov is approved:

Professor Gustavo Ponce

Professor Hector Ceniceros

Professor Thomas C. Sideris, Committee Chairperson

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VITA OF BOYAN YAVOROV JONOV

EDUCATION

Ph.D., Mathematics — September 2014 (expected)
University of California, Santa Barbara (UCSB)

M.S., Mathematics — May 2008
California State University, Northridge (CSUN)

B.S., Computer Science — May 2005
California State University, Northridge (CSUN)

PROFESSIONAL EMPLOYMENT

2008-2014: Teaching assistant, department of mathematics, UCSB

Summer 2013: Teaching associate, department of mathematics, UCSB

2005-2008: Graduate assistant, department of mathematics, CSUN

Fall 2006: Teaching associate, department of mathematics, CSUN

PUBLICATIONS

B. Jonov, Thomas C. Sideris, *Global and almost global existence of small solutions to a dissipative wave equation in 3D with nearly null nonlinear terms* (To appear in Commun. Pure Appl. Anal.)

Sudhir R. Ghorpade, B. Jonov, B.A. Sethuraman, *Hilbert series of certain jet schemes of determinantal varieties* (To appear in Pacific Journal of Mathematics)

B. Jonov, *Initial complex associated to a jet scheme of a determinantal variety*, Journal of Pure and Applied Algebra, Volume 215, Issue 5, May 2011, Pages 806-811

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Major Field: Partial Differential Equations
Studies in Nonlinear Partial Differential Equations with Professor Thomas C. Sideris

# Abstract

## Longtime behavior of small solutions to viscous perturbations of nonlinear hyperbolic systems in 3D

Boyan Yavorov Jonov

The first result in this dissertation concerns wave equations in three space dimensions with small $\mathcal{O}(\nu)$ viscous dissipation and $\mathcal{O}(\delta)$ non-null quadratic nonlinearities. Small $\mathcal{O}(\varepsilon)$ solutions are shown to exist globally provided that $\varepsilon\delta/\nu \ll 1$. When this condition is not met, small solutions exist "almost globally", and in certain parameter ranges, the addition of dissipation enhances the lifespan. We study next a system of nonlinear partial differential equations modeling the motion of incompressible Hookean isotropic viscoelastic materials. The nonlinearity inherently satisfies a null condition and our second result establishes global solutions with small initial data independent of viscosity. In the proofs we use vector fields, energy estimates, $L^\infty - L^2$ and weighted $L^2$-decay estimates.

# Contents

# Chapter 1

# Introduction

Extensive research has been devoted to the study of the long time behavior of solutions to evolutionary nonlinear Partial Differential Equation (PDEs). A main objective in this field of research is to provide an estimate of the largest interval on which a solution exists. Such an interval can extend to infinity ( the corresponding solution is referred to as global ) or the interval can be bounded by a finite time singularity at which the solution blows up in some sense.

In this dissertation, we first analyze the existence interval of the solution to a quadratically nonlinear damped wave equation in three space dimensions (see (1.0.1)). The techniques used are then adapted to the equations of motion of an incompressible Hookean viscoelastic material (see (1.0.2a) - (1.0.2c)). We focus on the connection between the longtime behavior of the solutions and the interaction of certain key parameters of the PDEs (size of initial data, viscosity, and nonlinear perturbation from the null condition).

Both (1.0.1) and (1.0.2a) - (1.0.2c) can be classified as perturbations of quasilinear hyperbolic systems. Although we exploit the dissipative nature of those PDEs, we still resort to many of the standard hyperbolic methods and techniques. We provide next a brief summary of the relevant advances in quasilinear hyperbolic systems in the past decades.

The first results date back to the seventies when Friedrichs showed through energy estimates that small size $\varepsilon$ solutions of positive definite symmetric hyperbolic systems exist on an interval of order $\mathcal{O}(1/\varepsilon)$. In 1976, Fritz John [5] improved this result to $\mathcal{O}((\varepsilon \log 1/\varepsilon)^{-4})$ for quasilinear waves in $3D$. In dimensions four and higher, John showed in the same work that the existence interval is $\mathcal{O}(1/\varepsilon^2)$. Estimates of the solution of the linear wave equation are key ingredients in John's approach. Klainerman was able to prove in [9] that solutions exist globally in dimension six and higher. The same result was later obtained in [13] by Klainerman and Ponce with simplified arguments involving energy estimates and the fundamental solution. Global results were further established for dimensions four and five (Klainerman [11]).

Klainerman's results raised the question whether global solutions can be expected in the physically important three dimensional case. John [4] and Sideris [18], however, provided examples of quadratic nonlinearities that develop finite time singularities. Even though these results demonstrate that in $3D$ solutions in general do not exist globally, John and Klainerman [8], with the use of their knowledge of the fundamental solution, were still able to extend the existence interval to $\mathcal{O}(\exp(C/\varepsilon))$ - a lifespan referred to as almost global existence.

The advances discussed so far rely on estimates of the fundamental solution of the wave equation. The techniques used are not easily adaptable to elasticity where the fundamental solutions tend to be much more complicated. Klainerman addressed that issue by developing a new method (we will refer to it as the generalized energy method) to handle quasilinear waves without relying on the solutions to the linear equations. Exploiting the scaling ($S = t\partial_t + x \cdot \nabla$), rotational ($\Omega = x \wedge \nabla$), and Lorentz boost ($L = t\nabla + x\partial_t$) invariance of the wave equation, Klainerman defined a generalized Sobolev space in terms of these new vector fields together with the standard space-time derivatives $\partial = (\partial_t, \nabla)$. With this new structure, Klainerman [11] improved the standard Sobolev Lemma by developing new $L^\infty - L^2$ decay estimates without any references to the fundamental solution.

Already known that quadratically nonlinear waves in $3D$ can develop finite time singularities even for small initial data, the next important question posed was whether there were nonlinearities for which global solutions were possible. Klainerman [12] and Christodoulou [2] independently identified a condition on the quadratic portion of the nonlinearity, called null condition, which allowed for global existence for small data. The corresponding proofs avoided estimates of the fundamental solution.

The advances in elastodynamics in $3D$ followed a path similar to the one of quasilinear waves. With the use of the fundamental solution in [6], John showed that the interval of existence of isotropic elastic materials is $\mathcal{O}((\varepsilon \log 1/\varepsilon)^{-4})$. In [3], he improved this result to an almost global existence interval for small solutions.

Analogous to the wave equation, solutions in nonlinear elasticity generally do not exist globally. In [7], John presents genuine nonlinearities for which arbitrary small spherically symmetric displacements to isotropic elastic materials develop singularities. Another result comes from Sideris [17] who, borrowing some of the techniques from his earlier work [18] on nonlinear wave equation, established that solutions to ceratin compressible fluids break down in finite time regardless of the size of the initial data.

Klainerman and Sideris [14] were able to simplify John's almost global result by avoiding any use of the fundamental solution. The authors used Klainerman's generalized energy method to prove that solutions to quadratically nonlinear waves exist almost globally. This result has immediate applications in isotropic hyperelasticity. A major obstacle in adapting the generalized energy approach was that motions of elasticity are not Lorentz invariant. The smaller symmetry group implied weaker Klainerman's inequalities. To compensate for this deficiency, the authors obtained an additional set of weighted $L^2$-estimates which, in combination with the $L^\infty - L^2$ inequalities, were successfully used in many other applications (see [19], [20], [23]).

The transition from almost global to global solution in nonlinear elastodynamics requires two conditions: (1) the initial deformation must be a small displacement from equilibrium and (2) the nonlinear terms must satisfy a nonresonance (null) condition. If either of these conditions fails, solutions breakdown in finite time. As discussed above, John [7] and Sideris [17] provided examples of such breakdowns in the absence of nonresonance. On the other hand, Tahvildar-Zadeh [25] showed formation of singularities for large displacements.

The exact formulation of the nonresonance condition was first given by Sideris in [19] and then in [20] the author weakened the condition and showed that it is physically realistic. Nonresonance essentially requires the cancelation of nonlinear interactions among the same wave families along the characteristic cone. Compressible elastodynamics, for example, is characterized by nonlinear shear and pressure waves interactions. In the isotropic case, however, shear waves interactions are linearly degenerate, and so null condition is imposed only on the pressure waves (see [20] ).

Under the assumption of nonresonance condition, Sideris [20] further developed the ideas associated with the energy method from [14] and [19] to prove global existence for small solutions in compressible, isotropic, nonlinear elastodynamics.

The approach was adapted in [24] to a system of coupled quadratically nonlinear waves in $3D$ with multiple propagations speeds. Agemi [1] used the same null conditions from [20] to discuss existence of solutions near unstressed reference configuration.

Incompressible elasticity naturally satisfies a null condition in the isotropic case and therefore global existence is expected for small initial displacement. The result was confirmed by Sideris and Thomases in [21]. The proof relies on the generalized energy method and strong dispersive estimates. Key ingredients in the argument are also local decay estimates which were further shown in [23] to be applicable to a wide class of certain isotropic symmetric hyperbolic systems. Sideris and Thomases [22] proved similar global small data existence for an isotropic incompressible material regarded as the limit of slightly compressible materials. In this setting, the shear waves are already null (by isotropy) and the pressure ways vanish in the limit.

We will consider first the nonlinear PDE:

$$\partial_t^2\varphi - \Delta\varphi - \nu\partial_t\Delta\varphi = \sum_{\alpha,\beta=0}^{3}\sum_{\ell=1}^{3} C^{\ell}_{\alpha,\beta}\partial_\alpha\varphi\partial_\ell\partial_\beta\varphi \quad (t,x)\in\mathbb{R}^+\times\mathbb{R}^3, \qquad (1.0.1)$$

where $\partial_0 = \partial_t$, $\partial_\ell = \partial_{x_\ell}$, $\ell = 1, 2, 3$, and $\mathbb{R}^+ = [0,\infty)$. The summation notation will be employed hereafter.

The viscosity parameter $\nu$ is assumed to satisfy $0 < \nu \leq 1$. We will define a second parameter $\delta$ to measure the deviation of the nonlinearity from being null. Furthermore, we will assume that the size of the initial data is controlled by a third parameter $\varepsilon$. Our main objective is to analyze how the interaction of those three parameters $\nu$, $\delta$, and $\varepsilon$ influences the long time behavior of the solution to the given PDE.

What is known already is that in the hyperbolic case ($\nu = 0$) under a null conditions ($\delta = 0$) we have a small global solution (see [12], [2]). Moreover, Ponce showed in [16] that (1.0.1) generally admits global solution from small initial data. In his proof, Ponce relied on the dissipative properties of the linear equation. Also, initial data was assumed small relative to the viscosity parameter although this relationship was not quantified explicitly.

Intuitively, one would expect that a global solution of (1.0.1) will be attained for large values of $\nu$ that make the dispersive effect of the linear term more pronounced and for small values of $\delta$ which enhance the effect of the null term. The precise interaction of the three parameters is captured in the two main theorems 2.3.1 and 2.3.2.

In Theorem 2.3.1 we show that the size of the initial data must be roughly $\nu/\delta$ in order to obtain a global result. If this condition is not met, then Theorem 2.3.2 gives lower bounds for the lifespan of the solution. In the hyperbolic case, it is well known ([8], [10]) that the solution exists almost globally. Theorem 2.3.1 shows that dissipation can improve the almost global result of the hyperbolic case if $\nu$ is large enough relative to the size of the data.

The main tools we use in the above results are the generalized energy method, $L^\infty - L^2$ and weighted $L^2$- decay estimates . In order to capture the dissipative nature of the PDEs, we incorporate the viscous terms in the energy definition (see (2.2.3)). This allows for terms arising in bootstrapping arguments to be controlled by the energy.

The PDE that we are studying is not invariant under the Lorentz boost $L$ and as a result this vector field is not included in our energy definition. Consequently, we can only use a weaker version of Klainerman's original $L^\infty - L^2$ decay estimates. To obtain the extra decay needed, we provide additional dispersive estimates by extending the weighted $L^2$- estimates approach introduced in [14] and further generalized in [23]. Those decay estimates are derived in two space-time regions and subsequently we obtain interior and exterior weighted $L^2$-estimates (see Theorem (2.8.1) and Theorem (2.8.2) correspondingly). It is convenient to work in the framework of [23] and therefore we express (1.0.1) as a first order system in (2.1.1a) and (2.1.1b).

Although the scaling operator $S$ does not commute with the linear part of (1.0.1), it is still included in the energy definition. However, we need to keep an explicit track of the occurrences of the scaling operator (see (2.2.3)). The additional linear terms that appear as a result of the lack of scaling invariance are handled through an inductive argument (see, for example, Theorem (2.7.3)).

Next we study the motion of incompressible Hookean isotropic viscoelastic materials given by:

$$\partial_t G - \nabla v = \nabla v G - v \cdot \nabla G \tag{1.0.2a}$$

$$\partial_t v - \nabla \cdot G - \nu \Delta v = \nabla \cdot (G G^T) - v \cdot \nabla v - \nabla p \tag{1.0.2b}$$

with constraints

$$\begin{aligned} &\nabla \cdot v = 0 \\ &\nabla \cdot G^T = 0 \qquad\qquad (1.0.2c) \\ &\partial_k G^{ij} - \partial_j G^{ik} = G^{\ell j} \partial_\ell G^{ik} - G^{\ell k} \partial_\ell G^{ij} \equiv Q_k^{ij}(G, \nabla G), \end{aligned}$$

where $G \in \mathbb{R}^3 \otimes \mathbb{R}^3$, $v \in \mathbb{R}^3$, and $p \in \mathbb{R}$. The precise derivation of the system is given in section 3.1.

The incompressible and isotropic assumptions imply that the quadratic nonlinearity of (1.0.2a) and (1.0.2b) inherently satisfies a null condition. The expected global existence result is verified in Theorem 3.3.1. The smallness of the initial data is shown to be uniform with respect to viscosity.

The proof of this global existence result shares the main features we discussed earlier in the damped wave equation case. In viscoelasticity, however, it is enough to establish only interior weighted $L^2$-estimates unlike the damped wave equation case in which those estimates are needed in both the interior and exterior regions. This major simplification comes from the special nonlinear structure associated with the Hookean assumption on the strain energy function (see (3.1.4)) and the incompressible constraints (1.0.2c).

Another difference in the viscoelastic case is the presence of the pressure term $\nabla p$ in (1.0.2b). It is treated as a part of the nonlinearity since its $L^2$ norm can be controlled by quadratic terms (see Lemma 3.9.1).

# Chapter 2

# Damped Wave Equation

## 2.1 PDEs

We will rewrite the PDE (1.0.1) as a first order system. We introduce the new unknowns:

$$u = u^\alpha e_\alpha = \partial_\alpha \varphi e_\alpha$$

where $e_0, \ldots, e_3$ are the standard basis column vectors in $\mathbb{R}^4$. We denote the spatial and the spatial-time gradients of $u$ as $\nabla u$ and $\partial u$ where

$$(\nabla u)_{\alpha k} = \partial_k u^\alpha \quad \textit{and} \quad (\partial u)_{\alpha\beta} = \partial_\beta u^\alpha.$$

Notice that we have the relation:

$$\partial u = \partial u^\top.$$

With the above notation, we can rewrite (1.0.1) as the following evolutionary system with constraints:

$$Lu \equiv \partial_t u - A^j \partial_j u - \nu B \Delta u = N(u, \nabla u) \tag{2.1.1a}$$

$$\partial_j u^k = \partial_k u^j. \tag{2.1.1b}$$

The coefficients are given by:

$$A^j = e_0 \otimes e_j + e_j \otimes e_0,\ j = 1, 2, 3; \quad B = e_0 \otimes e_0 \tag{2.1.1c}$$

and the nonlinearity is of the form:

$$N(u, \nabla u) = N^0(u, \nabla u) e_0 \quad \text{with} \quad N^0(u, \nabla u) = C^{\ell}_{\alpha,\beta} u^\alpha \partial_\ell v^\beta. \tag{2.1.1d}$$

With the notation $\bar{u} = u^j e_j$ ($\bar{u}$ is a vector in $\mathbb{R}^3$), the system (2.1.1a) - (2.1.1d) can also be written as:

$$\partial_t u^0 - \nabla \cdot \bar{u} - \nu \Delta u^0 = N^0(u, \nabla u) \tag{2.1.2a}$$

$$\partial_t \bar{u} - (\nabla u^0)^\top = 0 \tag{2.1.2b}$$

$$\nabla \wedge \bar{u} = 0. \tag{2.1.2c}$$

## 2.2 Notation

We will use the following vector fields:

$$\nabla, \quad \Omega = x \wedge \nabla, \quad S = t\partial_t + r\partial_r, \quad S_0 = r\partial_r.$$

We will also employ the following modified version of the rotational operators $\Omega$:

$$\widetilde{\Omega}_i = I\Omega_i + Z_i,$$

where

$$Z_1 = e_2 \otimes e_3 - e_3 \otimes e_2,\ Z_2 = e_3 \otimes e_1 - e_1 \otimes e_3,\ Z_3 = e_1 \otimes e_2 - e_2 \otimes e_1.$$

We note that, as defined, $\widetilde{\Omega}$ commutes with linear part of the system (2.1.1a). In particular, for any scalar function $\varphi$, we have:

$$\partial(\Omega_i\varphi) = \widetilde{\Omega}_i\partial\varphi.$$

On the other hand, the linear system 2.1.1a is not scale ($S$) invariant. For this reason, we will need to explicitly keep track of the number of occurrences of the vector field $S$ in the Hilbert space $X^{p,q}$ and energy $\mathcal{E}_{p,q}[u](t)$ definitions (see (2.2.2) and (2.2.3)).

We will also rely on the following spatial gradient decomposition:

$$\nabla = \omega\partial_r - \frac{w}{r} \wedge \Omega \tag{2.2.1}$$

with

$$\omega = \frac{x}{r}, \quad r = |x|, \quad \partial_r = \omega^j\partial_j.$$

For any $0 \leq q \leq p$ and with the following abbreviation:

$$\Gamma = \{\nabla, \widetilde{\Omega}\},$$

we can define the space:

$$X^{p,q} = \{u \in H^p(\mathbb{R}^3;\mathbb{R}^4) : \|S_0^k\Gamma^a u\|_{L^2} < \infty, \text{for all } |a| + k \leq p,\ k \leq q\}. \tag{2.2.2}$$

In the above definition $p$ denotes the total number of derivatives allowed, while $q$ stands for the number of occurrences of the scaling operator $S_0$. We emphasize again that it is the lack of commutativity of $S_0$ with the linear operator in (2.1.1a) that dictates this special consideration for the scaling operator. As defined, $X^{p,q}$ is as Hilbert Space with inner product:

$$\langle u, v\rangle_{X^{p,q}} = \sum_{\substack{|a|+k\leq p\\ k\leq q}} \langle S_0^k\Gamma^a u, S_0^k\Gamma^a v\rangle_{L^2}.$$

The initial data of the PDE under consideration will be defined in $X^{p,q}$.

The energy associated with the solution of the system (2.1.1a) - (2.1.1d) is defined as:

$$\mathcal{E}_{p,q}[u](t) = \sum_{\substack{|a|+k\leq p\\ k\leq q}} \left[\frac{1}{2}\|S^k\Gamma^a u(t)\|_{L^2}^2 + \nu\int_0^t \|\nabla S^k(\Gamma^a u)^0(s)\|_{L^2}^2\, ds\right]. \tag{2.2.3}$$

For $u_0 = u(0)$ we write:

$$\mathcal{E}_{p,q}[u_0] \equiv \mathcal{E}_{p,q}[u](0) = \frac{1}{2}\|u_0\|_{X^{p,q}}^2.$$

In order to obtain bounds for the energy, we need to establish dispersive estimates. These will be derived using weighted $L^2$-estimates in two space-time regions referred to as interior and exterior. We define those two regions with the following cut-off functions:

$$\zeta(t,x) = \psi\left(\frac{|x|}{\sigma\langle t\rangle}\right) \quad \text{and} \quad \eta(t,x) = 1 - \psi\left(\frac{2|x|}{\sigma\langle t\rangle}\right), \tag{2.2.4a}$$

where $\psi$ is given by:

$$\psi \in C^{\infty}(\mathbb{R}), \quad \psi(s) = \begin{cases} 1, & s \leq 1/2 \\ 0, & s \geq 1 \end{cases}, \quad \psi' \leq 0. \tag{2.2.4b}$$

We use the notation $\langle t \rangle = (1+t^2)^{1/2}$. The parameter $\sigma$ would be chosen to be small enough ($\sigma \ll 1$). We notice that:

$$1 \leq \zeta + \eta \quad \text{and} \quad 1 - \eta \leq \zeta^2. \tag{2.2.4c}$$

We also have the following property:

$$\langle r+t \rangle \Big[ |\partial \zeta(t,x)| + |\partial \eta(t,x)| \Big] \lesssim 1. \tag{2.2.4d}$$

In the interior region, we will derive estimates for the following quantities:

$$\mathcal{Y}^{\text{int}}_{p,q}[u](t) = \sum_{\substack{|a|+k \leq p-1 \\ k \leq q}} \|\zeta \nabla S^k \Gamma^a u(t)\|^2_{L^2}$$

and

$$\mathcal{Z}^{\text{int}}_{p,q}[u](t) = \sum_{\substack{|a|+k \leq p-1 \\ k \leq q}} \|\zeta \Delta S^k (\Gamma^a u)^0(t)\|^2_{L^2},$$

for $q < p$ and $0 \leq \theta \leq 1$.

In the exterior region, we need to decompose the solution into its orthogonal and tangential components along the light cone. We use the following projections:

$$\mathbb{P}u(t,x) = \frac{1}{2}\, \hat{\omega} \otimes \hat{\omega}\; u(t,x) = \frac{1}{2} \left[ u^0(t,x) - \omega \cdot \bar{u}(t,x) \right] \hat{\omega} \tag{2.2.5a}$$

$$\mathbb{Q}u(t,x) = (I - \mathbb{P})u(t,x),$$

in which

$$\hat{\omega} = \begin{bmatrix} 1 \\ \\ -\omega \end{bmatrix} \in \mathbb{R}^4, \quad \omega = \frac{x}{|x|}, \quad 0 \neq x \in \mathbb{R}^3. \tag{2.2.5b}$$

We have the following commutation properties:

$$[\widetilde{\Omega}_j, \mathbb{P}] = [\partial_r, \mathbb{P}] = 0 \quad \text{and} \quad [\widetilde{\Omega}_j, \mathbb{Q}] = [\partial_r, \mathbb{Q}] = 0 \tag{2.2.6}$$

due to the fact that $\widetilde{\Omega}_j \omega = 0$ and $\partial_r \omega = 0$.

The quantities to be estimated in the exterior region are:

$$\begin{aligned} \mathcal{Y}^{\text{ext}}_{p,q}[u](t) \;=\; & \sum_{\substack{|a|+k \leq p-1 \\ k \leq q}} \sum_{j=1}^{3} \big[ \|\eta \, \langle t-r \rangle \, \mathbb{P}\partial_j S^k \Gamma^a u(t)\|^2_{L^2} \\ & \qquad + \|\eta \, \langle t+r \rangle \, \mathbb{Q}\partial_j S^k \Gamma^a u(t)\|^2_{L^2} \big] \end{aligned}$$

and

$$\mathcal{Z}^{\text{ext}}_{p,q}[u](t) = t^2 \sum_{\substack{|a|+k \leq p-1 \\ k \leq q}} \|\eta \Delta S^k (\Gamma^a u)^0(t)\|^2_{L^2},$$

again for $q < p$.

We define a cubic polynomial $P_N(y)$ associated with the coefficients of a quadratic nonlinearity of the form 2.1.1d in the following way:

$$P_N(y) = C^{\ell}_{\alpha,\beta} y^{\alpha} y^{\beta} y^{\ell}, \quad y \in \mathbb{R}^4.$$

We say that the nonlinearity $N$ is null if

$$P_N(y) = 0 \quad \text{for all} \quad y \in \mathcal{N} = \{y \in \mathbb{R}^4 : y_0^2 - y_1^2 - y_2^2 - y_3^2 = 0\}.$$

The set of vectors in $\mathbb{R}^4$ belonging to $\mathcal{N}$ is called null vectors.

Finally, we remark that the expression $A \lesssim B$ would mean $A \leq CB$, where $C$ is a constant independent of $\nu, \delta, \varepsilon^2$, and the initial data. Furthermore, $\mathcal{O}(B)$ would denote a quantity that satisfies $\mathcal{O}(B) \lesssim B$.

## 2.3 Main Results

**Theorem 2.3.1** (Global existence)**.** *Choose* $(p,q)$ *such that* $p \geq 11$*, and* $p \geq q > p^*$*, where* $p^* = \left[\frac{p+5}{2}\right]$*. Define*

$$\delta = \max\{|\Omega^a P_N(y)| : y \in \mathcal{N},\ \|y\| = 1,\ |a| \leq p^*\} \tag{2.3.1}$$

*and assume that* $\delta \leq 1$*.*

*There are positive constants* $C_0,\ C_1 > 1$ *with the property that if the initial data* $u_0$ *satisfies*

$$C_0 \mathcal{E}_{p^*,p^*}[u_0]\left(1 + \mathcal{E}_{p,q}^{1/2}[u_0]\right) < \varepsilon^2, \tag{2.3.2a}$$

*for some* $\varepsilon^2 \ll 1$*, and*

$$C_0^3 \left(\frac{\delta}{\nu}\right)^2 \mathcal{E}_{p^*,p^*}[u_0]\left(1 + \mathcal{E}_{p,q}^{1/2}[u_0]\right) < 1, \tag{2.3.2b}$$

*then* (2.1.1a)-(2.1.1d) *has a unique global solution*

$$u \in C(\mathbb{R}^+; X^{p,q})$$

*with*

$$\sup_{0 \le t < \infty} \mathcal{E}_{p,q}[u](t) \le C_1 \mathcal{E}_{p,q}[u_0] \langle t \rangle^{C_1 \varepsilon}$$

*and*

$$\sup_{0 \le t < \infty} \mathcal{E}_{p^*,p^*}[u](t) < \varepsilon^2.$$

*Outline of Proof.* To establish global existence it is enough to show that the energy $\mathcal{E}_{p,q}[u](t)$ remains finite. Let $\mathcal{T}$ be the set of times $T \in (0, \infty)$ satisfying the properties:

(P1) Equations (2.1.1a)-(2.1.1d) have a unique local solution

$u \in C([0,T), X^{p,q})$, with $u(0) = u_0$, and

(P2) $\mathcal{E}_{p^*,p^*}[u](t) < \varepsilon^2$, for $0 \le t < T$.

If $T \in \mathcal{T}$ then we have $(0,T) \in \mathcal{T}$ and hence $\mathcal{T}$ is connected. Since $C_0 > 1$, (2.3.2a) implies that $\mathcal{E}_{p^*,p^*}[u_0] < \varepsilon^2$ and therefore, by the the local existence result, the set $\mathcal{T}$ is nonempty. The set $\mathcal{T}$ is relatively closed in $(0, \infty)$.

We show next that $\mathcal{T}$ is relatively open in $(0, \infty)$. If $T \in \mathcal{T}$, then by (P2) and Proposition 2.9.1

$$\sup_{0 \le t < T} \mathcal{E}_{p,q}[u](t) \le C_1 \mathcal{E}_{p,q}[u_0] \langle T \rangle^{C_1 \varepsilon} < \infty,$$

so, by the local existence theorem, (P1) holds for some $T' > T$.

Using the assumptions (2.3.2b) and (P2), we apply Proposition 2.10.1 which, together with (2.3.2a), gives

$$\sup_{0 \le t < T} \mathcal{E}_{p^*,p^*}[u](t) \le C_0 \mathcal{E}_{p^*,p^*}[u_0] \left(1 + \mathcal{E}_{p,q}^{1/2}[u_0]\right) < \varepsilon^2,$$

and so we have by continuity that (P2) holds for $0 \le t < T''$, with $T < T'' \le T'$. This shows that $(0, T'') \subset \mathcal{T}$, and so $\mathcal{T}$ is open. The nonempty connected set $\mathcal{T}$ is both open and closed in $(0, \infty)$, and therefore equal to $(0, \infty)$. □

The next result establishes "almost global" existence of small solutions in the case when the second smallness condition (2.3.2b) does not hold.

**Theorem 2.3.2** (Almost global existence)**.** *Choose* $(p, q)$ *with* $p \ge 11$ *and* $p \ge q > p^*$, *where* $p^* = \left[\frac{p+5}{2}\right]$. *Define* $\delta \le 1$ *by* (2.3.1).

*There are positive constants* $C_0$, $C_1 > 1$ *with the property that if the initial data* $u_0$ *satisfies* (2.3.2a), *for some* $\varepsilon^2 \ll 1$, *then* (2.1.1a)-(2.1.1d) *has a unique solution*

$$u \in C([0, T_0); X^{p,q})$$

*with* $T_0$ *defined by*

$$C_1 \langle T_0 \rangle^{C_1 \varepsilon} = \left( \frac{2 \ \max\{\nu, C_1 \varepsilon\}}{C_0 \ \delta \ \mathcal{E}_{p^*,p^*}^{1/2}[u_0]} \right)^2$$

*and*

$$\sup_{0\le t<T_0} \mathcal{E}_{p^*,p^*}[u](t) < \varepsilon^2.$$

*Proof.* Suppose that $u_0$ satisfies (2.3.2a), for $\varepsilon^2 \ll 1$. Consider the set

$$\mathcal{T} = \{T \in (0,T_0) : \text{(P1) and (P2) hold}\}.$$

The set $\mathcal{T}$ is nonempty, connected, and closed relative to $(0,T_0)$.

If $T \in \mathcal{T}$, then Propositions (2.10.2) and (2.3.2a) imply that

$$\sup_{0\le t<T} \mathcal{E}_{p^*,p^*}[u](t) \le C_0 \mathcal{E}_{p^*,p^*}[u_0] \left(1 + \mathcal{E}_{p,q}^{1/2}[u_0]\right) < \varepsilon^2.$$

Thus, $\mathcal{T}$ is open relative to $(0,T_0)$. By connectness, $\mathcal{T} = (0,T_0)$. □

*Remarks.*

- The following table summarizes the results of the Theorems. The basic smallness restriction (2.3.2a) must always be enforced.

| | | |
|---|---|---|
| $\dfrac{\varepsilon\,\delta}{\nu} < \dfrac{1}{C_0}$ | Global existence | (2.3.5a) |
| $\dfrac{1}{C_0} < \dfrac{\varepsilon\,\delta}{\nu} < \dfrac{\delta}{C_1}$ | Almost global existence with diffusion enhanced lifespan | (2.3.5b) |
| $\dfrac{\delta}{C_1} < \dfrac{\varepsilon\,\delta}{\nu}$ | Almost global existence with hyperbolic lifespan | |

- The cases (2.3.5a), (2.3.5b) show that diffusive effects are important when $\nu \geq C_1\varepsilon$.

For the remainder of the article, we assume that properties (P1) and (P2) hold. In the following sections, we are going to establish a series of *a priori* estimates culminating in Propositions 2.9.1, 2.10.1, and 2.10.2.

## 2.4 Commutation

For the linear terms we have the following commutation properties:

$$LS^k\Gamma^a u = (S+1)^k\Gamma^a Lu - \nu B\Delta[S^k - (S-1)^k]\Gamma^a u \tag{2.4.1a}$$

$$\nabla \wedge S^k\Gamma^a\bar{u} = (S+1)^k\Gamma^a\nabla \wedge \bar{u}, \tag{2.4.1b}$$

where $L = I\partial_t - A^j\partial_j - \nu B\nabla$, as defined in 2.1.1a, while $a$ in any multi-index and $k > 0$ is an integer.

For the nonlinear terms 2.1.1d we define the commutators as:

$$[\partial_i, N](u, \nabla v) = \partial_i N(u, \nabla v) - N(\partial_i u, \nabla v) - N(u, \nabla\partial_i v)$$

$$[(S+1), N](u, \nabla v) = SN(u, \nabla v) - N(Su, \nabla v) - N(u, \nabla Sv)$$

$$[\Omega_i, N](u, \nabla v) = \Omega_i N(u, \nabla v) - N(\widetilde{\Omega}_i u, \nabla v) - N(u, \nabla\widetilde{\Omega}_i v)$$

and we have the following commutation properties:

**Lemma 2.4.1.** *The nonlinear commutators satisfy the relations*

$$[\partial, N] = [S, N] = 0$$

*and*

$$[\Omega_i, N](u, \nabla v) = \widetilde{C}^{i,j}_{\alpha,\beta} u^\alpha \partial_j v^\beta, \tag{2.4.2}$$

*with*

$$\widetilde{C}^{i,j}_{\alpha,\beta} = C^j_{\lambda,\beta}(Z_i)_{\alpha\lambda} + C^j_{\alpha,\lambda}(Z_i)_{\beta\lambda} + C^\lambda_{\alpha,\beta}(Z_i)_{j\lambda}.$$

The higher order commutators are obtained inductively according to the Leibnitz-type formula:

**Lemma 2.4.2.**

$$(S+1)^k \Gamma^a N(u, \nabla v) = \sum_{\substack{a_1+a_2+a_3=a \\ k_1+k_2=k}} \frac{a!}{a_1!\, a_2!\, a_3!} \frac{k!}{k_1!\, k_2!} [\Gamma^{a_3}, N](S^{k_1}\Gamma^{a_1} u, \nabla S^{k_2}\Gamma^{a_2} v).$$

**Lemma 2.4.3.** *For any quadratic nonlinearity of the form* (2.1.1d)

$$\Omega_i P_N(y) = P_{[\Omega_i,N]}(y), \quad i = 1, 2, 3.$$

*If $N$ is null, then $[\Omega_i, N]$ is also null.*

*Proof.* We have $\widetilde{\Omega}_i y = \Omega_i y + Z_i y = 0$, for all $y \in \mathbb{R}^4$. Thus, from the chain rule and (2.4.2) we obtain the first statement:

$$\Omega_i P_N(y) = D_y P_N(y)[\Omega_i y] = D_y P_N(y)[-Z_i y] = P_{[\Omega_i,N]}(y).$$

Suppose that $N$ is null. The one-parameter family of rotations $U(s) = \exp(-sZ_i)$ leaves the set of null vectors $\mathcal{N}$ invariant. Thus, for any $y \in \mathcal{N}$, we have

$$0 = \frac{d}{ds} P_N(U(s)y)\Big|_{s=0} = D_y P_N(y)[-Z_i y] = P_{[\Omega_i,N]}(y).$$

This shows that $P_{[\Omega_i,N]}$ is also null. ∎

**Lemma 2.4.4.** *For any $|a| \le p^*$, we have*

$$\langle [\widetilde{\Omega}^a, N](u(x), \nabla v(x)), w(x)\rangle_{\mathbb{R}^4}$$
$$= \tfrac{1}{4} P_{[\widetilde{\Omega}^a,N]}(\hat{\omega})\ \langle \hat{\omega}, u(x)\rangle_{\mathbb{R}^4}\ \langle \hat{\omega}, \partial_r v(x)\rangle_{\mathbb{R}^4}\ w^0(x) + \mathcal{O}(\mathcal{R}), \quad (2.4.3a)$$

*with*

$$\mathcal{R} = \Big[ |\mathbb{Q}u(x)|\ |\partial_r v(x)| + |u(x)|\ |\mathbb{Q}\partial_r v(x)| + r^{-1}|u(x)|\ |\Omega v(x)|\ \Big] |w^0(x)|,$$

*and also*

$$\Big| \tfrac{1}{4} P_{[\widetilde{\Omega}^a,N]}(\hat{\omega})\ \langle \hat{\omega}, u(x)\rangle_{\mathbb{R}^4}\ \langle \hat{\omega}, \partial_r v(x)\rangle_{\mathbb{R}^4}\ w^0(x) \Big|$$
$$\lesssim \delta\ |u(x)|\ |\partial_r v(x)|\ |w^0(x)|, \quad (2.4.3b)$$

*with $\delta$ defined in* (2.3.1).

*Proof.* By (2.1.1d), $N = N^0 e_0$, so using (2.2.1), we can write

$$\langle [\widetilde{\Omega}^a, N](u(x), \nabla v(x)), w(x)\rangle_{\mathbb{R}^4}$$
$$= \langle [\widetilde{\Omega}^a, N](u(x), \omega \otimes \partial_r v(x)), w(x)\rangle_{\mathbb{R}^4}$$
$$+ \mathcal{O}\left( r^{-1} |u(x)| |\Omega v(x)| |w^0(x)| \right).$$

With the projections defined in (2.2.5a), (2.2.5b), we obtain

$$[\widetilde{\Omega}^a, N](u(x), \omega \otimes \partial_r v(x)) = [\widetilde{\Omega}^a, N](\mathbb{P}u(x), \omega \otimes \mathbb{P}\partial_r v(x))$$
$$+ [\widetilde{\Omega}^a, N](\mathbb{Q}u(x), \omega \otimes \mathbb{P}\partial_r v(x)) + [\widetilde{\Omega}^a, N](u(x), \omega \otimes \mathbb{Q}\partial_r v(x)).$$

Denoting the coefficients associated with the quadratic nonlinearity $[\widetilde{\Omega}^a, N]$ as $\widetilde{C}^{a,\ell}_{\alpha,\beta}$, we can write the key term as:

$$
\begin{aligned}
[\widetilde{\Omega}^a, N](\mathbb{P}u(x),&\omega \otimes \mathbb{P}\partial_r v(x)) = \widetilde{C}^{a,\ell}_{\alpha,\beta}(\mathbb{P}u(x))^\alpha(\omega \otimes \mathbb{P}\partial_r v(x))_{\beta\ell} \\
&= \widetilde{C}^{a,\ell}_{\alpha,\beta}\left(\frac{1}{2}\langle\hat{\omega}, u(x)\rangle_{\mathbb{R}^4}\hat{\omega}\right)^\alpha \left(\omega \otimes \frac{1}{2}\langle\hat{\omega}, \partial_r v(x)\rangle_{\mathbb{R}^4}\hat{\omega}\right)_{\beta\ell} \\
&= \frac{1}{4}\widetilde{C}^{a,\ell}_{\alpha,\beta}\hat{\omega}^\alpha\omega^\beta\omega^\ell\langle\hat{\omega}, u(x)\rangle_{\mathbb{R}^4}\langle\hat{\omega}, \partial_r v(x)\rangle_{\mathbb{R}^4} \\
&= \frac{1}{4}P_{[\widetilde{\Omega}^a,N]}(\hat{\omega})\langle\hat{\omega}, u(x)\rangle_{\mathbb{R}^4}\langle\hat{\omega}, \partial_r v(x)\rangle_{\mathbb{R}^4},
\end{aligned}
$$

from which (2.4.3a) now easily follows.

Notice that Lemma 2.4.3 gives

$$P_{[\widetilde{\Omega}^a,N]}(\hat{\omega}) = \Omega^a P_N(\hat{\omega}).$$

Now $\hat{\omega}/\sqrt{2}$ belongs to $\{\|y\|_{\mathbb{R}^4} = 1\} \cap \mathcal{N}$, so by homogeneity we have

$$|\Omega^a P_N(\hat{\omega})| \leq 2^{3/2}\delta,$$

and (2.4.3b) follows. □

## 2.5 Sobolev Inequalities

**Lemma 2.5.1.** *Suppose that $u \in X^{2,0}$. Set $r = |x|$. Then*

$$\|u\|_{L^\infty} \lesssim \sum_{|a|\leq 2} \|\nabla^a u\|_{L^2} \tag{2.5.1a}$$

$$\|r^{-1}u\|_{L^2} \lesssim \|\partial_r u\|_{L^2} \tag{2.5.1b}$$

$$\|r^{1/2}u\|_{L^\infty} \lesssim \sum_{|a|\le 1} \|\nabla\widetilde{\Omega}^a u\|_{L^2} \tag{2.5.1c}$$

$$\|ru\|_{L^\infty} \lesssim \left(\sum_{|a|\le 1} \|\partial_r\widetilde{\Omega}^a u\|_{L^2(|y|\ge r)} \sum_{|a|\le 2} \|\widetilde{\Omega}^a u\|_{L^2(|y|\ge r)}\right)^{1/2}. \tag{2.5.1d}$$

*Proof.* The inequality (2.5.1a) is the standard Sobolev lemma, and (2.5.1b) is Hardy's inequality. Inequalities (2.5.1c) and (2.5.1d) were proven in Lemma 3.3 of [20]. □

**Proposition 2.5.2.** *Suppose that* $u : [0, T) \times \mathbb{R}^3 \to \mathbb{R}^4$ *satisfies*

$$\mathcal{Y}^{int}_{2,0}[u](t) + \mathcal{Y}^{ext}_{2,0}[u](t) + \mathcal{E}_{2,0}[u](t) < \infty.$$

*Then using the weights* (2.2.4a)*, we have*

$$\|\ \zeta\ u(t)\ \|_{L^\infty} \lesssim \left(\mathcal{Y}^{int}_{2,0}[u](t)\right)^{1/2} + \langle t\rangle^{-1}\mathcal{E}^{1/2}_{1,0}[u](t) \tag{2.5.2a}$$

$$\|\ r\zeta\ \nabla u(t)\ \|_{L^\infty} \lesssim \left(\mathcal{Y}^{int}_{3,0}[u](t)\right)^{1/2} + \langle t\rangle^{-1}\mathcal{E}^{1/2}_{2,0}[u](t) \tag{2.5.2b}$$

$$\|\ r^{-1}\zeta\ u(t)\ \|_{L^2} \lesssim \left(\mathcal{Y}^{int}_{1,0}[u](t)\right)^{1/2} + \langle t\rangle^{-1}\mathcal{E}^{1/2}_{0,0}[u](t) \tag{2.5.2c}$$

$$\|\ \eta\ u(t)\|_{L^\infty} \lesssim \langle t\rangle^{-1}\mathcal{E}^{1/2}_{2,0}[u](t) \tag{2.5.2d}$$

$$\|\ \eta\ \mathbb{Q}u(t)\|_{L^\infty} \lesssim \langle t\rangle^{-3/2}\left(\left(\mathcal{Y}^{ext}_{2,0}[u](t)\right)^{1/2} + \mathcal{E}^{1/2}_{1,0}[u](t)\right). \tag{2.5.2e}$$

*Proof.* Using the cutoff function $\psi$ defined in (2.2.4b), apply (2.5.1a) to $\psi u(t)$, $\psi = \psi(|y|)$. This produces

$$\|\psi u(t)\|_{L^\infty} \lesssim \sum_{|a|=1,2} \|\nabla^a u(t)\|_{L^2} + \|u(t)\|_{L^2(|y|\le 1)},$$

where we have used that on the support of derivatives of $\psi$ we have $\frac{1}{2} \leq |y| \leq 1$. We apply (2.5.1c) to the second integral

$$\|u(t)\|_{L^2(|y|\leq 1)} \lesssim \|r^{1/2}u(t)\|_{L^\infty}\|r^{-1/2}\|_{L^2(|y|\leq 1)} \lesssim \sum_{|a|\leq 1} \|\nabla\widetilde{\Omega}^a u(t)\|_{L^2}.$$

Thus, we see that

$$\|\psi u(t)\|_{L^\infty} \lesssim \sum_{|a|\leq 1} \|\nabla\Gamma^a u(t)\|_{L^2}.$$

On the other hand, we have using (2.5.1c) again

$$\|(1-\psi)u(t)\|_{L^\infty} \lesssim \|u(t)\|_{L^\infty\left(|y|\geq\frac{1}{2}\right)} \lesssim \|r^{1/2}u(t)\|_{L^\infty} \lesssim \sum_{|a|\leq 1} \|\nabla\widetilde{\Omega}^a u(t)\|_{L^2}.$$

This shows that

$$\|u(t)\|_{L^\infty} \lesssim \sum_{|a|\leq 1} \|\nabla\Gamma^a u(t)\|_{L^2}. \tag{2.5.3}$$

To prove (2.5.2a), apply (2.5.3) to the function $\zeta u(t)$. We also use (2.2.4d), the identity $\Omega_i\zeta = 0$, and the fact that on the support of $\psi'$ we have $\frac{1}{2} \leq \frac{|y|}{\sigma\langle t\rangle} \leq 1$.

Applying (2.5.1d) to $\zeta\nabla u(t)$ yields (2.5.2b).

The inequality (2.5.2c) follows by applying (2.5.1b) to $\zeta u(t)$.

Since

$$\langle t\rangle\|\eta u(t)\|_{L^\infty} \lesssim \|r\eta u(t)\|_{L^\infty},$$

we can get (2.5.2d), by applying (2.5.1d) to $\eta u(t)$.

Finally, we prove (2.5.2e). By (2.5.1d) applied to $\eta\mathbb{Q}u(t)$, we have

$$\langle t\rangle\|\eta\mathbb{Q}u(t)\|_{L^\infty} \lesssim \|r\eta\mathbb{Q}u\|_{L^\infty}$$

$$\lesssim \left( \sum_{|a|\le 1} \|\partial_r \widetilde{\Omega}^a \eta \mathbb{Q} u(t)\|_{L^2} \sum_{|a|\le 2} \|\widetilde{\Omega}^a \eta \mathbb{Q} u(t)\|_{L^2} \right)^{1/2}. \quad (2.5.4)$$

Using (2.2.4d) and the commutation property (2.2.6), we see that

$$\sum_{|a|\le 1} \|\partial_r \widetilde{\Omega}^a \eta \mathbb{Q} u(t)\|_{L^2} \lesssim \sum_{|a|\le 1} \|\eta \mathbb{Q} \partial_r \widetilde{\Omega}^a u(t)\|_{L^2} + \langle t\rangle^{-1} \sum_{|a|\le 1} \|\mathbb{Q} \widetilde{\Omega}^a u(t)\|_{L^2}.$$

By linearity, we have $\mathbb{Q}\partial_r = \mathbb{Q}\omega^j \partial_j = \omega^j \mathbb{Q}\partial_j$, so

$$\sum_{|a|\le 1} \|\eta \mathbb{Q} \partial_r \widetilde{\Omega}^a u(t)\|_{L^2} \lesssim \sum_{|a|\le 1} \sum_{j=1}^{3} \|\eta \mathbb{Q} \partial_j \widetilde{\Omega}^a u(t)\|_{L^2} \lesssim \langle t\rangle^{-1} \mathcal{Y}^{\text{ext}}_{2,0}[u](t)^{1/2}.$$

Since

$$\sum_{|a|\le 1} \|\mathbb{Q} \widetilde{\Omega}^a u(t)\|_{L^2} \lesssim \sum_{|a|\le 1} \|\widetilde{\Omega}^a u(t)\|_{L^2} \le \mathcal{E}^{1/2}_{1,0}[u](t),$$

we obtain the bound

$$\sum_{|a|\le 1} \|\partial_r \widetilde{\Omega}^a \eta \mathbb{Q} u(t)\|_{L^2} \lesssim \langle t\rangle^{-1} (\mathcal{Y}^{\text{ext}}_{2,0}[u](t)^{1/2} + \mathcal{E}^{1/2}_{1,0}[u](t)).$$

Noting that

$$\sum_{|a|\le 2} \|\widetilde{\Omega}^a \eta \mathbb{Q} u(t)\|_{L^2} = \sum_{|a|\le 2} \|\eta \mathbb{Q} \widetilde{\Omega}^a u(t)\|_{L^2} \lesssim \sum_{|a|\le 2} \|\widetilde{\Omega}^a u(t)\|_{L^2} \lesssim \mathcal{E}^{1/2}_{2,0}[u](t),$$

we deduce from (2.5.4)

$$\langle t\rangle \|\eta \mathbb{Q} u(t)\|_{L^\infty} \lesssim \left( \langle t\rangle^{-1} \left( \mathcal{Y}^{\text{ext}}_{2,0}[u](t)^{1/2} + \mathcal{E}^{1/2}_{2,0}[u](t) \right) \mathcal{E}^{1/2}_{2,0}[u](t) \right)^{1/2},$$

from which (2.5.2e) follows by Young's inequality. □

## 2.6 Calculus Inequalities

**Lemma 2.6.1.** *Suppose that* $u : [0,T) \times \mathbb{R}^3 \to \mathbb{R}^4$. *If*

$$k_1 + k_2 + |a_1| + |a_2| \le \bar{p} \quad \textit{and} \quad k_1 + k_2 \le \bar{q},$$

*then we have*

$$\|\zeta(S^{k_1}\Gamma^{a_1}u(t))^\alpha(S^{k_2}\Gamma^{a_2+1}u(t))^\beta\|_{L^2}$$
$$\lesssim \left(\left(\mathcal{Y}^{int}_{[\frac{\bar p+5}{2}],[\frac{\bar p}{2}]}[u](t)\right)^{1/2} + \langle t\rangle^{-1}\mathcal{E}^{1/2}_{[\frac{\bar p+3}{2}],[\frac{\bar p}{2}]}[u](t)\right)\mathcal{E}^{1/2}_{\bar p+1,\bar q}[u](t),$$

*provided the right-hand side is finite.*

*In the special case when* $k_2+|a_2|<\bar p$, *we have*

$$\|\zeta(S^{k_1}\Gamma^{a_1}u(t))^\alpha(S^{k_2}\Gamma^{a_2+1}u(t))^\beta\|_{L^2}$$
$$\lesssim \left(\left(\mathcal{Y}^{int}_{[\frac{\bar p+5}{2}],[\frac{\bar p}{2}]}[u](t)\right)^{1/2} + \langle t\rangle^{-1}\mathcal{E}^{1/2}_{[\frac{\bar p+3}{2}],[\frac{\bar p}{2}]}[u](t)\right)\mathcal{E}^{1/2}_{\bar p,\bar q}[u](t),$$

*provided the right-hand side is finite.*

*Proof.* In the case $k_1+|a_1|<k_2+|a_2|+1$, i.e. $k_1+|a_1|\le\left[\frac{\bar p}{2}\right]$, using the Sobolev inequality (2.5.2a) we have the following bound:

$$\|\zeta(S^{k_1}\Gamma^{a_1}u(t))^\alpha(S^{k_2}\Gamma^{a_2+1}u(t))^\beta\|_{L^2} \lesssim \|\zeta S^{k_1}\Gamma^{a_1}u(t)\|_{L^\infty}\|S^{k_2}\Gamma^{a_2+1}u(t)\|_{L^2}$$
$$\lesssim \left(\left(\mathcal{Y}^{\mathrm{int}}_{[\frac{\bar p+4}{2}],[\frac{\bar p}{2}]}[u](t)\right)^{1/2} + \langle t\rangle^{-1}\mathcal{E}^{1/2}_{[\frac{\bar p+2}{2}],[\frac{\bar p}{2}]}[u](t)\right)\mathcal{E}^{1/2}_{\bar p+1,\bar q}[u](t).$$

And in the case $k_2+|a_2|+1\le k_1+|a_1|$, i.e. $k_2+|a_2|\ \le\left[\frac{\bar p-1}{2}\right]$, we likewise have:

$$\|\zeta(S^{k_1}\Gamma^{a_1}u(t))^\alpha(S^{k_2}\Gamma^{a_2+1}u(t))^\beta\|_{L^2} \lesssim \|S^{k_2}\Gamma^{a_2+1}u(t)\|_{L^\infty}\|S^{k_1}\Gamma^{a_1}u(t)\|_{L^2}$$
$$\lesssim \left(\left(\mathcal{Y}^{\mathrm{int}}_{[\frac{\bar p+5}{2}],[\frac{\bar p-1}{2}]}[u](t)\right)^{1/2} + \langle t\rangle^{-1}\mathcal{E}^{1/2}_{[\frac{\bar p+3}{2}],[\frac{\bar p-1}{2}]}[u](t)\right)\mathcal{E}^{1/2}_{\bar p,\bar q}[u](t).$$

The second statement of the lemma follows similarly from the preceding arguments. □

**Lemma 2.6.2.** *Suppose that* $u : [0, T) \times \mathbb{R}^3 \to \mathbb{R}^4$. *If*

$$k_1 + k_2 + |a_1| + |a_2| \le \bar{p} \quad \textit{and} \quad k_1 + k_2 \le \bar{q},$$

*then we have*

$$\|\eta(S^{k_1}\Gamma^{a_1}u(t))^\alpha(S^{k_2}\Gamma^{a_2+1}u(t))^\beta\|_{L^2} \lesssim \langle t\rangle^{-1}\mathcal{E}^{1/2}_{\left[\frac{\bar{p}+5}{2}\right],\left[\frac{\bar{p}}{2}\right]}[u](t)\mathcal{E}^{1/2}_{\bar{p}+1,\bar{q}}[u](t),$$

*provided the right-hand side is finite.*

*In the special case when* $k_2 + |a_2| < \bar{p}$, *we have*

$$\|\eta(S^{k_1}\Gamma^{a_1}u(t))^\alpha(S^{k_2}\Gamma^{a_2+1}u(t))^\beta\|_{L^2} \lesssim \langle t\rangle^{-1}\mathcal{E}^{1/2}_{\left[\frac{\bar{p}+5}{2}\right],\left[\frac{\bar{p}}{2}\right]}[u](t)\mathcal{E}^{1/2}_{\bar{p},\, qbar}[u](t),$$

*provided the right-hand side is finite.*

*Proof.* In the case $k_1 + |a_1| < k_2 + |a_2| + 1$, i.e. $k_1 + |a_1| \le \left[\frac{\bar{p}}{2}\right]$, using the Sobolev inequality (2.5.2d) we have the following bound:

$$\begin{aligned}
&\|\eta(S^{k_1}\Gamma^{a_1}u(t))^\alpha(S^{k_2}\Gamma^{a_2+1}u(t))^\beta\|_{L^2} \\
&\qquad \lesssim \|\eta S^{k_1}\Gamma^{a_1}u(t)\|_{L^\infty}\|S^{k_2}\Gamma^{a_2+1}u(t)\|_{L^2} \\
&\qquad \lesssim \langle t\rangle^{-1}\mathcal{E}^{1/2}_{\left[\frac{\bar{p}+4}{2}\right],\left[\frac{\bar{p}}{2}\right]}[u](t)\mathcal{E}^{1/2}_{\bar{p}+1,\bar{q}}[u](t).
\end{aligned}$$

And in the case $k_2 + |a_2| + 1 \le k_1 + |a_1|$, i.e. $k_2 + |a_2| \le \left[\frac{\bar{p}-1}{2}\right]$, we similarly have:

$$\begin{aligned}
&\|\eta(S^{k_1}\Gamma^{a_1}u(t))^\alpha(S^{k_2}\Gamma^{a_2+1}u(t))^\beta\|_{L^2} \\
&\qquad \lesssim \|\eta S^{k_2}\Gamma^{a_2+1}u(t)\|_{L^\infty}\|S^{k_1}\Gamma^{a_1}u(t)\|_{L^2} \\
&\qquad \lesssim \langle t\rangle^{-1}\mathcal{E}^{1/2}_{\left[\frac{\bar{p}+5}{2}\right],\left[\frac{\bar{p}-1}{2}\right]}[u](t)\mathcal{E}^{1/2}_{\bar{p},\bar{q}}[u](t).
\end{aligned}$$

The second statement of the lemma follows analogously. □

Two slightly more specialized instances of this basic argument occur in the proof of Proposition 2.10.1.

## 2.7 Estimates of the Linear Equation

In this section we provide estimates on the solutions of the linear system (2.1.2a), (2.1.2b), and (2.1.2c):

$$\partial_t u^0 - \nabla \cdot \bar{u} - \nu \Delta u^0 = G \tag{2.7.1a}$$

$$\partial_t \bar{u} - (\nabla u^0)^\top = 0 \tag{2.7.1b}$$

$$\nabla \wedge \bar{u} = 0. \tag{2.7.1c}$$

**Lemma 2.7.1.** *Assume that $\sigma$ in* (2.2.4a) *is sufficiently small and that $\nu \leq 1$. Let $G \in L^2([0,T]; L^2(\mathbb{R}^3))$, for some $0 < T < \infty$. If $u = (u^0, \bar{u})$ is a solution of* (2.7.1a), (2.7.1b), (2.7.1c) *such that*

$$\sup_{0 \leq t \leq T} \mathcal{E}_{1,1}[u](t) < \infty,$$

*then for any $0 \leq \theta \leq 1$,*

$$\int_0^T \langle t \rangle^\theta \left[ \|\zeta \nabla u(t)\|_{L^2}^2 + \nu^2 \|\zeta \Delta u^0(t)\|_{L^2}^2 \right] dt \lesssim \nu \langle T \rangle^{\theta-2} \mathcal{E}_{1,0}[u](T) + \int_0^T \langle t \rangle^{\theta-2} \mathcal{E}_{1,1}[u](t) dt + \int_0^T \langle t \rangle^\theta \|\zeta G(t)\|_{L^2}^2 dt.$$

Multiplying (2.7.1a) and (2.7.1b) by $t$ and recalling that $S = t\partial_t + r\partial_r$ we have:

$$t(\nabla \cdot \bar{u} + \nu \Delta u^0) = S u^0 - r \partial_r u^0 - tG$$

$$t\nabla u^0 = S\bar{u} - r\partial_r \bar{u}.$$

Multiplying by $\zeta^2$ and taking the $L^2$-inner product we obtain:

$$\begin{aligned} t^2[\|\zeta\nabla\cdot\bar{u}\|_{L^2}^2 + 2\nu\langle\zeta\nabla\cdot\bar{u}, \zeta\Delta u^0\rangle_{L^2} \\ + \nu^2\|\zeta\Delta u^0\|_{L^2}^2 + \|\zeta\nabla u^0\|_{L^2}^2] \\ \lesssim \|\zeta r\partial_r u\|_{L^2}^2 + \mathcal{E}_{1,1}[u](t) + t^2\|\zeta G\|_{L^2}^2, \end{aligned} \quad (2.7.2)$$

where we have used that $\|\zeta S u\|_{L^2}^2$ is bounded by $\mathcal{E}_{1,1}[u](t)$. By (2.7.1b) we have:

$$\Delta u^0 = \nabla\cdot\nabla u^0 = \nabla\cdot\partial_t\bar{u},$$

so we can estimate the cross term as:

$$\begin{aligned} 2\nu\langle\zeta\nabla\cdot\bar{u}, \zeta\Delta u^0\rangle_{L^2} &= 2\nu\langle\zeta\nabla\cdot\bar{u}, \zeta\partial_t(\nabla\cdot\bar{u})\rangle_{L^2} = \nu\int_{\mathbb{R}^3}\zeta^2\partial_t(\nabla\cdot\bar{u})^2dx \\ &= \nu\partial_t\|\zeta\nabla\cdot\bar{u}\|_{L^2}^2 - \nu\int_{\mathbb{R}^3}\partial_t(\zeta^2)(\nabla\cdot\bar{u})^2dx \\ &\geq \nu\partial_t\|\zeta\nabla\cdot\bar{u}\|_{L^2}^2 - \nu\int_{\mathbb{R}^3}C_1\zeta\langle t\rangle^{-1}(\nabla\cdot\bar{u})^2dx \\ &\geq \nu\partial_t\|\zeta\nabla\cdot\bar{u}\|_{L^2}^2 - \frac{1}{2}\|\zeta\nabla\cdot\bar{u}\|_{L^2}^2 - \frac{C_2\nu^2}{\langle t\rangle^2}\mathcal{E}_{1,0}[u](t). \end{aligned}$$

We have applied above Young's inequality and have used that $\partial_t\zeta^2 \leq C_1\zeta\langle t\rangle^{-1}$, for some positive constant $C_1$. Substituting in (2.7.2) we get:

$$\begin{aligned} t^2[\nu\partial_t\|\zeta\nabla\cdot\bar{u}\|_{L^2}^2 + \frac{1}{2}\|\zeta\nabla\cdot\bar{u}\|_{L^2}^2 \\ + \nu^2\|\zeta\Delta u^0\|_{L^2}^2 + \|\zeta\nabla u^0\|_{L^2}^2] \\ \lesssim \|\zeta r\partial_r u\|_{L^2}^2 + \mathcal{E}_{1,1}[u](t) + t^2\|\zeta G\|_{L^2}^2. \end{aligned}$$

Choosing $0 \le \theta \le 1$, multiplying by $\langle t\rangle^{\theta-2}$, and integrating in time, we get:

$$\int_0^T t^2\langle t\rangle^{\theta-2}\left[\frac{1}{2}\|\zeta\nabla\cdot\bar{u}\|_{L^2}^2+\nu^2\|\zeta\Delta u^0\|_{L^2}^2+\|\zeta\nabla u^0\|_{L^2}^2\right]dt$$
$$\lesssim \int_0^T \langle t\rangle^{\theta-2}[\|\zeta r\partial_r u\|_{L^2}^2+\mathcal{E}_{1,1}[u](t)+t^2\|\zeta G\|_{L^2}^2]dt$$
$$-\int_0^T t^2\langle t\rangle^{\theta-2}\nu\partial_t\|\zeta\nabla\cdot\bar{u}\|_{L^2}^2 dt.$$

We next treat the time derivative term. We start with:

$$\begin{aligned}\partial_t(\nu t^2\langle t\rangle^{\theta-2}) &= 2\nu t\langle t\rangle^{\theta-2}+\nu t^2(\theta-2)\langle t\rangle^{\theta-3}t\langle t\rangle^{-1}\\ &\le 2\nu t\langle t\rangle^{\theta-2}+\nu\theta t^3\langle t\rangle^{\theta-4}\le 4\nu t\langle t\rangle^{\theta-2}\\ &\le \frac{1}{4}t^2\langle t\rangle^{\theta-2}+C\nu^2\langle t\rangle^{\theta-2},\end{aligned} \tag{2.7.3}$$

and then, by integration by parts, we obtain:

$$-\int_0^T t^2\langle t\rangle^{\theta-2}\nu\partial_t\|\zeta\nabla\cdot\bar{u}\|_{L^2}^2 dt$$
$$\le \int_0^T\left(\frac{1}{4}t^2\langle t\rangle^{\theta-2}+C\nu^2\langle t\rangle^{\theta-2}\right)\|\zeta\nabla\cdot\bar{u}\|_{L^2}^2 dt$$
$$\le \int_0^T \frac{1}{4}t^2\langle t\rangle^{\theta-2}\|\zeta\nabla\cdot\bar{u}\|_{L^2}^2 dt + C\int_0^T\langle t\rangle^{\theta-2}\mathcal{E}_{1,0}[u](t)dt.$$

Substituting into 2.7, we obtain:

$$\int_0^T t^2\langle t\rangle^{\theta-2}\left[\frac{1}{4}\|\zeta\nabla\cdot\bar{u}\|_{L^2}^2+\nu^2\|\zeta\Delta u^0\|_{L^2}^2+\|\zeta\nabla u^0\|_{L^2}^2\right]dt$$
$$\lesssim \int_0^T \langle t\rangle^{\theta-2}[\|\zeta r\partial_r u\|_{L^2}^2+\mathcal{E}_{1,1}[u](t)+t^2\|\zeta G\|_{L^2}^2]dt.$$

Thanks to Lemma 2.7.2, we can control the full gradient on the left:

$$\int_0^T t^2\langle t\rangle^{\theta-2}\left[\|\zeta\nabla u\|_{L^2}^2+\nu^2\|\zeta\Delta u^0\|_{L^2}^2\right]dt$$

$$\lesssim \int_0^T \langle t\rangle^{\theta-2}[\|\zeta r\partial_r u\|_{L^2}^2 + \mathcal{E}_{1,1}[u](t) + t^2\|\zeta G\|_{L^2}^2]dt.$$

Using that $t^2 = \langle t\rangle^2 - 1$, we can write:

$$\int_0^T \langle t\rangle^\theta \left[\|\zeta\nabla u\|_{L^2}^2 + \nu^2\|\zeta\Delta u^0\|_{L^2}^2\right] dt$$
$$\lesssim \int_0^T \langle t\rangle^{\theta-2}[\|\zeta r\partial_r u\|_{L^2}^2 + \nu^2\|\zeta\Delta u^0\|_{L^2}^2 + \mathcal{E}_{1,1}[u](t) + t^2\|\zeta G\|_{L^2}^2]dt.$$

Since $r \leq \sigma\langle t\rangle$ on the support of $\zeta$, we have that

$$\int_0^T \langle t\rangle^{\theta-2}\|\zeta r\partial_r u\|_{L^2}^2 \lesssim \int_0^T \sigma^2\langle t\rangle^\theta\|\zeta\partial_r u\|_{L^2}^2$$

and for sufficiently small $\sigma$ the above term can be absorbed on the left. This key steps yields:

$$\int_0^T \langle t\rangle^\theta \left[\|\zeta\nabla u\|_{L^2}^2 + \nu^2\|\zeta\Delta u^0\|_{L^2}^2\right] dt$$
$$\lesssim \int_0^T \langle t\rangle^{\theta-2}[\nu^2\|\zeta\Delta u^0\|_{L^2}^2 + \mathcal{E}_{1,1}[u](t) + t^2\|\zeta G\|_{L^2}^2]dt. \quad (2.7.4)$$

Finally, the first term on the right can be estimated as follows:

$$\int_0^T \langle t\rangle^{\theta-2}\nu^2\|\zeta\Delta u^0\|_{L^2}^2 dt = \int_0^T \langle t\rangle^{\theta-2}\nu^2\frac{d}{dt}\int_0^t \|\zeta\Delta u^0\|_{L^2}^2 ds dt$$
$$= \langle T\rangle^{\theta-2}\nu^2\int_0^T \|\zeta\Delta u^0\|_{L^2}^2 dt + (2-\theta)\int_0^T t\langle t\rangle^{\theta-4}\nu^2\int_0^t \|\zeta\Delta u^0\|_{L^2}^2 ds dt$$
$$\lesssim \nu\langle T\rangle^{\theta-2}\mathcal{E}_{1,0}[u](T) + \nu\int_0^T \langle t\rangle^{\theta-2}\mathcal{E}_{1,0}[u](t)dt.$$

Substituting into (2.7.4) and recalling that $\nu \leq 1$, we get the desired estimate:

$$\int_0^T \langle t\rangle^\theta \left[\|\zeta\nabla u\|_{L^2}^2 + \nu^2\|\zeta\Delta u^0\|_{L^2}^2\right] dt$$

$$\lesssim \nu \langle T\rangle^{\theta-2}\mathcal{E}_{1,0}[u](T) + \int_0^T \langle t\rangle^{\theta-2}\mathcal{E}_{1,1}[u](t)dt$$

$$+ \int_0^T \langle t\rangle^{\theta}\|\zeta G\|_{L^2}^2]dt.$$

In the proof of Lemma 2.7.1, we used the following estimate:

**Lemma 2.7.2.** *If $w \in H^1(\mathbb{R}^3, \mathbb{R}^3)$ and $\nabla \wedge w = 0$, then*

$$\frac{1}{2}\|\zeta\nabla w\|_{L^2}^2 - \|\zeta\nabla\cdot w\|_{L^2}^2 \lesssim \langle t\rangle^{-2}\|w\|_{L^2}^2.$$

*Proof.* The constraint $\nabla \wedge w = 0$ implies that:

$$\begin{aligned} -(\nabla\cdot w)^2 &= -\partial_i w^i \partial_j w^j = -\partial_j(\partial_i w^i w^j) + \partial_i\partial_j w^i w^j \\ &= -\partial_j(\partial_i w^i w^j) + \partial_i\partial_i w^j w^j \\ &= -\partial_j(\partial_i w^i w^j) + \partial_i(\partial_i w^j w^j) - \partial_i w^j \partial_i w^j \\ &= -\partial_j(\partial_i w^i w^j) + \partial_i(\partial_i w^j w^j) - |\nabla w|^2. \end{aligned}$$

Therefore, we have:

$$|\nabla w|^2 - (\nabla\cdot w)^2 = \partial_i(\partial_i w^j w^j) - \partial_j(\partial_i w^i w^j).$$

After multiplying by $\zeta^2$ and integrating, we have:

$$\begin{aligned} &\|\zeta\nabla w\|_{L^2}^2 - \|\zeta\nabla\cdot w\|_{L^2}^2 \\ &\qquad = \int_{\mathbb{R}^3} \zeta^2[\partial_i(\partial_i w^j w^j) - \partial_j(\partial_i w^i w^j)]dx \\ &\qquad \lesssim \int_{\mathbb{R}^3} \zeta\langle t\rangle^{-1}|\nabla w||w|dx \\ &\qquad \le \frac{1}{2}\|\zeta\nabla w\|_{L^2}^2 + C\langle t\rangle^{-2}\|w\|_{L^2}^2, \end{aligned}$$

where we have used Young's inequality and the fact that $|\nabla\zeta^2| \lesssim \zeta\langle t\rangle^{-1}$. □

We now establish a higher order version of Lemma 2.7.1.

**Proposition 2.7.3.** *Assume that $\sigma$ in (2.2.4a) is sufficiently small and that $\nu \leq 1$. Fix $0 \leq q < p$. Suppose that*

$$S^k G \in L^2([0,T]; X^{p-k-1,0}), \quad k = 0, \dots, q$$

*for some $0 < T < \infty$. If $u$ is a solution of (2.7.1a), (2.7.1b), (2.7.1c) such that*

$$\sup_{0 \leq t \leq T} \mathcal{E}_{p,q+1}[u](t) < \infty,$$

*then for any $0 \leq \theta \leq 1$,*

$$\begin{aligned}\int_0^T \langle t\rangle^\theta \left[\mathcal{Y}^{int}_{p,q}[u](t) + \nu^2 \mathcal{Z}^{int}_{p,q}[u](t)\right] dt \\ \lesssim \nu \langle T\rangle^{\theta-2}\mathcal{E}_{p,q}[u](T) + \int_0^T \langle t\rangle^{\theta-2}\mathcal{E}_{p,q+1}[u](t)dt \\ + \sum_{\substack{|a|+k\leq p-1\\ k\leq q}} \int_0^T \langle t\rangle^\theta \|\zeta S^k\Gamma^a G(t)\|^2_{L^2} dt.\end{aligned}$$

*Proof.* We prove the result by induction on $q$. Recall the commutation properties (2.4.1a) and (2.4.1a):

$$LS^k\Gamma^a u = (S+1)^k\Gamma^a Lu - \nu B\Delta[S^k - (S-1)^k]\Gamma^a u,$$

$$\nabla \wedge S^k\Gamma^a \bar{u} = (S+1)^k \Gamma^a \nabla \wedge \bar{u}.$$

For the case $q = 0$, first fix $|a| \leq p-1$. Next, from the commutation properties, we notice that $\Gamma^a u$ solves (2.7.1a) and (2.7.1b) with $\Gamma^a G$ on the right. Applying Lemma 2.7.1 with $\Gamma^a u$, we obtain:

$$\int_0^T \langle t\rangle^\theta \left[\|\zeta\nabla\Gamma^a u(t)\|_{L^2}^2 + \nu^2\|\zeta\Delta\Gamma^a u^0(t)\|_{L^2}^2\right] dt$$

$$\lesssim \nu\langle T\rangle^{\theta-2}\mathcal{E}_{1,0}[\Gamma^a u](T) + \int_0^T \langle t\rangle^{\theta-2}\mathcal{E}_{1,1}[\Gamma^a u](t)dt$$

$$+ \int_0^T \langle t\rangle^\theta \|\zeta\Gamma^a G(t)\|_{L^2}^2 dt,$$

which after summation over $|a| \le p-1$ gives the result for $q = 0$:

$$\int_0^T \langle t\rangle^\theta \left[\mathcal{Y}_{p,0}^{\text{int}}[u](t) + \nu^2\mathcal{Z}_{p,0}^{\text{int}}[u](t)\right] dt$$

$$\lesssim \nu\langle T\rangle^{\theta-2}\mathcal{E}_{p,0}[u](T) + \int_0^T \langle t\rangle^{\theta-2}\mathcal{E}_{p,1}[u](t)dt$$

$$+ \sum_{|a|\le p-1}\int_0^T \langle t\rangle^\theta \|\zeta\Gamma^a G(t)\|_{L^2}^2 dt.$$

Next, we take $1 \le r < p$ and assume the result holds for $q = r-1$. We choose $a$ and $k$ such that $|a| + k \le p-1$ and $k \le r$. By the commutation properties, we see that $S^k\Gamma^a u$ solves (2.7.1a) and (2.7.1b) with

$$(S+1)^k\Gamma^a G - \nu B\Delta[S^k - (S-1)^k]\Gamma^a u$$

on the right. Therefore, application of Lemma 2.7.1 gives:

$$\int_0^T \langle t\rangle^\theta \left[\|\zeta\nabla S^k\Gamma^a u(t)\|_{L^2}^2 + \nu^2\|\zeta\Delta S^k\Gamma^a u^0(t)\|_{L^2}^2\right] dt$$

$$\lesssim \nu\langle T\rangle^{\theta-2}\mathcal{E}_{1,0}[S^k\Gamma^a u](T) + \int_0^T \langle t\rangle^{\theta-2}\mathcal{E}_{1,1}[S^k\Gamma^a u](t)dt$$

$$+ \int_0^T \langle t\rangle^\theta\nu^2\|\zeta\Delta[S^k - (S-1)^k](\Gamma^a u)^0\|_{L^2}^2 dt$$

$$+ \int_0^T \langle t\rangle^\theta \|\zeta(S+1)^k\Gamma^a G(t)\|_{L^2}^2 dt$$

$$\lesssim \nu\langle T\rangle^{\theta-2}\mathcal{E}_{1,0}[S^k\Gamma^a u](T) + \int_0^T \langle t\rangle^{\theta-2}\mathcal{E}_{1,1}[S^k\Gamma^a u](t)dt$$

$$+\int_0^T \langle t\rangle^\theta[\nu^2\mathcal{Z}^{\text{int}}_{p,r-1}[u](t)+\|\zeta(S+1)^k\Gamma^a G(t)\|^2_{L^2}]dt.$$

Summing over $|a|+k\le p-1$ and $k\le r$, we get:

$$\begin{aligned}\int_0^T [\mathcal{Y}^{\text{int}}_{p,r}[u](t)+\nu^2\mathcal{Z}^{\text{int}}_{p,r}[u](t)]dt \\ \lesssim \nu\langle T\rangle^{\theta-2}\mathcal{E}_{p,r}[u](T)+\int_0^T\langle t\rangle^{\theta-2}\mathcal{E}_{p,r+1}[u](t)dt \\ +\sum_{\substack{|a|+k\le p-1\\ k\le r}}\int_0^T\langle t\rangle^\theta[\nu^2\mathcal{Z}^{\text{int}}_{p,r-1}[u](t)+\|\zeta S^k\Gamma^a G(t)\|^2_{L^2}]dt.\end{aligned}$$

□

We next consider the exterior region:

**Lemma 2.7.4.** *Let $G\in C([0,T];L^2(\mathbb{R}^3))$, for some $0<T<\infty$. If $u=(u^0,\bar u)$ is a solution of* (2.7.1a), (2.7.1b), (2.7.1c) *such that*

$$\sup_{0\le t\le T}\mathcal{E}_{1,1}[u](t)<\infty,$$

*then for all $0\le t\le T$,*

$$\|\eta(r\partial_r u^0+t\nabla\cdot\bar u)\|^2_{L^2}+\|\eta(r\partial_r\bar u+t\nabla u^0)\|^2_{L^2}+(\nu t)^2\|\eta\Delta u^0\|^2_{L^2} \lesssim \mathcal{E}_{1,1}[u](t)+t^2\|\eta G\|^2_{L^2}.$$

*Proof.* We start again by multiplying (2.7.1a) and (2.7.1b) by $t$. Using that $S=t\partial_t+r\partial_r$, we get:

$$r\partial_r u^0+t\nabla\cdot\bar u+t\nu\Delta u^0=Su^0-tG^0$$

$$r\partial_r\bar u+t\nabla u^0=S\bar u.$$

Next, we multiply the above equations by $\eta$ and take the $L^2$-inner product:

$$\|\eta(r\partial_r u^0 + t\nabla\cdot\bar{u})\|_{L^2}^2 + \|\eta(r\partial_r\bar{u} + t\nabla u^0)\|_{L^2}^2 + (\nu t)^2\|\eta\Delta u^0\|_{L^2}^2$$
$$+ 2\langle\eta(r\partial_r u^0 + t\nabla\cdot\bar{u}), \eta t\nu\Delta u^0\rangle_{L^2}$$
$$\lesssim \|\eta S u\|_{L^2}^2 + t^2\|\eta G\|_{L^2}^2.$$

We will estimate the cross term on the left which we denote as:

$$I \;\equiv\; 2\langle\eta(r\partial_r u^0 + t\nabla\cdot\bar{u}), \eta t\nu\Delta u^0\rangle_{L^2} \;=\; 2\nu t\int \eta^2(r\partial_r u^0 + t\nabla\cdot\bar{u})\Delta u^0 dx. \quad (2.7.5)$$

To establish the result it suffices to show that:

$$|I| \le \mu(\nu t)^2\|\eta\Delta u^0\|_{L^2}^2 + C\mathcal{E}_{1,1}[u](t) \quad (2.7.6)$$

for some small enough $\mu$, say $\mu \le 1/2$ (so that the Laplacian term can be absorbed). Using the definition $S = t\partial_t + r\partial_r$, the gradient decomposition $\nabla = \omega\partial_r - \frac{\omega}{r}\wedge\Omega$, and the constraint $\partial_t\bar{u} = \nabla u^0$, we obtain:

$$\begin{aligned}
\nabla\cdot\bar{u} \;&=\; \omega\cdot\partial_r\bar{u} - \left(\frac{\omega}{r}\wedge\Omega\right)\cdot\bar{u} \\
&= \omega\cdot\left(\frac{1}{r}S\bar{u} - \frac{t}{r}\partial_t\bar{u}\right) - \left(\frac{\omega}{r}\wedge\Omega\right)\cdot\bar{u} \\
&= -\omega\cdot\frac{t}{r}\nabla u^0 + \mathcal{O}\left(\frac{1}{r}(|\Omega u| + |Su|)\right) \\
&= -\frac{t}{r}\partial_r u^0 + \mathcal{O}\left(\frac{1}{r}(|\Omega u| + |Su|)\right).
\end{aligned}$$

So we can write the cross term $I$ from (2.7.5) as

$$2\nu t\int \eta^2\left[r\partial_r u^0 - \frac{t^2}{r}\partial_r u^0 + t\mathcal{O}\left(\frac{1}{r}(|\Omega u| + |Su|)\right)\right]\Delta u^0 dx$$

$$= 2\nu t \int \eta^2 \left[ \left(1 - \frac{t^2}{r^2}\right) r\partial_r u^0 \Delta u^0 + \mathcal{O}\left(\frac{t}{r}(|\Omega u| + |Su|)\right) \Delta u^0 \right] dx. \quad (2.7.7)$$

We have the following identity:

$$\begin{aligned} r\partial_r u^0 \Delta u^0 &= r\partial_r u^0 \nabla \cdot \nabla u^0 \qquad (2.7.8) \\ &= \nabla \cdot (r\partial_r u^0 \nabla u^0) - \nabla(r\partial_r u^0) \cdot \nabla u^0. \end{aligned}$$

The last term on the right can be written as:

$$\begin{aligned} &- \nabla(r\partial_r u^0) \cdot \nabla u^0 = -\partial_i(x_k \partial_k u^0)\partial_i u^0 \\ &= (-x_k \partial_i \partial_k u^0 - \delta_{ik}\partial_k u^0)\partial_i u^0 = -x_k \partial_i \partial_k u^0 \partial_i u^0 - |\nabla u^0|^2 \\ &= -\frac{1}{2} x_k \partial_k |\nabla u^0|^2 - |\nabla u^0|^2 = -\partial_k \left(\frac{1}{2} x_k |\nabla u^0|^2\right) + \frac{3}{2}|\nabla u^0|^2 - |\nabla u^0|^2 \\ &= -\nabla \cdot \left(\frac{1}{2}|\nabla u^0|^2 x\right) + \frac{1}{2}|\nabla u^0|^2. \end{aligned}$$

Substituting into (2.7.8), we get:

$$r\partial_r u^0 \Delta u^0 = \nabla \cdot \left(r\partial_r u^0 \nabla u^0 - \frac{1}{2} x |\nabla u^0|^2\right) + \frac{1}{2}|\nabla u^0|^2.$$

Insertion into (2.7.7) and integration by parts give:

$$\begin{aligned} I \;=\; &-2\nu t \int \nabla \left[\eta^2 \left(1 - \frac{t^2}{r^2}\right)\right] \cdot \left[r\partial_r u^0 \nabla u^0 - \frac{1}{2} x |\nabla u^0|^2\right] dx \\ &+ 2\nu t \int \eta^2 \left[\left(\frac{t^2}{r^2} - 1\right) \frac{1}{2}|\nabla u^0|^2 + \mathcal{O}\left(\frac{t}{r}(|\Omega u| + |Su|)\right) \Delta u^0\right] dx. \end{aligned}$$

We have that $\nabla \eta^2 \leq \mathcal{O}(\eta |\psi'| \langle t \rangle^{-1})$ and $\nabla \left(1 - \frac{t^2}{r^2}\right) \leq \mathcal{O}(\frac{t^2}{r^3})$. Also by the fact that $r \gtrsim \langle t \rangle$ on the support of $\eta$ and $\langle t \rangle \lesssim r \lesssim \langle t \rangle$ on the support of $\psi'$, we have:

$$r \left| \nabla \left[\eta^2 \left(1 - \frac{t^2}{r^2}\right)\right] \right| \lesssim \eta$$

and hence

$$|I| \lesssim \nu t \int \eta |\nabla u^0|^2 dx + \nu t \int \eta(|\Omega u| + |Su|)|\Delta u^0| dx.$$

Using integration by parts and the fact that $\nabla\eta \lesssim \langle t\rangle^{-1}$, we get:

$$\begin{aligned}\nu t \int \eta |\nabla u^0|^2 dx &= -\nu t \int (\eta u^0 \Delta u^0 + u^0 \nabla\eta \cdot \nabla u^0)\, dx \\ &\lesssim \nu t \int \eta |u^0 \Delta u^0| dx + \nu \mathcal{E}_{1,1}[u](t),\end{aligned}$$

so

$$|I| \lesssim \nu t \int \eta(|u^0| + |\Omega u| + |Su|)|\Delta u^0| dx + C\mathcal{E}_{1,1}[u](t).$$

□

The estimate (2.7.6) follows from Young's inequality.

**Proposition 2.7.5.** *Fix $0 \le q \le p$. Suppose that*

$$S^k G \in C([0,T], X^{p-k-1,0}), \quad k = 0, \dots, q-1,$$

*for some $0 < T < \infty$. If $u = (u^0, \bar{u})$ is a solution of* (2.7.1a), (2.7.1b), (2.7.1c) *such that*

$$\sup_{0 \le t \le T} \mathcal{E}_{p,q+1}[u](t) < \infty,$$

*then for all $0 \le t \le T$,*

$$\mathcal{Y}^{ext}_{p,q}[u](t) + \nu^2 \mathcal{Z}^{ext}_{p,q}[u](t) \lesssim \mathcal{E}_{p,q+1}[u](t) + \sum_{\substack{|a|+k \le p-1 \\ k \le q}} t^2 \|\eta S^k \Gamma^a G\|^2_{L^2}.$$

*Proof.* From from the projection definition (2.2.5a) and the gradient decomposition (2.2.1), for each $j$, we have:

$$|\mathbb{P}\partial_j u|_{\mathbb{R}^4}^2 = \frac{1}{4}(\partial_j u^0 - \omega\cdot\partial_j\bar{u})^2 \leq \frac{1}{4}(\partial_r u^0 - \omega\cdot\partial_r\bar{u})^2 + \mathcal{O}(\frac{1}{r^2}|\Omega u|^2),$$

and by (2.7.1c),

$$\begin{aligned}
|\mathbb{Q}\partial_j u|_{\mathbb{R}^4}^2 &= |(I-\mathbb{P})\partial_j u|_{\mathbb{R}^4}^2 \\
&= \frac{1}{4}(\partial_j u^0 + \omega\cdot\partial_j\bar{u})^2 + |\omega\wedge\partial_j\bar{u}|_{\mathbb{R}^3}^2 \\
&= \frac{1}{4}(\partial_j u^0 + \omega\cdot\partial_j\bar{u})^2 + |\omega\wedge\nabla u^j|_{\mathbb{R}^3} \\
&\leq \frac{1}{4}(\partial_r u^0 + \omega\cdot\partial_r\bar{u})^2 + \mathcal{O}(\frac{1}{r^2}|\Omega u|^2).
\end{aligned}$$

Therefore, since $r \gtrsim \langle t+r\rangle \geq \langle t-r\rangle$ on the support of $\eta$, we obtain

$$\begin{aligned}
\mathcal{Y}_{1,0}^{\text{ext}}[u](t) &= \sum_{j=1}^{3}[\|\eta\langle t-r\rangle\mathbb{P}\partial_j u\|_{L^2}^2 + \|\eta\langle t+r\rangle\mathbb{Q}\partial_j u\|_{L^2}^2] \\
&\leq \frac{1}{4}\|\eta\langle t-r\rangle(\partial_r u^0 - \omega\cdot\partial_r\bar{u})\|_{L^2}^2 \\
&\qquad + \frac{1}{4}\|\eta\langle t+r\rangle(\partial_r u^0 + \omega\cdot\partial_r\bar{u})\|_{L^2}^2 + C\|\Omega u\|_{L^2}^2 \\
&\leq \frac{1}{4}\|\eta(t-r)(\partial_r u^0 - \omega\cdot\partial_r\bar{u})\|_{L^2}^2 \\
&\qquad + \frac{1}{4}\|\eta(t+r)(\partial_r u^0 + \omega\cdot\partial_r\bar{u})\|_{L^2}^2 + C[\|\nabla u\|_{L^2}^2 + \|\Omega u\|_{L^2}^2].
\end{aligned}$$

An algebraic manipulation gives:

$$\begin{aligned}
&\frac{1}{4}(t-r)^2(\partial_r u^0 - \omega\cdot\partial_r\bar{u})^2 + \frac{1}{4}(t+r)^2(\partial_r u^0 + \omega\cdot\partial_r\bar{u})^2 \\
&\qquad\qquad = \frac{1}{2}(r\partial_r u^0 + t\omega\cdot\partial_r\bar{u})^2 + \frac{1}{2}(r\omega\cdot\partial_r\bar{u} + t\partial_r u^0)^2.
\end{aligned}$$

Thus, we obtain:

$$\mathcal{Y}^{\text{ext}}_{1,0}[u](t) \ \le \ \frac{1}{2}\|\eta(r\partial_r u^0 + t\omega\cdot\partial_r\bar{u})\|^2_{L^2} + \frac{1}{2}\|\eta(r\omega\cdot\partial_r\bar{u} + t\partial_r u^0)\|^2_{L^2} + C[\|\nabla u\|^2_{L^2} + \|\Omega u\|^2_{L^2}].$$

By the gradient decomposition (2.2.1), we have:

$$\eta|r\partial_r u^0 + t\omega\cdot\partial_r\bar{u}| \le \eta|r\partial_r u^0 + t\nabla\cdot\bar{u}| + \mathcal{O}(|\Omega\bar{u}|)$$

and

$$\eta|r\omega\cdot\partial_r\bar{u} + t\partial_r u^0| = \eta|r\omega\cdot\partial_r\bar{u} + t\partial_r u^0\omega\cdot\omega| = \eta|\omega\cdot(r\partial_r\bar{u} + t\partial_r u^0\omega)| \le \eta|r\partial_r\bar{u} + t\nabla u^0|_{\mathbb{R}^3} + \mathcal{O}(|\Omega u^0|),$$

which gives us:

$$\mathcal{Y}^{\text{ext}}_{1,0}[u](t) \ \le \ \frac{1}{2}\|\eta(r\partial_r u^0 + t\nabla\cdot\bar{u})\|^2_{L^2} + \frac{1}{2}\|\eta(r\partial_r\bar{u} + t\nabla u^0)\|^2_{L^2} + C[\|\nabla u\|^2_{L^2} + \|\Omega u\|^2_{L^2}].$$

Application of Lemma 2.7.4 gives:

$$\mathcal{Y}^{\text{ext}}_{1,0}[u](t) + \nu^2\mathcal{Z}^{\text{ext}}_{1,0}[u](t) \lesssim \mathcal{E}_{1,1}[u](t) + t^2\|\eta G\|^2_{L^2}.$$

Now take any multi-index $a$ with $|a| \le p-1$. By the commutation property (2.4.1a), we can apply the preceding inequality to $\Gamma^a u$ to get:

$$\mathcal{Y}^{\text{ext}}_{1,0}[\Gamma^a u](t) + \nu^2\mathcal{Z}^{\text{ext}}_{1,0}[\Gamma^a u](t) \lesssim \mathcal{E}_{1,1}[\Gamma^a u](t) + t^2\|\eta\Gamma^a G\|^2_{L^2}.$$

Summation over $|a| \leq p-1$ yields

$$\mathcal{Y}^{\text{ext}}_{p,0}[u](t) + \nu^2 \mathcal{Z}^{\text{ext}}_{p,0}[u](t) \lesssim \mathcal{E}_{p,1}[u](t) + \sum_{|a|\leq p-1} t^2 \|\eta \Gamma^a G\|^2_{L^2},$$

which proves the result in the case $q = 0$.

The result for $0 < q < p$ follows from (2.4.1b) and induction, as in the proof of Propostion 2.7.3. □

## 2.8 Decay Estimates

In this section we establish the dispersive estimates for the nonlinear equation using a bootstrap argument and an application of Propositions 2.7.3 and 2.7.5.

**Theorem 2.8.1.** *Choose* $(p, q)$ *so that* $p^* = \left[\frac{p+5}{2}\right] < q \leq p$. *Suppose that* $u \in C([0,T), X^{p,q})$ *is a solution of* (2.1.1a), (2.1.1b) *with*

$$\sup_{0\leq t\leq T} \mathcal{E}_{p,q}[u](t) < \infty,$$

*and*

$$\sup_{0\leq t\leq T} \mathcal{E}_{p^*,p^*}[u](t) \leq \varepsilon^2 \ll 1. \tag{2.8.1}$$

*Then*

$$\int_0^T \langle t\rangle^\theta \left[\mathcal{Y}^{int}_{p^*,p^*-1}[u](t) + \nu^2 \mathcal{Z}^{int}_{p^*,p^*-1}[u](t)\right] dt$$

$$\lesssim \begin{cases} \sup_{0\le t\le T} \langle t\rangle^{-\gamma}\mathcal{E}_{p^*,p^*}[u](t), & 0<\theta+\gamma<1 \\ \\ \log(e+T) \sup_{0\le t\le T} \mathcal{E}_{p^*,p^*}[u](t), & \theta=1 \end{cases}, \quad (2.8.2a)$$

*and*

$$\int_0^T \langle t\rangle^{\theta} \left[\mathcal{Y}^{int}_{p^*+1,p^*}[u](t) + \nu^2 \mathcal{Z}^{int}_{p^*+1,p^*}[u](t)\right] dt$$

$$\lesssim \sup_{0\le t\le T} \langle t\rangle^{-\gamma}\mathcal{E}_{p,q}[u](t), \quad 0<\theta+\gamma<1.$$

*Proof.* Consider an application of Proposition 2.7.3 with $G = N(u,\nabla u)$ and a fixed pair $(\bar{p},\bar{q})$ with $\bar{q} = \bar{p}-1$, $2\le \bar{p}\le p$:

$$\int_0^T \langle t\rangle^{\theta} l[\mathcal{Y}^{\text{int}}_{\bar{p},\bar{q}}[u](t) + \nu^2 \mathcal{Z}^{\text{int}}_{\bar{p},\bar{q}}[u](t)]dt$$

$$\lesssim \nu\langle T\rangle^{\theta-2}\mathcal{E}_{\bar{p},\bar{q}}[u](T) + \int_0^T \langle t\rangle^{\theta-2}\mathcal{E}_{\bar{p},\bar{q}+1}[u](t)dt$$

$$+ \sum_{\substack{|a|+k\le\bar{p}-1 \\ k\le\bar{q}}} \int_0^T \langle t\rangle^{\theta}\|\zeta S^k\Gamma^a N(u,\nabla u)(t)\|^2_{L^2}dt. \quad (2.8.3)$$

Note the range of the indices: $|a|+k\le\bar{p}-1$ and $k\le\bar{q}=\bar{p}-1$. By Lemmas 2.4.2 and 2.4.1, we have that $\|\zeta S^k\Gamma^a N(u,\nabla u)\|^2_{L^2}$ is bounded by a sum of terms of the form:

$$\|\zeta S^{k_1}\Gamma^{a_1}u\ S^{k_2}\Gamma^{a_2+1}u\|^2_{L^2},$$

with $|a_1|+|a_2|\le|a|$ and $k_1+k_2\le k$. Therefore, we have:

$$k_1+k_2+|a_1|+|a_2|\le k+|a|\le\bar{p}-1 \quad \text{and} \quad k_1+k_2\le\bar{p}-1.$$

Application of Lemma 2.6.1 shows that

$$\|\zeta S^{k_1}\Gamma^{a_1}u\, S^{k_2}\Gamma^{a_2+1}u\|_{L^2}^2 \lesssim \left[\mathcal{Y}^{\rm int}_{\bar p',\bar q'}[u](t) + \langle t\rangle^{-2}\mathcal{E}_{\bar p'-1,\bar q'}[u](t)\right] \mathcal{E}_{\bar p,\bar q}[u](t)$$

with $\bar p' = [\frac{(\bar p-1)+5}{2}] = [\frac{\bar p}{2}] + 2$ and $\bar q' = [\frac{\bar p-1}{2}]$.

Therefore, from (2.8.3) we have:

$$\begin{aligned}\int_0^T &\langle t\rangle^\theta l[\mathcal{Y}^{\rm int}_{\bar p,\bar q}[u](t) + \nu^2\mathcal{Z}^{\rm int}_{\bar p,\bar q}[u](t)]dt \\ &\lesssim \nu\langle T\rangle^{\theta-2}\mathcal{E}_{\bar p,\bar q}[u](T) + \int_0^T \langle t\rangle^{\theta-2}\mathcal{E}_{\bar p,\bar q+1}[u](t)dt \\ &\quad + \int_0^T \langle t\rangle^\theta \left[\mathcal{Y}^{\rm int}_{\bar p',\bar q'}[u](t) + \langle t\rangle^{-2}\mathcal{E}_{\bar p'-1,\bar q'}[u](t)\right] \mathcal{E}_{\bar p,\bar q}[u](t)dt, \quad (2.8.4)\end{aligned}$$

for any $0 < \theta \le 1$. We are going to apply this for two pairs $(\bar p,\bar q)$.

First, let $(\bar p,\bar q) = (p^*, p^*-1)$. Since $\bar p = p^* \ge 5$, we get

$$\bar p' = \left[\frac{p^*}{2}\right] + 2 \le p^*, \quad \bar q' = \left[\frac{p^*-1}{2}\right] \le p^*-1.$$

In this case, (2.8.4) and (2.8.1) yield:

$$\begin{aligned}\int_0^T &\langle t\rangle^\theta l[\mathcal{Y}^{\rm int}_{p^*,p^*-1}[u](t) + \nu^2\mathcal{Z}^{\rm int}_{p^*,p^*-1}[u](t)]dt \qquad (2.8.5)\\ &\lesssim \nu\langle T\rangle^{\theta-2}\mathcal{E}_{p^*,p^*-1}[u](T) + \int_0^T \langle t\rangle^{\theta-2}\mathcal{E}_{p^*,p^*}[u](t)dt \\ &\quad + \int_0^T \langle t\rangle^\theta \left[\mathcal{Y}^{\rm int}_{p^*,p^*-1}[u](t) + \langle t\rangle^{-2}\mathcal{E}_{p^*-1,p^*-1}[u](t)\right] \mathcal{E}_{p^*,p^*-1}[u](t)dt \\ &\lesssim \nu\langle T\rangle^{\theta-2}\mathcal{E}_{p^*,p^*}[u](T) + \int_0^T \langle t\rangle^{\theta-2}\mathcal{E}_{p^*,p^*}[u](t)dt \\ &\quad + \varepsilon^2\int_0^T \langle t\rangle^\theta \mathcal{Y}^{\rm int}_{p^*,p^*-1}[u](t)dt.\end{aligned}$$

Choose $\gamma \geq 0$ such that $0 < \theta + \gamma \leq 1$. We have that

$$\langle t\rangle^{\theta-2}\mathcal{E}_{p^*,p^*}[u](t) = \langle t\rangle^{\theta-2}\langle t\rangle^{\gamma}\langle t\rangle^{-\gamma}\mathcal{E}_{p^*,p^*}[u](t)$$
$$\leq \langle t\rangle^{\theta+\gamma-2} \sup_{0\leq t\leq T}[\langle t\rangle^{-\gamma}\mathcal{E}_{p^*,p^*}[u](t)]$$

for all $0 \leq t \leq T$. Thus, from (2.8.5)we have:

$$\int_0^T \langle t\rangle^{\theta} l[\mathcal{Y}^{\text{int}}_{p^*,p^*-1}[u](t) + \nu^2\mathcal{Z}^{\text{int}}_{p^*,p^*-1}[u](t)]dt$$
$$\lesssim \sup_{0\leq t\leq T}\langle t\rangle^{-\gamma}\mathcal{E}_{p^*,p^*}[u](t)\left[\langle T\rangle^{\theta+\gamma-2} + \int_0^T \langle t\rangle^{\theta+\gamma-2}dt\right]$$
$$+ \varepsilon^2 \int_0^T \langle t\rangle^{\theta}\mathcal{Y}^{\text{int}}_{p^*,p^*-1}[u](t)dt.$$

For $\varepsilon^2$ sufficiently small, the last term above can be absorbed on the left and then the inequalities (2.8.2a) follow immediately.

Next, we use the pair $(\bar{p},\bar{q}) = (p^*+1, p^*)$ in (2.8.4). Again since $p^* \geq 5$, we have

$$\bar{p}' = \left[\frac{p^*+1}{2}\right] + 2 \leq p^* \quad \text{and} \quad \bar{q}' = \left[\frac{p^*}{2}\right] \leq p^* - 1.$$

We obtain from (2.8.4):

$$\int_0^T \langle t\rangle^{\theta}\left[\mathcal{Y}^{\text{int}}_{p^*+1,p^*}[u](t) + \nu^2\mathcal{Z}^{\text{int}}_{p^*+1,p^*}[u](t)\right]dt$$
$$\lesssim \nu\langle T\rangle^{\theta-2}\mathcal{E}_{p^*+1,p^*}[u](T) + \int_0^T \langle t\rangle^{\theta-2}\mathcal{E}_{p^*+1,p^*+1}[u](t)dt$$
$$+ \int_0^T \langle t\rangle^{\theta}\left[\mathcal{Y}^{\text{int}}_{p^*,p^*-1}[u](t) + \langle t\rangle^{-2}\mathcal{E}_{p^*-1,p^*-1}[u](t)\right]\mathcal{E}_{p^*+1,p^*}[u](t)dt.$$

Choose $\gamma \geq 0$ such that $0 < \theta + \gamma < 1$. We have:

$$\langle t\rangle^{\theta-2}\mathcal{E}_{p,q}[u](t) \leq \langle t\rangle^{\theta+\gamma-2}\sup_{0\leq t\leq T}[\langle t\rangle^{-\gamma}\mathcal{E}_{p,q}[u](t)] \quad \text{and}$$

$$\langle t\rangle^{\theta}\mathcal{E}_{p,q}[u](t) \le \ \langle t\rangle^{\theta+\gamma}\sup_{0\le t\le T}[\langle t\rangle^{-\gamma}\mathcal{E}_{p,q}[u](t)]$$

for all $0 \le t \le T$. Since $p^* + 1 \le q \le p$, we have the estimate:

$$\begin{aligned}\int_0^T \langle t\rangle^{\theta}\left[\mathcal{Y}^{\rm int}_{p^*+1,p^*}[u](t) + \nu^2\mathcal{Z}^{\rm int}_{p^*+1,p^*}[u](t)\right]dt \\ \lesssim \sup_{0\le t\le T}\langle t\rangle^{-\gamma}\mathcal{E}_{p,q}[u](t)\Bigg[\langle T\rangle^{\theta+\gamma-2} + \int_0^T\langle t\rangle^{\theta+\gamma-2}dt \\ \int_0^T\langle t\rangle^{\theta+\gamma}\left[\mathcal{Y}^{\rm int}_{p^*,p^*-1}[u](t) + \langle t\rangle^{-2}\mathcal{E}_{p^*-1,p^*-1}[u](t)\right]dt\Bigg], \end{aligned}\quad (2.8.6)$$

where by (2.8.2a) and (2.8.1) the last integral is bounded by:

$$\begin{aligned}\int_0^T \langle t\rangle^{\theta+\gamma}\left[\mathcal{Y}^{\rm int}_{p^*,p^*-1}[u](t) + \langle t\rangle^{-2}\mathcal{E}_{p^*-1,p^*-1}[u](t)\right]dt \\ \lesssim \int_0^T\langle t\rangle^{\theta+\gamma}\mathcal{Y}^{\rm int}_{p^*,p^*-1}[u](t)dt + \varepsilon^2\int_0^T\langle t\rangle^{\theta+\gamma-2}dt \ \lesssim\ C\varepsilon^2.\end{aligned}$$

So from (2.8.6) we arrive at:

$$\int_0^T \langle t\rangle^{\theta}\left[\mathcal{Y}^{\rm int}_{p^*+1,p^*}[u](t) + \nu^2\mathcal{Z}^{\rm int}_{p^*+1,p^*}[u](t)\right]dt \ \lesssim\ \sup_{0\le t\le T}\langle t\rangle^{-\gamma}\mathcal{E}_{p,q}[u](t).$$

□

**Theorem 2.8.2.** *Fix $p \ge 11$. Assume that $p^* = \left[\frac{p+5}{2}\right] < q \le p$. Suppose that $u \in C([0,T), X^{p,q})$ is a solution of* (2.1.1a), (2.1.1b) *with*

$$\sup_{0\le t\le T}\mathcal{E}_{p,q}[u](t) < \infty$$

*and*

$$\sup_{0\le t\le T}\mathcal{E}_{p^*,p^*}[u](t) \le 1. \quad (2.8.7)$$

*Then*

$$\mathcal{Y}^{ext}_{p,q-1}[u](t) + \nu^2 \mathcal{Z}^{ext}_{p,q-1}[u](t) \lesssim \mathcal{E}_{p,q}[u](t), \quad 0 \le t \le T.$$

*Proof.* Consider an application of Proposition 2.7.5 with $G = N(u, \nabla u)$:

$$\mathcal{Y}^{\text{ext}}_{p,q}[u](t) + \nu^2 \mathcal{Z}^{\text{ext}}_{p,q}[u](t) \lesssim \mathcal{E}_{p,q+1}[u](t) + \sum_{\substack{|a|+k\le p-1\\ k\le q}} t^2 \|\eta S^k \Gamma^a N(u, \nabla u)\|^2_{L^2}.$$

Note the range of the indices: $|a| + k \le p - 1$, $k \le q$. By Lemmas 2.4.2 and 2.4.1, we have that $\|\eta S^k \Gamma^a N(u, \nabla u)\|^2_{L^2}$ is bounded by a sum of terms of the form

$$\|\eta S^{k_1} \Gamma^{a_1} u \; S^{k_2} \Gamma^{a_2+1} u\|^2_{L^2},$$

with $|a_1| + |a_2| \le |a|$ and $k_1 + k_2 \le k$. Therefore, we have:

$$k_1 + k_2 + |a_1| + |a_2| \le k + |a| \le p - 1 \quad \text{and} \quad k_1 + k_2 \le q.$$

Application of Lemma 2.6.2 shows that

$$\|\eta S^{k_1} \Gamma^{a_1} u \; S^{k_2} \Gamma^{a_2+1} u\|^2_{L^2} \;\lesssim\; \langle t\rangle^{-2} \, \mathcal{E}_{p',q'}[u](t) \, \mathcal{E}_{p,q}[u](t) \;\lesssim\; \langle t\rangle^{-2} \, \mathcal{E}_{p,q}[u](t),$$

where $p' = [\frac{(p-1)+5}{2}] = [\frac{p}{2}] + 2 \le p^*$ and $q' = [\frac{p-1}{2}] < p^*$ and therefore $\mathcal{E}_{p',q'}[u](t) \le \mathcal{E}_{p^*,p^*}[u](t) \le 1$, by (2.8.7). □

## 2.9 High Energy Estimates

**Proposition 2.9.1.** *Choose $(p,q)$ so that $5 \le p^* = \left[\frac{p+5}{2}\right] \le q \le p$. Suppose that $u \in C([0,T_0), X^{p,q})$ is a solution of* (2.1.1a), (2.1.1b) *with*

$$\sup_{0 \le t < T_0} \mathcal{E}_{p^*,p^*}[u](t) \le \varepsilon^2 \ll 1. \tag{2.9.1}$$

*Then there exists a constant $C_1 > 1$ such that*

$$\mathcal{E}_{p,q}[u](t) \le C_1 \mathcal{E}_{p,q}[u_0] \langle t \rangle^{C_1 \varepsilon}$$

$$\mathcal{E}_{p^*,p^*}[u](t) \le C_1 \mathcal{E}_{p^*,p^*}[u_0] \langle t \rangle^{C_1 \varepsilon},$$

*for $0 \le t < T_0$.*

*Proof.* Taking the $L^2$-dot product of

$$Lu \equiv \partial_t u - A^j \partial_j u - \nu B \Delta u$$

with $u(t)$, we obtain:

$$\langle \partial_t u(t), u(t) \rangle_{L^2} - \langle A^j \partial_j u(t), u(t) \rangle_{L^2} - \langle \nu B \Delta u(t), u(t) \rangle_{L^2}$$
$$= \langle Lu(t), u(t) \rangle_{L^2}. \tag{2.9.2}$$

Integration by parts and the symmetry of the coefficient matrices (2.1.1c) give:

$$\langle A^j \partial_j u(t), u(t) \rangle_{L^2} = -\langle u(t), A^j \partial_j u(t) \rangle_{L^2},$$

which implies that $\langle A^j \partial_j u(t), u(t) \rangle_{L^2} = 0$. Furthermore, again by integration by parts and the definition of the coefficient matrix $B$ in (2.1.1c), we get:

$$\langle \nu B \Delta u(t), u(t) \rangle_{L^2} = \int_{\mathbb{R}} \nu \Delta u^0(t) u^0(t) dx$$

$$= \int_{\mathbb{R}}^{3} -\nu|\nabla u^0(t)|^2 dx = -\nu\|\nabla u^0(t)\|_{L^2}^2.$$

Thus, (3.10.1) becomes:

$$\frac{1}{2}\partial_t\|u(t)\|_{L^2}^2 + \nu\|\nabla u^0(t)\|_{L^2}^2 = \langle Lu(t), u(t)\rangle_{L^2}.$$

Integration over time gives:

$$\frac{1}{2}\|u(T)\|_{L^2}^2 \ + \ \nu\int_0^T \|\nabla u^0(t)\|_{L^2}^2 dt \ = \ \frac{1}{2}\|u(0)\|_{L^2}^2 + \int_0^T \langle Lu(t), u(t)\rangle_{L^2} dt,$$

which implies that

$$\mathcal{E}_{0,0}[u](T) = \mathcal{E}_{0,0}[u_0] + \int_0^T \langle Lu(t), u(t)\rangle_{L^2} dt, \quad 0 \le T < T_0.$$

For $p \ge q \ge 0$, we apply the above estimate to higher order vector fields and together with the commutation property (2.4.1a) we get:

$$\mathcal{E}_{p,q}[u](T) = \mathcal{E}_{p,q}[u_0] + I + \sum_{\substack{|a|+k\le p \\ k\le q}} \int_0^T \langle (S+1)^k\Gamma^a Lu(t), S^k\Gamma^a u(t)\rangle_{L^2} dt, \qquad (2.9.3)$$

with

$$I = -\sum_{\substack{|a|+k\le p \\ k\le q}} \int_0^T \langle \nu B\Delta[S^k - (S-1)^k]\Gamma^a u(t), S^k\Gamma^a u(t)\rangle_{L^2} dt.$$

For $q > 0$, using the definition $B = e_0 \otimes e_0$ and integration by parts, we get the bound

$$\begin{aligned}
&\int_0^T \langle \nu B\Delta[S^k - (S-1)^k]\Gamma^a u(t), S^k\Gamma^a u(t)\rangle_{L^2} dt \\
&\qquad = \int_0^T \int_{\mathbb{R}^3} \nu\Delta[S^k - (S-1)^k](\Gamma^a u(t))^0 S^k(\Gamma^a u(t))^0 dx dt \\
&\qquad = -\int_0^T \int_{\mathbb{R}^3} \nu\nabla[S^k - (S-1)^k](\Gamma^a u(t))^0 \cdot \nabla S^k(\Gamma^a u(t))^0 dx dt
\end{aligned}$$

$$\lesssim \int_0^T \nu \|\nabla[S^k - (S-1)^k](\Gamma^a u(t))^0\|_{L^2} \|\nabla S^k (\Gamma^a u(t))^0\|_{L^2} dt$$

$$\lesssim \left(\nu \int_0^T \|\nabla[S^k - (S-1)^k](\Gamma^a u(t))^0\|_{L^2}^2 dt\right)^{1/2} \times \left(\nu \int_0^T \|\nabla S^k (\Gamma^a u(t))^0\|_{L^2}^2 dt\right)^{1/2}.$$

Therefore, we have:

$$I \;\lesssim\; \mathcal{E}_{p,q-1}^{1/2}[u](T)\mathcal{E}_{p,q}^{1/2}[u](T).$$

Applying Young's inequality and inserting into (2.9.3) result in:

$$\mathcal{E}_{p,q}[u](T) \lesssim \mathcal{E}_{p,q}[u_0] + \mathcal{E}_{p,q-1}[u](T) + \mu \mathcal{E}_{p,q}[u](T) + \sum_{\substack{|a|+k\le p \\ k\le q}} \left| \int_0^T \langle (S+1)^k \Gamma^a L u(t), S^k \Gamma^a u(t) \rangle_{L^2} dt \right|,$$

where $\mu$ can be chosen sufficiently small so that the corresponding energy term can be absorbed on the left. It follows from induction on $q$ that

$$\mathcal{E}_{p,q}[u](T) \;\lesssim\; \mathcal{E}_{p,q}[u_0] + \sum_{\substack{|a|+k\le p \\ k\le q}} \left| \int_0^T \langle (S+1)^k \Gamma^a L u(t), S^k \Gamma^a u(t) \rangle_{L^2} dt \right|. \qquad (2.9.4)$$

If we combine (2.9.4) with Lemma 2.4.2, we arrive at

$$\mathcal{E}_{p,q}[u](T) \lesssim \mathcal{E}_{p,q}[u_0] + \sum_{\substack{a_1+a_2+a_3=a \\ k_1+k_2=k \\ |a|+k\le p \\ k\le q}} \left| \int_0^T \langle [\Gamma^{a_3}, N](S^{k_1}\Gamma^{a_1} u, \nabla S^{k_2}\Gamma^{a_2} u), S^k \Gamma^a u \rangle_{L^2} dt \right|. \qquad (2.9.5)$$

Special care must be taken for the terms in the sum with $|a_2| + k_2 = |a| + k = p$. To simplify the notation when analyzing these terms, set $v = S^k \Gamma^a u$. Then using the fact that $\partial_\alpha v^\beta = \partial_\beta v^\alpha$, we may write:

$$\begin{aligned}\langle N(u, \nabla S^k\Gamma^a u), S^k\Gamma^a u\rangle_{\mathbb{R}^4} &= \langle N(u, \nabla v), v\rangle_{\mathbb{R}^4} \\ &= N^0(u, \nabla v)v^0 \\ &= C^\ell_{\alpha,\beta} u^\alpha \partial_\ell v^\beta v^0 \\ &= C^\ell_{\alpha,0} u^\alpha \partial_\ell v^0 v^0 + C^\ell_{\alpha,m} u^\alpha \partial_\ell v^m v^0 \\ &= \frac{1}{2} C^\ell_{\alpha,0} u^\alpha \partial_\ell (v_0)^2 + \\ &\qquad \frac{1}{2} C^\ell_{\alpha,\beta} u^\alpha [\partial_\ell v^m v^0 + \partial_\ell v^m v^0].\end{aligned} \tag{2.9.6}$$

Furthermore,

$$\begin{aligned}\partial_\ell v^m v^0 + \partial_\ell v^m v^0 &= \partial_\ell (v^m v^0) - v^m \partial_\ell v^0 + \partial_m v^\ell v^0 \\ &= \partial_\ell (v^m v^0) - v^m \partial_0 v^\ell + \partial_m (v^\ell v^0) - v^\ell \partial_m v^0 \\ &= \partial_\ell (v^m v^0) + \partial_m (v^\ell v^0) - v^m \partial_0 v^\ell - v^\ell \partial_0 v^m \\ &= \partial_\ell (v^m v^0) + \partial_m (v^\ell v^0) - \partial_0 (v^m v^\ell).\end{aligned}$$

From (2.9.6) we have:

$$\begin{aligned}&\langle N(u, \nabla S^k\Gamma^a u), S^k\Gamma^a u\rangle_{\mathbb{R}^4} \\ &\quad = \frac{1}{2}\left[C^\ell_{\alpha,0}\partial_\ell (u^\alpha (v^0)^2) + C^\ell_{\alpha,m}\left(\partial_\ell (u^\alpha v^m v^0) + \partial_m (u^\alpha v^\ell v^0) - \partial_0 (u^\alpha v^m v^\ell)\right)\right] \\ &\qquad - \mathcal{O}(|\partial u||v|^2).\end{aligned}$$

Integration over $[0, T] \times \mathbb{R}^3$ yields:

$$\int_0^T \langle N(u, \nabla S^k\Gamma^a u), S^k\Gamma^a u\rangle_{L^2} dt \tag{2.9.7}$$

$$
\begin{aligned}
&= -\frac{1}{2}\int_0^T \int_{\mathbb{R}^3} C^{\ell}_{\alpha,m}\partial_0 \left(u^\alpha(t)(S^k\Gamma^a u(t))^m (S^k\Gamma^a u(t))^\ell\right) dxdt \\
&\qquad + \int_0^T \int_{\mathbb{R}^3} \mathcal{O}(|\partial u||S^k\Gamma^a u|^2) dxdt \\
&= -\frac{1}{2}\int_{\mathbb{R}^3} C^{\ell}_{\alpha,m} u^\alpha(T)(S^k\Gamma^a u(T))^m (S^k\Gamma^a u(T))^\ell dx \\
&\qquad + \frac{1}{2}\int_{\mathbb{R}^3} C^{\ell}_{\alpha,m} u^\alpha(0)(S^k\Gamma^a u(0))^m (S^k\Gamma^a u(0))^\ell dx \\
&\qquad + \mathcal{O}\left(\int_0^T \int_{\mathbb{R}^3} |\partial u||S^k\Gamma^a u|^2 dxdt\right) \\
&\lesssim \|u(T)\|_{L^\infty(\mathbb{R}^3)} \int_{\mathbb{R}^3} |S^k\Gamma^a u(T)|^2 dx \\
&\qquad + \|u(0)\|_{L^\infty(\mathbb{R}^3)} \int_{\mathbb{R}^3} |S^k\Gamma^a u(0)|^2 dx \\
&\qquad + \mathcal{O}\left(\int_0^T \int_{\mathbb{R}^3} |\partial u||S^k\Gamma^a u|^2 dxdt\right).
\end{aligned}
$$

By (2.5.1a) and the assumption (2.9.1), we have

$$
\|u\|_{L^\infty(\mathbb{R}^3)} \lesssim \|u\|_{H^2(\mathbb{R}^3)} \le \mathcal{E}^{1/2}_{2,0}[u] < \varepsilon \ll 1. \tag{2.9.8}
$$

Using the PDE (2.1.1a), we can estimate the time derivative:

$$
|\partial_0| \lesssim |\nabla u| + |\Delta u| + |u||\nabla u|
$$

and, together with the smallness condition (2.9.8), we have

$$
|\partial u| \lesssim |\partial_0 u| + |\nabla u| \lesssim (1+|u|)\,|\nabla u| + |\Delta u| \lesssim |\nabla u| + |\Delta u|. \tag{2.9.9}
$$

It follows that the right-hand side of (2.9.7) is bounded by

$$
\varepsilon\Big(\mathcal{E}_{p,q}[u](T) + \mathcal{E}_{p,q}[u_0]\Big) \;+\; \mathcal{O}\left(\int_0^T \int_{\mathbb{R}^3} (|\nabla u| + |\Delta u|)|S^k\Gamma^a u|^2 dxdt\right).
$$

The remaining terms in (2.9.5) satisfy:

$$\sum_{\substack{|a|+k\le p\\ k\le q}} \sum_{\substack{a_1+a_2+a_3=a\\ k_1+k_2=k\\ |a_2|+k_2\le p}} \left| \int_0^T \langle [\Gamma^{a_3}, N](S^{k_1}\Gamma^{a_1}u, \nabla S^{k_2}\Gamma^{a_2}u), S^k\Gamma^a u\rangle_{L^2} dt \right|$$

$$\lesssim \sum_{\substack{|a|+k\le p\\ k\le q}} \sum_{\substack{|a_1+a_2|\le|a|\\ k_1+k_2\le k\\ |a_2|+k_2<p}} \int_0^T \int_{\mathbb{R}^3} |S^{k_1}\Gamma^{a_1}u||\nabla S^{k_2}\Gamma^{a_2}u)||S^k\Gamma^a u| dxdt.$$

This leads to:

$$\mathcal{E}_{p,q}[u](T) \lesssim \mathcal{E}_{p,q}[u_0] + \sum_{|a|+k=p} \int_0^T (\|\nabla u(t)\|_{L^\infty} + \|\Delta u(t)\|_{L^\infty})\mathcal{E}_{p,q}[u](t)dt$$

$$+ \sum_{\substack{|a_1+a_2|+k_1+k_2\le p\\ k_1+k_2\le q\\ |a_2|+k_2<p}} \int_0^T \|\, |S^{k_1}\Gamma^{a_1}u(t)|\, |S^{k_2}\Gamma^{a_2+1}u(t)|\, \|_{L^2} \mathcal{E}^{1/2}_{p,q}[u](t)dt. \quad (2.9.10)$$

Using property (2.2.4c) of the cut-off functions, the Sobolev inequalities (2.5.2a), (2.5.2d), and the fact that $5 \le \left[\frac{p+5}{2}\right]$, we obtain:

$$\|\nabla u(t)\|_{L^\infty} + \|\Delta u(t)\|_{L^\infty}$$

$$\lesssim \|\zeta\nabla u(t)\|_{L^\infty} + \|\zeta\Delta u(t)\|_{L^\infty} + \|\eta\nabla u(t)\|_{L^\infty} + \|\eta\Delta u(t)\|_{L^\infty}$$

$$\lesssim \left(\mathcal{Y}^{\rm int}_{4,0}[u](t)\right)^{1/2} + \langle t\rangle^{-1}\mathcal{E}^{1/2}_{4,0}[u](t)$$

$$\lesssim \left(\mathcal{Y}^{\rm int}_{\left[\frac{p+5}{2}\right],\left[\frac{p+3}{2}\right]}[u](t)\right)^{1/2} + \langle t\rangle^{-1}\mathcal{E}^{1/2}_{\left[\frac{p+5}{2}\right],\left[\frac{p+5}{2}\right]}[u](t).$$

By (2.2.4c) and Lemmas 2.6.1 and 2.6.2, we get the bound

$$\sum_{\substack{|a_1+a_2|+k_1+k_2\le p\\ k_1+k_2\le q\\ |a_2|+k_2<p}} \|\, |S^{k_1}\Gamma^{a_1}u(t)|\, |S^{k_2}\Gamma^{a_2+1}u(t)|\, \|_{L^2}$$

$$\lesssim \sum_{\substack{|a_1+a_2|+k_1+k_2\le p\\ k_1+k_2\le q\\ |a_2|+k_2<p}} \|\zeta\, |S^{k_1}\Gamma^{a_1}u(t)|\, |S^{k_2}\Gamma^{a_2+1}u(t)|\, \|_{L^2}$$

$$+ \sum_{\substack{|a_1+a_2|+k_1+k_2\le p\\ k_1+k_2\le q\\ |a_2|+k_2<p}} \|\eta\, |S^{k_1}\Gamma^{a_1}u(t)|\, |S^{k_2}\Gamma^{a_2+1}u(t)|\, \|_{L^2}$$

$$\lesssim \left[\left(\mathcal{Y}^{\text{int}}_{\left[\frac{p+5}{2}\right],\left[\frac{p+3}{2}\right]}[u](t)\right)^{1/2} + \langle t\rangle^{-1}\mathcal{E}^{1/2}_{\left[\frac{p+5}{2}\right],\left[\frac{p+5}{2}\right]}[u](t)\right] \mathcal{E}^{1/2}_{p,q}[u](t).$$

Inserting the two bounds above into the previous energy inequality yields

$$\mathcal{E}_{p,q}[u](T) \lesssim \mathcal{E}_{p,q}[u_0] + \int_0^T \left[\left(\mathcal{Y}^{\text{int}}_{\left[\frac{p+5}{2}\right],\left[\frac{p+3}{2}\right]}[u](t)\right)^{1/2} + \langle t\rangle^{-1}\mathcal{E}^{1/2}_{\left[\frac{p+5}{2}\right],\left[\frac{p+5}{2}\right]}[u](t)\right] \mathcal{E}_{p,q}[u](t)dt.$$

Gronwall's inequality further implies that:

$$\mathcal{E}_{p,q}[u](T) \lesssim \mathcal{E}_{p,q}[u_0] \exp \int_0^T \left[\left(\mathcal{Y}^{\text{int}}_{\left[\frac{p+5}{2}\right],\left[\frac{p+3}{2}\right]}[u](t)\right)^{1/2} + \langle t\rangle^{-1}\mathcal{E}^{1/2}_{\left[\frac{p+5}{2}\right],\left[\frac{p+5}{2}\right]}[u](t)\right] dt. \quad (2.9.11)$$

Recalling the definition $p^* = \left[\frac{p+5}{2}\right]$ and using Theorem 2.8.1, we obtain:

$$\begin{aligned}\int_0^T \left(\mathcal{Y}^{\text{int}}_{\left[\frac{p+5}{2}\right],\left[\frac{p+3}{2}\right]}[u](t)\right)^{1/2} dt &\le \left(\int_0^T \langle t\rangle \mathcal{Y}^{\text{int}}_{p^*,p^*-1}[u](t)dt\right)^{1/2} \left(\int_0^T \langle t\rangle^{-1}dt\right)^{1/2}\\ &\lesssim \left(\sup_{0\le t\le T} \mathcal{E}_{p^*,p^*}[u](t)\log(e+T)\right)^{1/2} (\log(e+T))^{1/2}\\ &\lesssim \sup_{0\le t\le T} \mathcal{E}^{1/2}_{p^*,p^*}[u](t)\log(e+T).\end{aligned}$$

Similarly, we have the bound

$$\int_0^T \langle t\rangle^{-1}\mathcal{E}^{1/2}_{\left[\frac{p+5}{2}\right],\left[\frac{p+5}{2}\right]}[u](t) \lesssim \sup_{0\le t\le T} \mathcal{E}^{1/2}_{p^*,p^*}[u](t)\log(e+T).$$

With the smallness assumption (2.9.1), the energy inequality (2.9.11) becomes:

$$\mathcal{E}_{p,q}[u](T) \lesssim \mathcal{E}_{p,q}[u_0] \exp\left[C\varepsilon \log(e+T)\right] \le \mathcal{E}_{p,q}[u_0]\langle T\rangle^{C_1\varepsilon}.$$

Returning to (2.9.11), we can repeat this argument with the pair $(p,q) = (p^*,p^*)$, because $p^* \geq 5$ implies that $p^* \geq \left[\frac{p*+5}{2}\right]$. Therefore, we obtain also the bound

$$\mathcal{E}_{p^*,p^*}[u](T) \lesssim \mathcal{E}_{p^*,p^*}[u_0]\langle T\rangle^{C_1\varepsilon}.$$

□

**Corollary 2.9.2.** *Under the hypotheses of Proposition 2.9.1, we have*

$$\mathcal{E}_{p^*,p^*}[u](T) \lesssim \mathcal{E}_{p^*,p^*}[u_0] \\ + \sum_{\substack{a_1+a_2+a_3=a \\ k_1+k_2=k \\ |a|+k\leq p^* \\ k\leq p^*}} \left| \int_0^T \langle [\Gamma^{a_3}, N](S^{k_1}\Gamma^{a_1}u, \nabla S^{k_2}\Gamma^{a_2}u), S^k\Gamma^a u\rangle_{L^2} dt \right|.$$

*Proof.* This is simply (2.9.5) from the proof of Proposition 2.9.1 in the case when $(p,q) = (p^*,p^*)$. □

## 2.10 Low Energy Estimates

**Proposition 2.10.1.** *Choose $(p,q)$ such that $p \geq 11$, and $p \geq q > p^*$, where $p^* = \left[\frac{p+5}{2}\right]$. Let $\delta \leq 1$ be defined as in* (2.3.1)*. Suppose that $u \in C([0,T_0), X^{p,q})$ is a solution of* (2.1.1a), (2.1.1b) *with*

$$\sup_{0\leq t<T_0} \mathcal{E}_{p^*,p^*}[u](t) \leq \varepsilon^2 \ll 1. \tag{2.10.1}$$

*There exists a constant $C_0 > 1$ such that if*

$$C_0^3 \left(\frac{\delta}{\nu}\right)^2 \mathcal{E}_{p^*,p^*}[u_0]\left(1 + \mathcal{E}_{p,q}^{1/2}[u_0]\right) < 1, \tag{2.10.2}$$

*then*

$$\sup_{0\le t<T_0} \mathcal{E}_{p^*,p^*}[u](t) \le C_0\mathcal{E}_{p^*,p^*}[u_0]\left(1+\mathcal{E}_{p,q}^{1/2}[u_0]\right). \tag{2.10.3}$$

*Proof.* We start with the inequality in Corollary 2.9.2

$$\begin{aligned}\mathcal{E}_{p^*,p^*}[u](T) \lesssim \mathcal{E}_{p^*,p^*}[u_0]& \\ &+ \sum_{\substack{a_1+a_2+a_3=a\\ k_1+k_2=k\\ |a|+k\le p^*\\ k\le p^*}} \left|\int_0^T \langle[\Gamma^{a_3},N](S^{k_1}\Gamma^{a_1}u,\nabla S^{k_2}\Gamma^{a_2}u),S^k\Gamma^a u\rangle_{L^2}dt\right|.\end{aligned}$$

Using the cut-off function defined in (2.2.4a) and satisfying the property (2.2.4c), we can bound the nonlinear term by:

$$\begin{aligned}&\sum_{\substack{a_1+a_2+a_3=a\\ k_1+k_2=k\\ |a|+k\le p^*\\ k\le p^*}} \left|\int_0^T \langle[\Gamma^{a_3},N](S^{k_1}\Gamma^{a_1}u,\nabla S^{k_2}\Gamma^{a_2}u),S^k\Gamma^a u\rangle_{L^2}dt\right| \\
&\qquad\lesssim \sum_{\substack{|a_1+a_2|\le|a|\\ k_1+k_2=k\\ |a|+k\le p^*\\ k\le p^*}} \int_0^T\int_{\mathbb{R}^3} \zeta^2|S^{k_1}\Gamma^{a_1}u|\,|\nabla S^{k_2}\Gamma^{a_2}u|\,|S^k\Gamma^a u|\,dxdt \\
&\qquad+ \sum_{\substack{a_1+a_2+a_3=a\\ k_1+k_2=k\\ |a|+k\le p^*\\ k\le p^*}} \left|\int_0^T\int_{\mathbb{R}^3} \eta\langle[\Gamma^{a_3},N](S^{k_1}\Gamma^{a_1}u,\nabla S^{k_2}\Gamma^{a_2}u),S^k\Gamma^a u\rangle_{\mathbb{R}^4}dxdt\right| \\
&\qquad\equiv I_1+I_2, \end{aligned} \tag{2.10.4}$$

where $I_1$ and $I_2$ are defined correspondingly as the first and second term on the right-hand side of the above inequality. The time variable $T$ is in the range $0\le T<T_0$.

**Interior Low Energy**

The interior intergral $I_1$ can be bounded by:

$$I_1 = \sum_{\substack{|a_1+a_2|\le|a|\\ k_1+k_2=k\\ |a|+k\le p^*\\ k\le p^*}} \int_0^T \int_{\mathbb{R}^3} \zeta^2 |S^{k_1}\Gamma^{a_1}u|\ |\nabla S^{k_2}\Gamma^{a_2}u|\ |S^k\Gamma^a u|\ dxdt$$

$$\lesssim \sum_{\substack{k_1+k_2+|a_1|+|a_2|\le p^*\\ k_1+k_2\le p^*}} \int_0^T \|\ \zeta^2|S^{k_1}\Gamma^{a_1}u|\ |\nabla S^{k_2}\Gamma^{a_2}u|\ \|_{L^2}\ \mathcal{E}^{1/2}_{p^*,p^*}[u](t)dt.$$

In the case $k_1+|a_1| < k_2+|a_2|+1$, i.e. $k_1+|a_1| \le \left[\frac{p^*}{2}\right]$, using (2.5.2a), we have:

$$\|\zeta^2|S^{k_1}\Gamma^{a_1}u||\nabla S^{k_2}\Gamma^{a_2}u|\ \|_{L^2}$$

$$\lesssim \|\zeta S^{k_1}\Gamma^{a_1}u\|_{L^\infty}\|\zeta\nabla S^{k_2}\Gamma^{a_2}u\|_{L^2}$$

$$\lesssim \left[\left(\mathcal{Y}^{\rm int}_{\left[\frac{p^*+4}{2}\right],\left[\frac{p^*}{2}\right]}[u](t)\right)^{1/2} + \langle t\rangle^{-1}\mathcal{E}^{1/2}_{\left[\frac{p^*+2}{2}\right],\left[\frac{p^*}{2}\right]}[u](t)\right]\left(\mathcal{Y}^{\rm int}_{p^*+1,p^*}[u](t)\right)^{1/2}.$$

We next consider the case when $k_2+|a_2|+1 \le k_1+|a_1|$, i.e. $k_2+|a_2| \le \left[\frac{p^*-1}{2}\right]$. With the use of Hardy's inequality (2.5.2c) and the Sobolev inequality (2.5.2b), we have:

$$\|\zeta^2\ |S^{k_1}\Gamma^{a_1}u|\ |\nabla S^{k_2}\Gamma^{a_2}u|\ \|_{L^2}$$

$$\lesssim \|r^{-1}\zeta S^{k_1}\Gamma^{a_1}u\|_{L^2}\|r\zeta\nabla S^{k_2}\Gamma^{a_2}u\|_{L^\infty}$$

$$\lesssim \left[\left(\mathcal{Y}^{\rm int}_{p^*+1,p^*}[u](t)\right)^{1/2} + \langle t\rangle^{-1}\mathcal{E}^{1/2}_{p^*,p^*}[u](t)\right]$$

$$\times\left[\left(\mathcal{Y}^{\rm int}_{\left[\frac{p^*+5}{2}\right],\left[\frac{p^*-1}{2}\right]}[u](t)\right)^{1/2} + \langle t\rangle^{-1}\mathcal{E}^{1/2}_{\left[\frac{p^*+3}{2}\right],\left[\frac{p^*-1}{2}\right]}[u](t)\right].$$

Recall that $\left[\frac{p^*+5}{2}\right] \leq p^*$ since $p \geq 11$. Overall, for the interior low energy we have:

$$
\begin{aligned}
I_1 \lesssim & \int_0^T \left[ \left( \mathcal{Y}^{\text{int}}_{\left[\frac{p^*+5}{2}\right],\left[\frac{p^*}{2}\right]}[u](t) \right)^{1/2} + \langle t \rangle^{-1} \mathcal{E}^{1/2}_{\left[\frac{p^*+3}{2}\right],\left[\frac{p^*}{2}\right]}[u](t) \right] \\
& \qquad \times \left[ \left( \mathcal{Y}^{\text{int}}_{p^*+1,p^*}[u](t) \right)^{1/2} + \langle t \rangle^{-1} \mathcal{E}^{1/2}_{p^*,p^*}[u](t) \right] \mathcal{E}^{1/2}_{p^*,p^*}[u](t) dt \\
\lesssim & \int_0^T \left[ \left( \mathcal{Y}^{\text{int}}_{p^*,p^*-1}[u](t) \right)^{1/2} + \langle t \rangle^{-1} \mathcal{E}^{1/2}_{p^*,p^*}[u](t) \right] \\
& \qquad \times \left[ \left( \mathcal{Y}^{\text{int}}_{p^*+1,p^*}[u](t) \right)^{1/2} + \langle t \rangle^{-1} \mathcal{E}^{1/2}_{p^*,p^*}[u](t) \right] \mathcal{E}^{1/2}_{p^*,p^*}[u](t) dt \\
\lesssim & \int_0^T \left( \mathcal{Y}^{\text{int}}_{p^*,p^*-1}[u](t) \right)^{1/2} \left( \mathcal{Y}^{\text{int}}_{p^*+1,p^*}[u](t) \right)^{1/2} \mathcal{E}^{1/2}_{p^*,p^*}[u](t) dt \\
& \qquad + \int_0^T \langle t \rangle^{-1} \left( \mathcal{Y}^{\text{int}}_{p^*+1,p^*}[u](t) \right)^{1/2} \mathcal{E}_{p^*,p^*}[u](t) dt \\
& \qquad + \int_0^T \langle t \rangle^{-2} \mathcal{E}^{3/2}_{p^*,p^*}[u](t) dt.
\end{aligned}
$$

Next, we are going to estimate the three integrals from above. We will use Theorem 2.8.1 and Proposition 2.9.1. Furthermore, we will require that $2C_1\varepsilon < 1$.

The first integral can be estimated as follows:

$$
\begin{aligned}
& \int_0^T \left( \mathcal{Y}^{\text{int}}_{p^*,p^*-1}[u](t) \right)^{1/2} \left( \mathcal{Y}^{\text{int}}_{p^*+1,p^*}[u](t) \right)^{1/2} \mathcal{E}^{1/2}_{p^*,p^*}[u](t) dt \\
& \lesssim \left( \sup_{0 \leq t \leq T} \langle t \rangle^{-C_1\varepsilon} \mathcal{E}_{p^*,p^*}[u](t) \right)^{1/2} \\
& \qquad \times \int_0^T \langle t \rangle^{C_1\varepsilon/2} \left( \mathcal{Y}^{\text{int}}_{p^*,p^*-1}[u](t) \right)^{1/2} \left( \mathcal{Y}^{\text{int}}_{p^*+1,p^*}[u](t) \right)^{1/2} dt \\
& \lesssim \mathcal{E}^{1/2}_{p^*,p^*}[u_0] \left( \int_0^T \langle t \rangle^{C_1\varepsilon/2} \mathcal{Y}^{\text{int}}_{p^*,p^*-1}[u](t) dt \right)^{1/2} \left( \int_0^T \langle t \rangle^{C_1\varepsilon/2} \mathcal{Y}^{\text{int}}_{p^*+1,p^*}[u](t) dt \right)^{1/2} \\
& \lesssim \mathcal{E}^{1/2}_{p^*,p^*}[u_0] \left( \sup_{0 \leq t \leq T} \langle t \rangle^{-C_1\varepsilon} \mathcal{E}_{p^*,p^*}[u](t) \right)^{1/2} \left( \sup_{0 \leq t \leq T} \langle t \rangle^{-C_1\varepsilon} \mathcal{E}_{p,q}[u](t) \right)^{1/2} \\
& \lesssim \mathcal{E}_{p^*,p^*}[u_0] \mathcal{E}^{1/2}_{p,q}[u_0].
\end{aligned}
$$

For the second integral, we have:

$$
\begin{aligned}
&\int_0^T \langle t\rangle^{-1} \left(\mathcal{Y}^{\text{int}}_{p^*+1,p^*}[u](t)\right)^{1/2} \mathcal{E}_{p^*,p^*}[u](t)dt \\
&\qquad \lesssim \sup_{0\le t\le T} \langle t\rangle^{-C_1\varepsilon}\mathcal{E}_{p^*,p^*}[u](t) \int_0^T \langle t\rangle^{-1+C_1\varepsilon} \left(\mathcal{Y}^{\text{int}}_{p^*+1,p^*}[u](t)\right)^{1/2} dt \\
&\qquad \lesssim \mathcal{E}_{p^*,p^*}[u_0] \int_0^T \langle t\rangle^{-1+C_1\varepsilon/2}\langle t\rangle^{C_1\varepsilon/2} \left(\mathcal{Y}^{\text{int}}_{p^*+1,p^*}[u](t)\right)^{1/2} dt \\
&\qquad \lesssim \mathcal{E}_{p^*,p^*}[u_0] \left(\int_0^T \langle t\rangle^{-2+C_1\varepsilon} dt\right)^{1/2} \left(\int_0^T \langle t\rangle^{C_1\varepsilon}\mathcal{Y}^{\text{int}}_{p^*+1,p^*}[u](t)dt\right)^{1/2} \\
&\qquad \lesssim \mathcal{E}_{p^*,p^*}[u_0] \left(\sup_{0\le t\le T} \langle t\rangle^{-C_1\varepsilon}\mathcal{E}_{p,q}[u](t)\right)^{1/2} \\
&\qquad \lesssim \mathcal{E}_{p^*,p^*}[u_0]\mathcal{E}^{1/2}_{p,q}[u_0].
\end{aligned}
$$

And finally, the third integral is bounded by:

$$
\begin{aligned}
\int_0^T \langle t\rangle^{-2}\mathcal{E}^{3/2}_{p^*,p^*}[u](t)dt &\lesssim \left(\sup_{0\le t\le T} \langle t\rangle^{-C_1\varepsilon}\mathcal{E}_{p^*,p^*}[u](t)\right)^{3/2} \int_0^T \langle t\rangle^{-2+\frac{3}{2}C_1\varepsilon} dt \\
&\lesssim \mathcal{E}^{3/2}_{p^*,p^*}[u_0] \\
&\lesssim \mathcal{E}_{p^*,p^*}[u_0]\mathcal{E}^{1/2}_{p,q}[u_0].
\end{aligned}
$$

Combining these estimates, we have:

$$
I_1 \lesssim \mathcal{E}_{p^*,p^*}[u_0]\mathcal{E}^{1/2}_{p,q}[u_0]. \tag{2.10.5}
$$

**Exterior Low Energy**

Recall the definition of the second integral $I_2$ in (2.10.4):

$$
I_2 = \sum_{\substack{a_1+a_2+a_3=a \\ k_1+k_2=k \\ |a|+k\le p^* \\ k\le p^*}} \left| \int_0^T \int_{\mathbb{R}^3} \eta\langle [\Gamma^{a_3}, N](S^{k_1}\Gamma^{a_1}u, \nabla S^{k_2}\Gamma^{a_2}u), S^k\Gamma^a u\rangle_{\mathbb{R}^4} dxdt \right|.
$$

Using projection decompositions (Lemma 2.4.4), we can write:

$$I_2 = I_2' + I_2\text{''}, \tag{2.10.6a}$$

where

$$I_2' = \sum_{\substack{a_1+a_2+a_3=a \\ k_1+k_2=k \\ |a|+k\le p^* \\ k\le p^*}} \left| \int_0^T \int_{\mathbb{R}^3} \tfrac{1}{4}\ \eta\ P_{[\widetilde{\Omega}^a, N]}(\hat{\omega})\ \langle \hat{\omega}, S^{k_1}\Gamma^{a_1}u\rangle_{\mathbb{R}^4} \right.$$

$$\left. \times\ \langle \hat{\omega}, \partial_r S^{k_2}\Gamma^{a_2}u\rangle_{\mathbb{R}^4}\ (S^k\Gamma^a u)^0\ dxdt \right| \tag{2.10.6b}$$

and

$$I_2'' = \sum_{\substack{k_1+k_2+|a_1|+|a_2|\le p^* \\ k_1+k_2\le p^*}} \int_0^T \Big[ \|\ \eta |\mathbb{Q} S^{k_1}\Gamma^{a_1}u|\ |\partial_r S^{k_2}\Gamma^{a_2}u|\ \|_{L^2} \tag{2.10.6c}$$

$$+ \|\ \eta |S^{k_1}\Gamma^{a_1}u|\ |\mathbb{Q}\partial_r S^{k_2}\Gamma^{a_2}u|\ \|_{L^2}$$

$$+ \|\ r^{-1}\eta |S^{k_1}\Gamma^{a_1}u|\ |S^{k_2}\widetilde{\Omega}\Gamma^{a_2}u|\ \|_{L^2} \Big] \mathcal{E}^{1/2}_{p^*,p^*}[u](t)\ dt.$$

We first estimate the terms $I_2''$. Using the notation

$$Q_1 = \|\ \eta |\mathbb{Q} S^{k_1}\Gamma^{a_1}u|\ |\partial_r S^{k_2}\Gamma^{a_2}u|\ \|_{L^2}$$

$$Q_2 = \|\ \eta |S^{k_1}\Gamma^{a_1}u|\ |\mathbb{Q}\partial_r S^{k_2}\Gamma^{a_2}u|\ \|_{L^2}$$

$$Q_3 = \|\ r^{-1}\eta |S^{k_1}\Gamma^{a_1}u|\ |S^{k_2}\widetilde{\Omega}\Gamma^{a_2}u|\ \|_{L^2},$$

we show that

$$Q_1 + Q_2 + Q_3 \lesssim \langle t\rangle^{-3/2}\ \mathcal{E}^{1/2}_{p^*,p^*}[u](t)\ \mathcal{E}^{1/2}_{p,q}[u](t). \tag{2.10.7}$$

We recall the facts that $p^* + 3 \le p$ and $\left[\frac{p^*+5}{2}\right] \le p^*$ (since $p \ge 11$) and also $p^* < q$.

With the use of (2.5.2e) and Theorem 2.8.2, we have:

$$
\begin{aligned}
Q_1 &= \| \, \eta |\mathbb{Q} S^{k_1} \Gamma^{a_1} u| \, |\partial_r S^{k_2} \Gamma^{a_2} u| \, \|_{L^2} \\
&\leq \|\eta \mathbb{Q} S^{k_1} \Gamma^{a_1} u\|_{L^\infty} \|\partial_r S^{k_2} \Gamma^{a_2} u\|_{L^2} \\
&\lesssim \langle t \rangle^{-3/2} \left( \left( \mathcal{Y}^{\text{ext}}_{k_1+|a_1|+2,k_1}[u](t) \right)^{1/2} + \mathcal{E}^{1/2}_{k_1+|a_1|+1,k_1}[u](t) \right) \mathcal{E}^{1/2}_{k_2+|a_2|+1,k_2}[u](t) \\
&\lesssim \langle t \rangle^{-3/2} \, \mathcal{E}^{1/2}_{k_1+|a_1|+2,k_1+1}[u](t) \, \mathcal{E}^{1/2}_{k_2+|a_2|+1,k_2}[u](t) \\
&\lesssim \langle t \rangle^{-3/2} \mathcal{E}^{1/2}_{p^*,p^*}[u](t) \mathcal{E}^{1/2}_{p,q}[u](t),
\end{aligned}
$$

where in the last line we have used that fact that since $k_1 + k_2 + |a_1| + |a_2| \leq p^*$, either

$$
k_1 + |a_1| + 2 \leq \left[ \frac{p^* + 3}{2} \right] \leq p^* \quad \text{and} \quad k_2 + |a_2| + 1 \leq p^* + 1 \leq p
$$

or

$$
k_1 + |a_1| + 2 \leq p^* + 2 \leq p \quad \text{and} \quad k_2 + |a_2| + 1 \leq \left[ \frac{p^* + 3}{2} \right] \leq p^*.
$$

For $Q_2$ we similarly have:

$$
\begin{aligned}
Q_2 &= \| \, \eta |S^{k_1} \Gamma^{a_1} u| \, |\mathbb{Q} \partial_r S^{k_2} \Gamma^{a_2} u| \, \|_{L^2} \\
&\leq \|S^{k_1} \Gamma^{a_1} u\|_{L^2} \|\eta \mathbb{Q} \partial_r S^{k_2} \Gamma^{a_2} u\|_{L^\infty} \\
&\lesssim \mathcal{E}^{1/2}_{k_1+|a_1|,k_1}[u](t) \langle t \rangle^{-3/2} \left( \left( \mathcal{Y}^{\text{ext}}_{k_2+|a_2|+3,k_2}[u](t) \right)^{1/2} + \mathcal{E}^{1/2}_{k_2+|a_2|+2,k_2}[u](t) \right) \\
&\lesssim \langle t \rangle^{-3/2} \, \mathcal{E}^{1/2}_{k_1+|a_1|,k_1}[u](t) \, \mathcal{E}^{1/2}_{k_2+|a_2|+3,k_2+1}[u](t) \\
&\lesssim \langle t \rangle^{-3/2} \, \mathcal{E}^{1/2}_{p^*,p^*}[u](t) \, \mathcal{E}^{1/2}_{p^*+3,p^*+1}[u](t) \\
&\lesssim \langle t \rangle^{-3/2} \, \mathcal{E}^{1/2}_{p^*,p^*}[u](t) \, \mathcal{E}^{1/2}_{p,q}[u](t),
\end{aligned}
$$

since $p^* + 3 \leq p$ and $p^* < q$.

Finally, to estimate $Q_3$, we use Lemma 2.6.2:

$$\begin{aligned} Q_3 &= \| \, r^{-1}\eta |S^{k_1}\Gamma^{a_1} u| \; |S^{k_2}\widetilde{\Omega}\Gamma^{a_2} u| \, \|_{L^2} \\ &\lesssim \langle t\rangle^{-1} \| \, \eta |S^{k_1}\Gamma^{a_1} u| \; |S^{k_2}\widetilde{\Omega}\Gamma^{a_2} u| \, \|_{L^2} \\ &\lesssim \langle t\rangle^{-2} \mathcal{E}^{1/2}_{p^*,p^*}[u](t) \mathcal{E}^{1/2}_{p^*+1,p^*}[u](t) \\ &\lesssim \langle t\rangle^{-3/2} \mathcal{E}^{1/2}_{p^*,p^*}[u](t) \mathcal{E}^{1/2}_{p,q}[u](t). \end{aligned}$$

Thus, from (2.10.6c) and (2.10.7), we have that

$$I''_2 \lesssim \int_0^T \langle t\rangle^{-3/2} \; \mathcal{E}_{p^*,p^*}[u](t) \; \mathcal{E}^{1/2}_{p,q}[u](t) \; dt. \tag{2.10.8}$$

Next, we return to $I'_2$.By (2.4.3b) we can write:

$$\begin{aligned} I'_2 \; &= \sum_{\substack{a_1+a_2+a_3=a \\ k_1+k_2=k \\ |a|+k\leq p^* \\ k\leq p^*}} \left| \int_0^T \int_{\mathbb{R}^3} \tfrac{1}{4} \; \eta \; P_{[\widetilde{\Omega}^a, N]}(\hat{\omega}) \; \langle \hat{\omega}, S^{k_1}\Gamma^{a_1} u\rangle_{\mathbb{R}^4} \right. \\ &\qquad\qquad \left. \times \; \langle \hat{\omega}, \partial_r S^{k_2}\Gamma^{a_2} u\rangle_{\mathbb{R}^4} \; (S^k \Gamma^a u)^0 \; dxdt \right| \\ &\lesssim \sum_{\substack{a_1+a_2+a_3=a \\ k_1+k_2=k \\ |a|+k\leq p^* \\ k\leq p^*}} \int_0^T \delta \, \| \, \eta \; |S^{k_1}\Gamma^{a_1} u| \; |\partial_r S^{k_2}\Gamma^{a_2} u| \; |S^k(\Gamma^a u)^0| \, \|_{L^1} dt. \end{aligned} \tag{2.10.9}$$

We denote

$$Q_0 = \delta \, \| \, \eta \; |S^{k_1}\Gamma^{a_1} u| \; |\partial_r S^{k_2}\Gamma^{a_2} u| \; |S^k(\Gamma^a u)^0| \, \|_{L^1}.$$

Using (2.1.2b) and the definition of the scaling operator $S$, we have the equality:

$$\partial_r u = \partial_r u^0 e_0 + \partial_r \bar{u} = \partial_r u^0 e_0 + \frac{1}{r} S\bar{u} - \frac{t}{r}\partial_t \bar{u} = \partial_r u^0 e_0 + \frac{1}{r} S\bar{u} - \frac{t}{r}(\nabla u^0)^\top,$$

and thus, on the support of $\eta$, we have:

$$\eta|\partial_r u| \lesssim \eta(|\nabla u^0| + \langle t\rangle^{-1}|Su|).$$

Therefore, we have the following estimate for $Q_0$:

$$Q_0 \lesssim \delta\| \, |S^{k_1}\Gamma^{a_1}u| \, |\nabla S^{k_2}(\Gamma^{a_2}u)^0| \, |S^k(\Gamma^a u)^0| \, \|_{L^1} + \delta\langle t\rangle^{-1}\| \, \eta \, |S^{k_1}\Gamma^{a_1}u| \, |S^{k_2+1}\Gamma^{a_2}u| \, \|_{L^2}\|S^k\Gamma^a u\|_{L^2}. \quad (2.10.10)$$

Using a slight variant of Lemma 2.6.2 with $(\bar{p},\bar{q}) = (p^*, p^*)$, we see that the second term in (2.10.10) satisfies

$$\delta\langle t\rangle^{-1}\| \, \eta \, |S^{k_1}\Gamma^{a_1}u| \, |S^{k_2+1}\Gamma^{a_2}u| \, \|_{L^2}\|S^k\Gamma^a u\|_{L^2} \quad (2.10.11)$$

$$\lesssim \delta\langle t\rangle^{-2} \, \mathcal{E}^{1/2}_{[\frac{p^*+5}{2}],[\frac{p^*+1}{2}]}[u](t) \, \mathcal{E}^{1/2}_{p^*+1,p^*+1}[u](t)\mathcal{E}^{1/2}_{p^*,p^*}[u](t)$$

$$\lesssim \delta\langle t\rangle^{-2} \, \mathcal{E}_{p^*,p^*}[u](t)\mathcal{E}^{1/2}_{p^*+1,p^*+1}[u](t)$$

$$\lesssim \delta\langle t\rangle^{-2} \, \mathcal{E}_{p^*,p^*}[u](t)\mathcal{E}^{1/2}_{p,q}[u](t).$$

The first term in (2.10.10) measures the deviation from the null condition and it will be estimated with the help of the diffusion term in the energy. Using (2.5.1b), we show that

$$\delta\| \, |S^{k_1}\Gamma^{a_1}u| \, |\nabla S^{k_2}(\Gamma^{a_2}u)^0| \, |S^k(\Gamma^a u)^0| \, \|_{L^1}$$

$$\lesssim \delta\| \, r \, |S^{k_1}\Gamma^{a_1}u| \, |\nabla S^{k_2}(\Gamma^{a_2}u)^0| \, \|_{L^2}\|r^{-1}S^k(\Gamma^a u)^0\|_{L^2} \quad (2.10.12)$$

$$\lesssim \delta\| \, r \, |S^{k_1}\Gamma^{a_1}u| \, |\nabla S^{k_2}(\Gamma^{a_2}u)^0| \, \|_{L^2}\|\nabla S^k(\Gamma^a u)^0\|_{L^2}.$$

In the case that $k_1 + |a_1| \leq \left[\frac{p^*}{2}\right]$ and $k_2 + |a_2| \leq p^*$, (2.5.1d) gives:

$$\| \, r \, |S^{k_1}\Gamma^{a_1}u| \, |\nabla S^{k_2}(\Gamma^{a_2}u)^0| \, \|_{L^2} \quad (2.10.13)$$

$$\lesssim \|rS^{k_1}\Gamma^{a_1}u\|_\infty \|\nabla S^{k_2}(\Gamma^{a_2}u)^0\|_{L^2}$$

$$\lesssim \mathcal{E}^{1/2}_{\left[\frac{p^*+4}{2}\right],\left[\frac{p^*}{2}\right]}[u](t)\|\nabla S^{k_2}(\Gamma^{a_2}u)^0\|_{L^2}$$

$$\lesssim \mathcal{E}^{1/2}_{p^*,p^*}[u](t) \sum_{k+|a|\leq p^*} \|\nabla S^{k}(\Gamma^{a}u)^0\|_{L^2}.$$

And in the case $k_2 + |a_2| \leq \left[\frac{p^*-1}{2}\right]$ and $k_1 + |a_1| \leq p^*$, again by (2.5.1d), we can estimate:

$$\begin{aligned} &\| \ r \ |S^{k_1}\Gamma^{a_1}u| \ |\nabla S^{k_2}(\Gamma^{a_2}u)^0| \ \|_{L^2} \\ &\qquad \lesssim \|S^{k_1}\Gamma^{a_1}u\|_{L^2}\|r\nabla S^{k_2}(\Gamma^{a_2}u)^0\|_\infty \\ &\qquad \lesssim \mathcal{E}^{1/2}_{p^*,p^*}[u](t) \sum_{|b|\leq 2} \|\nabla S^{k_2}(\Gamma^{a_2+b}u)^0\|_{L^2} \\ &\qquad \lesssim \mathcal{E}^{1/2}_{p^*,p^*}[u](t) \sum_{k+|a|\leq p^*} \|\nabla S^{k}(\Gamma^{a}u)^0\|_{L^2}, \end{aligned} \tag{2.10.14}$$

where in the last line we used that $k_2 + |a_2| + 2 \leq \left[\frac{p^*+3}{2}\right] \leq p^*$.

From (2.10.12), (2.10.13) and (2.10.14), we have:

$$\begin{aligned} &\delta\| \ |S^{k_1}\Gamma^{a_1}u| \ |\nabla S^{k_2}(\Gamma^{a_2}u)^0| \ |S^{k}(\Gamma^{a}u)^0| \ \|_{L^1} \\ &\qquad \lesssim \delta\mathcal{E}^{1/2}_{p^*,p^*}[u](t) \left[\sum_{k+|a|\leq p^*} \|\nabla S^{k}(\Gamma^{a}u)^0\|_{L^2}\right] \|\nabla S^{k}(\Gamma^{a}u)^0\|_{L^2} \\ &\qquad \lesssim \delta\mathcal{E}^{1/2}_{p^*,p^*}[u](t) \sum_{k+|a|\leq p^*} \|\nabla S^{k}(\Gamma^{a}u)^0\|^2_{L^2}. \end{aligned} \tag{2.10.15}$$

Therefore, from (2.10.9),(2.10.10), (2.10.11), and (2.10.15), we get:

$$\begin{aligned} I_2' \lesssim &\sum_{k+|a|\leq p^*} \int_0^T \delta \ \mathcal{E}^{1/2}_{p^*,p^*}[u](t) \ \|\nabla S^{k}(\Gamma^{a}u)^0\|^2_{L^2} \ dt \\ &+ \int_0^T \delta\langle t\rangle^{-2} \ \mathcal{E}_{p^*,p^*}[u](t)\mathcal{E}^{1/2}_{p,q}[u](t) \ dt. \end{aligned} \tag{2.10.16}$$

Inserting the estimates (2.10.16) and (2.10.8) into (2.10.6a), we find that

$$I_2 \lesssim \int_0^T \langle t\rangle^{-3/2}\, \mathcal{E}_{p^*,p^*}[u](t)\, \mathcal{E}_{p,q}^{1/2}[u](t)\, dt + \sum_{k+|a|\le p^*} \int_0^T \delta\, \mathcal{E}_{p^*,p^*}^{1/2}[u](t)\, \|\nabla S^k(\Gamma^a u)^0\|_{L^2}^2\, dt. \quad (2.10.17)$$

By Proposition 2.9.1, we get

$$\begin{aligned} I_2 &\lesssim \sup_{0\le t\le T} \left(\langle t\rangle^{-\frac{3}{2}C_1\varepsilon} \mathcal{E}_{p^*,p^*}[u](t)\mathcal{E}_{p,q}^{1/2}[u](t)\right) \int_0^T \langle t\rangle^{-\frac{3}{2}(1-C_1\varepsilon)} dt \quad (2.10.18)\\ &\quad + \delta \sup_{0\le t\le T} \mathcal{E}_{p^*,p^*}^{1/2}[u](t) \sum_{k+|a|\le p^*} \int_0^T \|\nabla S^k(\Gamma^a u)^0\|_{L^2}^2\, dt \\ &\lesssim\ \mathcal{E}_{p^*,p^*}[u_0]\mathcal{E}_{p,q}^{1/2}[u_0] + \frac{\delta}{\nu} \sup_{0\le t\le T}\ \mathcal{E}_{p^*,p^*}^{3/2}[u](t), \end{aligned}$$

provided that $C_1\varepsilon < 1/3$.

From (2.10.4), (2.10.5), and (2.10.18) we conclude that

$$\mathcal{E}_{p^*,p^*}[u](T)\ \lesssim\ \mathcal{E}_{p^*,p^*}[u_0]\left(1 + \mathcal{E}_{p,q}^{1/2}[u_0]\right) + \frac{\delta}{\nu} \sup_{0\le t\le T}\ \mathcal{E}_{p^*,p^*}^{3/2}[u](t),$$

for every $0 \le T < T_0$. Therefore, there is a constant $C_0 > 4$ such that

$$\mathcal{E}_{p^*,p^*}[u](T)\ \le\ \frac{C_0}{4}\left[\mathcal{E}_{p^*,p^*}[u_0]\left(1 + \mathcal{E}_{p,q}^{1/2}[u_0]\right) + \frac{\delta}{\nu} \sup_{0\le t\le T}\ \mathcal{E}_{p^*,p^*}^{3/2}[u](t)\right],$$

for every $0 \le T < T_0$. Denoting

$$S(T) = \sup_{0\le t\le T} \mathcal{E}_{p^*,p^*}[u](t)$$

and

$$A_0 = \frac{C_0}{4}\mathcal{E}_{p^*,p^*}[u_0]\left(1 + \mathcal{E}_{p,q}^{1/2}[u_0]\right) \quad \text{and} \quad B_0 = \frac{C_0\delta}{4\nu},$$

we have:

$$S(T) \leq A_0 + B_0 S(T)^{3/2}, \quad 0 \leq T < T_0. \tag{2.10.19}$$

Suppose now that $S(T) < 4A_0$ (note that $S(0) = \mathcal{E}_{p^*,p^*}[u_0] < 4A_0$). Then we have:

$$S(T) \leq A_0 + (4A_0)^{1/2} B_0 S(T).$$

If $(4A_0)^{1/2} B_0 < 1/2$, i.e. (2.10.2) holds, then

$$S(T) < 2A_0.$$

By continuation argument we have that $S(T) < 2A_0 < 4A_0$ for all $0 \leq T < T_0$, i.e. (2.10.3) holds. □

We now consider the situation when the condition (2.10.2) does not hold.

**Proposition 2.10.2.** *Choose* $(p, q)$ *with* $p \geq 11$ *and* $p \geq q > p^*$, *where* $p^* = \left[\frac{p+5}{2}\right]$. *Let* $\delta \leq 1$ *be defined by* (2.3.1). *Suppose that* $u \in C([0, T_0), X^{p,q})$ *is a solution of* (2.1.1a), (2.1.1b) *with*

$$\sup_{0 \leq t < T_0} \mathcal{E}_{p^*,p^*}[u](t) \leq \varepsilon^2 \ll 1.$$

*There exist constants* $C_0, C_1 > 1$ *such that if*

$$C_1 \langle T_0 \rangle^{C_1 \varepsilon} \leq \left( \frac{2 \ \max \{\nu, C_1 \varepsilon\}}{C_0 \ \delta \ \mathcal{E}^{1/2}_{p^*,p^*}[u_0]} \right)^2, \tag{2.10.20}$$

*then*

$$\sup_{0 \leq t < T_0} \mathcal{E}_{p^*,p^*}[u](t) \leq C_0 \mathcal{E}_{p^*,p^*}[u_0] \left(1 + \mathcal{E}^{1/2}_{p,q}[u_0]\right).$$

*Remark.* The constant $C_0$ may be assumed to be the same in Propositions 2.10.1 and 2.10.2. The constant $C_1$ is the one given by Proposition 2.9.1.

*Proof.* We continue with the same notation as used in the proof of Proposition 2.10.1. All of the estimates derived there up to and including (2.10.19) are valid under the current hypotheses, (note that (2.10.2) is used only in the final paragraph of the proof).

Using (2.10.19) with the energy estimate in Proposition 2.9.1, we have:

$$\begin{aligned} S(T) &\le A_0 + B_0 S(T)^{3/2} \\ &\le A_0 + B_0 (C_1 \mathcal{E}_{p^*,p^*}[u_0] \langle T_0 \rangle^{C_1 \varepsilon})^{1/2}\ S(T) \\ &\le A_0 + B_1 (C_1 \langle T_0 \rangle^{C_1 \varepsilon})^{1/2}\ S(T), \end{aligned} \tag{2.10.21}$$

where

$$B_1 = B_0 \mathcal{E}^{1/2}_{p^*,p^*}[u_0] = \frac{C_0 \delta}{4\nu} \mathcal{E}^{1/2}_{p^*,p^*}[u_0].$$

Recall the definition of $I_2'$ in (2.10.6b):

$$I_2' = \sum_{\substack{a_1+a_2+a_3=a \\ k_1+k_2=k \\ |a|+k\le p^* \\ k \le p^*}} \Bigg| \int_0^T \int_{\mathbb{R}^3} \tfrac{1}{4}\ \eta\ P_{[\widetilde{\Omega}^a, N]}(\hat{\omega})\ \langle \hat{\omega}, S^{k_1} \Gamma^{a_1} u \rangle_{\mathbb{R}^4} \\ \times\ \langle \hat{\omega}, \partial_r S^{k_2} \Gamma^{a_2} u \rangle_{\mathbb{R}^4}\ (S^k \Gamma^a u)^0\ dxdt \Bigg|.$$

The term $I_2'$ in (2.10.6b) can be estimated alternatively without using dissipation. We consider first the terms in $I_2'$ for which $k_2 + |a_2| \ne p^*$. Using (2.4.3b) and

Lemma 2.6.2, we can write:

$$\begin{aligned}
&\left| \int_0^T \int_{\mathbb{R}^3} \tfrac{1}{4}\ \eta\ P_{[\widetilde{\Omega}^a, N]}(\hat{\omega})\ \langle \hat{\omega}, S^{k_1}\Gamma^{a_1}u\rangle_{\mathbb{R}^4} \langle \hat{\omega}, \partial_r S^{k_2}\Gamma^{a_2}u\rangle_{\mathbb{R}^4}\ (S^k\Gamma^a u)^0\ dxdt \right| \\
&\qquad\qquad \lesssim \int_0^T \delta\ \|\ \eta\ |S^{k_1}\Gamma^{a_1}u|\ |\partial_r S^{k_2}\Gamma^{a_2}u|\ |S^k(\Gamma^a u)^0|\ \|_{L^1} dt \qquad (2.10.22) \\
&\qquad\qquad \lesssim \int_0^T \delta\|\ \eta |S^{k_1}\Gamma^{a_1}u|\ |\partial_r S^{k_2}\Gamma^{a_2}u|\ \|_{L^2}\ \|S^k\Gamma^a u\|_{L^2} dt \\
&\qquad\qquad \lesssim \delta \int_0^T \langle t\rangle^{-1} \mathcal{E}^{3/2}_{p^*,p^*}[u](t) dt.
\end{aligned}$$

The remaining terms in $I_2'$, i.e. the ones for which $k_2 + |a_2| = p^*$, have the form:

$$\begin{aligned}
&\sum_{\substack{k+|a|=p^* \\ k\le p^*}} \int_0^T \int_{\mathbb{R}^3} \tfrac{1}{4}\eta P_N(\hat{\omega})\langle \hat{\omega}, u\rangle_{\mathbb{R}^4}\langle \hat{\omega}, \partial_r S^k\Gamma^a u\rangle_{\mathbb{R}^4}(S^k\Gamma^a u)^0 dxdt \\
&\qquad = \sum_{\substack{k+|a|=p^* \\ k\le p^*}} \int_0^T \int_{\mathbb{R}^3} \tfrac{1}{4}\eta P_N(\hat{\omega})\hat{\omega}^\gamma\hat{\omega}^\mu\hat{\omega}^j u^\gamma \partial_j (S^k\Gamma^a u)^\mu (S^k\Gamma^a u)^0 dxdt.
\end{aligned}$$

By (2.4.1a) and (2.4.1b), $v = S^k\Gamma^a u$ satisfies $\partial_j v^\mu = \partial_\mu v^j$, so we can write:

$$\begin{aligned}
\partial_j v^\mu v^0 &= \frac{1}{2}(\partial_j v^\mu v^0 + \partial_j v^\mu v^0) \\
&= \frac{1}{2}(\partial_j v^\mu v^0 + \partial_\mu v^j v^0) \\
&= \frac{1}{2}\left[\partial_j(v^\mu v^0) + \partial_\mu(v^j v^0) - v^\mu\partial_j v^0 - v^j\partial_\mu v^0\right] \\
&= \frac{1}{2}\left[\partial_j(v^\mu v^0) + \partial_\mu(v^j v^0) - v^\mu\partial_0 v^j - v^j\partial_0 v^\mu\right] \\
&= \frac{1}{2}\left[\partial_j(v^\mu v^0) + \partial_\mu(v^j v^0) - \partial_0(v^\mu v^j)\right].
\end{aligned}$$

Thus, we have:

$$\begin{aligned}
\partial_j (S^k\Gamma^a u)^\mu (S^k\Gamma^a u)^0 &= \tfrac{1}{2}\partial_j \left[(S^k\Gamma^a u)^\mu (S^k\Gamma^a u)^0\right] \\
&\qquad + \tfrac{1}{2}\partial_\mu \left[(S^k\Gamma^a u)^j (S^k\Gamma^a u)^0\right]
\end{aligned}$$

$$- \tfrac{1}{2}\partial_0 \left[ (S^k\Gamma^a u)^\mu (S^k\Gamma^a u)^j \right].$$

Using integration by parts, we have:

$$\begin{aligned}
\int_0^T \int_{\mathbb{R}^3} \tfrac{1}{4}\eta P_N(\hat{\omega})\langle\hat{\omega}, u\rangle_{\mathbb{R}^4}\langle\hat{\omega}, \partial_r S^k\Gamma^a u\rangle_{\mathbb{R}^4}(S^k\Gamma^a u)^0 dxdt& \\
\lesssim \int_{\mathbb{R}^3} |u(T)|\, |S^k\Gamma^a u(T)|^2\, dx \ + \int_{\mathbb{R}^3} |u(0)|\, |S^k\Gamma^a u(0)|^2\, dx& \\
+ \int_0^T \int_{\mathbb{R}^3} \max_{\gamma,\mu,j} |\partial(\eta P_N(\hat{\omega})\hat{\omega}^\gamma\hat{\omega}^\mu\hat{\omega}^j)|\, |u|\, |S^k\Gamma^a u|^2\, dxdt& \\
+ \int_0^T \int_{\mathbb{R}^3} \eta\, |P_N(\hat{\omega})|\, |\partial u|\, |S^k\Gamma^a u|^2\, dxdt.&
\end{aligned}$$

Using (2.10.1), the bound $|P_N(\hat{\omega})| \le \delta$ from (2.3.1), and the fact that $\partial\eta \lesssim \langle t\rangle^{-1}$ and $|\partial\hat{\omega}| \lesssim \langle t\rangle^{-1}$, we can bound the above terms by:

$$\begin{aligned}
\varepsilon\Big[\mathcal{E}_{p^*,p^*}[u](T) + \mathcal{E}_{p^*,p^*}[u_0]\Big] + \int_0^T \langle t\rangle^{-3/2} \|r^{1/2}u\|_{L^\infty}\mathcal{E}_{p^*,p^*}[u](t)dt& \\
+ \delta \int_0^T \|\eta\partial u\|_{L^\infty}\mathcal{E}_{p^*,p^*}[u](t)dt.&
\end{aligned}$$

From the Sobolev inequalities (2.5.1c) and (2.5.2d) and the fact that $\|\eta\partial u\|_{L^\infty} \lesssim \|\eta(|\nabla u| + |\Delta u|)u\|_{L^\infty}$ by (2.9.9), the above is in turn estimated by

$$\begin{aligned}
\varepsilon\Big[\mathcal{E}_{p^*,p^*}[u](T) + \mathcal{E}_{p^*,p^*}[u_0]\Big] + \int_0^T \langle t\rangle^{-3/2}\mathcal{E}^{3/2}_{p^*,p^*}[u](t)dt& \\
+ \delta \int_0^T \langle t\rangle^{-1}\mathcal{E}^{3/2}_{p^*,p^*}[u](t)dt.&
\end{aligned}$$

Together with (2.10.22), we conclude that:

$$\begin{aligned}
I_2' \lesssim \varepsilon\Big[\mathcal{E}_{p^*,p^*}[u](T) + \mathcal{E}_{p^*,p^*}[u_0]\Big] + \int_0^T \langle t\rangle^{-3/2}\mathcal{E}^{3/2}_{p^*,p^*}[u](t)dt& \\
+ \delta \int_0^T \langle t\rangle^{-1}\mathcal{E}^{3/2}_{p^*,p^*}[u](t)dt.&
\end{aligned}$$

By Lemma 2.6.2, we have:

$$\begin{aligned}
I_2' \lesssim\; & \varepsilon\Big[\mathcal{E}_{p^*,p^*}[u](T) + \mathcal{E}_{p^*,p^*}[u_0]\Big] \\
& + (C_1\mathcal{E}_{p^*,p^*}[u_0])^{3/2} \int_0^T \langle t\rangle^{-3/2(1-C_1\varepsilon)} dt \\
& + \delta \sup_{0\le t\le T} \mathcal{E}_{p^*,p^*}[u](t)(C_1\mathcal{E}_{p^*,p^*}[u_0])^{1/2} \int_0^T \langle t\rangle^{-1+C_1\varepsilon/2} dt \\
\lesssim\; & \varepsilon\Big[\mathcal{E}_{p^*,p^*}[u](T) + \mathcal{E}_{p^*,p^*}[u_0]\Big] \\
& + (C_1\mathcal{E}_{p^*,p^*}[u_0])^{3/2} + \frac{\delta}{C_1\varepsilon} S(T)(C_1\mathcal{E}_{p^*,p^*}[u_0]\ \langle T\rangle^{C_1\varepsilon})^{1/2}.
\end{aligned}$$

From (2.9.1) and similarly to the estimates in (2.10.17) and (2.10.18), we have:

$$I_2'' \lesssim \int_0^T \langle t\rangle^{-3/2}\ \mathcal{E}_{p^*,p^*}[u](t)\ \mathcal{E}_{p,q}^{1/2}[u](t)\ dt \lesssim \mathcal{E}_{p^*,p^*}[u_0]\mathcal{E}_{p,q}^{1/2}[u_0].$$

Therefore, from (2.10.6a) we have:

$$\begin{aligned}
I_2 \lesssim I_2' + I_2'' \lesssim\; & \varepsilon\Big[\mathcal{E}_{p^*,p^*}[u](T) + \mathcal{E}_{p^*,p^*}[u_0]\Big] + \mathcal{E}_{p^*,p^*}[u_0]\mathcal{E}_{p,q}^{1/2}[u_0] \\
& + \frac{\delta}{C_1\varepsilon} S(T)(C_1\mathcal{E}_{p^*,p^*}[u_0]\ \langle T\rangle^{C_1\varepsilon})^{1/2}.
\end{aligned}$$

Combining the estimate $I_1 \lesssim \mathcal{E}_{p^*,p^*}[u_0]\mathcal{E}_{p,q}^{1/2}[u_0]$ in (2.10.5) with (2.10.4), we have:

$$\begin{aligned}
\mathcal{E}_{p^*,p^*}[u](T) \lesssim\; & \mathcal{E}_{p^*,p^*}[u_0] + I_1 + I_2 \\
\lesssim\; & \mathcal{E}_{p^*,p^*}[u_0] + \varepsilon\Big[\mathcal{E}_{p^*,p^*}[u](T) + \mathcal{E}_{p^*,p^*}[u_0]\Big] + \mathcal{E}_{p^*,p^*}[u_0]\mathcal{E}_{p,q}^{1/2}[u_0] \\
& + \frac{\delta}{C_1\varepsilon} S(T)(C_1\mathcal{E}_{p^*,p^*}[u_0]\ \langle T\rangle^{C_1\varepsilon})^{1/2}.
\end{aligned}$$

Thus, we arrive at:

$$\mathcal{E}_{p^*,p^*}[u](T) \lesssim \mathcal{E}_{p^*,p^*}[u_0] + \mathcal{E}_{p^*,p^*}[u_0]\mathcal{E}_{p,q}^{1/2}[u_0]$$

$$+\frac{\delta}{C_1\varepsilon}S(T)(C_1\mathcal{E}_{p^*,p^*}[u_0]\,\langle T\rangle^{C_1\varepsilon})^{1/2}.$$

Therefore, we have:

$$S(T)\le A_0+B_2\left(C_1\langle T_0\rangle^{C_1\varepsilon}\right)^{1/2}S(T), \tag{2.10.23}$$

with $A_0$ as above and

$$B_2=\frac{C_0\delta}{4C_1\varepsilon}\mathcal{E}^{1/2}_{p^*,p^*}[u_0].$$

From (2.10.21) and (2.10.23) we have:

$$S(T)\le A_0+\min\{B_1,B_2\}\left(C_1\langle T_0\rangle^{C_1\varepsilon}\right)^{1/2}S(T).$$

If (2.10.20) holds, i.e.

$$\min\{B_1,B_2\}(C_1\langle T_0\rangle^{C_1\varepsilon})^{1/2}\le 1/2,$$

we obtain the desired conclusion:

$$S(T)\le 2A_0\le 4A_0=C_0\mathcal{E}_{p^*,p^*}[u_0]\left(1+\mathcal{E}^{1/2}_{p,q}[u_0]\right),$$

for $0\le T<T_0$.

□

# Chapter 3

# Incompressible Hookean Viscoelasticity

## 3.1 Background and PDEs

In this section we will derive the equations of motion for an incompressible Hookean viscoelastic material in $3D$. We will assume that our viscoelastic material is distributed homogeneously in space and that, at rest, it occupies a region $\mathcal{B}$, where $\mathcal{B} \subseteq \mathbb{R}^3$ is a (possibly unbounded) subset with a smooth boundary. Each point $m = (m_1, m_2, m_3) \in \mathcal{B}$ corresponds to a point in the material. Those points are referred to as Lagrangian or material coordinates.

The motion of the viscoelastic material is modeled by a time dependent family of orientation preserving diffeomorphisms $x(t, m) : \mathbb{R}^+ \times \mathcal{B} \to \mathbb{R}^3$. Material points

$m$ in the reference configuration are deformed to a spatial point $x(t, m)$ at time $t$. The new coordinates $x = x(t, m)$ are known as Eulerian or spatial coordinates. The inverse function $m(t, x)$ returns the material point that has been deformed to the spatial coordinated $x$ at time $t$. Derivatives with respect to material and spatial coordinates are correspondingly denoted as $(D_t, D = \frac{\partial}{\partial m})$ and $(\partial_t, \nabla = \frac{\partial}{\partial x})$.

The deformation gradient of the motion in spatial coordinates is the matrix

$$F(t, x) = Dx(t, m(t, x))$$

and we have that $\det F(t, x) > 0$ because $x(t, m)$ is orientation preserving. Moreover, the incompressibility assumption implies that $\det F(t, x) = 1$. The deformation gradient also satisfies:

$$F^{\ell k} \partial_\ell F^{ij} = F^{\ell j} \partial_\ell F^{ik} \tag{3.1.1a}$$

$$\nabla \cdot F^T = 0, \tag{3.1.1b}$$

where (3.1.1a) is essentially the chain rule and (3.1.1b) follows from the incompressibility constraint and Piola's formula (see [15]).

The equations of motion for our viscoelastic material are derived from the conservation of mass and momentum balance laws:

$$\partial_t \rho + v \cdot \nabla \rho + \nabla \cdot v \rho = 0 \tag{3.1.2a}$$

$$\rho(\partial_t v + v \cdot \nabla v) - \nabla \cdot T = 0, \tag{3.1.2b}$$

where $\rho(t, x)$ is density, $v(t, x) = D_t x(t, m(t, x))$ is spatial velocity, and $T(t, x)$ is the Cauchy stress.

Assuming that the density of the material in the reference configuration is unity, i.e. $\rho_0(m) = 1$, and using the relation

$$\rho(t,x) = \frac{\rho_0(m(t,x))}{\det F(t,x)},$$

we see that the density function $\rho(t,x)$ must be unity as well (recall $\det F(t,x) = 1$). Thus, from (3.1.2a) we obtain:

$$\nabla \cdot v = 0. \tag{3.1.3}$$

The Cauchy stress tensor $T$ encodes information about the internal self-interaction of the material and it depends on an unknown pressure $p(t,x)$, on $F(t,x)$, and on $\nabla v(t,x)$, i.e.

$$T = T(p, F, \nabla v).$$

Objectivity of the material implies that $T$ depends on $\nabla v$ thought the rate of strain tensor $D = \frac{1}{2}[\nabla v + (\nabla v)^T]$ (this dependence provides information about the internal frictional forces within the material). If we further assume that the dependence of the Cauchy stress tensor on the pressure $p(t,x)$ and the rate of strain tensor $D(t,x)$ is linear, then we can write:

$$T = -pI + \nu_0 D + \widetilde{T}(F).$$

The tensor $\widetilde{T}(F)$ contains information about the elastic nature of the material. Elastic forces come from an isotropic and objective strain energy function $W(F)$. That function satisfies $W(F) = W(FU) = W(UF)$ for every proper orthogonal matrix $U$, which implies that the strain energy function depends on $F$ through the principal invariants of $FF^T$. The Hookean property of our material further implies that

$$W(F) = \frac{1}{2}\mathrm{Tr}(FF^T). \tag{3.1.4}$$

The strain energy function $W(F)$ is related to the tensor $\widetilde{T}(F)$ through the Piola-Kirchoff tensor $S(F)$, which in the incompressible case satisfies:

$$S(F) = \frac{\partial W}{\partial F} \quad \text{and} \quad \widetilde{T}(F) = S(F)F^T.$$

Therefore, simple calculations shows that

$$\widetilde{T}(F) = FF^T.$$

Altogether, the Cauchy stress tensor can be written as:

$$T = -pI + \nu_0 D + FF^T.$$

Taking the divergence of the above expression and recalling (3.1.3) and that $\rho = 1$, we can rewrite (3.1.2a) and (3.1.2b) as:

$$\nabla \cdot v = 0 \tag{3.1.5a}$$

$$\partial_t v + v \cdot \nabla v + \nabla p - \frac{\nu_0}{2}\Delta v - \nabla \cdot (FF^T) = 0. \tag{3.1.5b}$$

Furthermore, the deformation gradient $F(t, x)$ satisfies the following transport equation:

$$\partial_t F + v \cdot \nabla F - \nabla v F = 0. \tag{3.1.5c}$$

If we consider a perturbation of the deformation gradient defined as:

$$G = F - I,$$

we can rewrite (3.1.1a),(3.1.1b),(3.1.5a), (3.1.5b), and (3.1.5c) in terms of $G$. Denoting $\nu = \frac{1}{2}\nu_0$, we arrive at the following system of equations:

$$\partial_t G - \nabla v = \nabla v G - v \cdot \nabla G \tag{3.1.6a}$$

$$\partial_t v - \nabla \cdot G - \nu \Delta v = \nabla \cdot (GG^T) - v \cdot \nabla v - \nabla p \tag{3.1.6b}$$

with constraints:

$$\nabla \cdot v = 0 \tag{3.1.6c}$$

$$\nabla \cdot G^T = 0 \tag{3.1.6d}$$

$$\partial_k G^{ij} - \partial_j G^{ik} = G^{\ell j}\partial_\ell G^{ik} - G^{\ell k}\partial_\ell G^{ij} \equiv Q_k^{ij}(G, \nabla G). \tag{3.1.6e}$$

Using the following notation:

$$U = (G, v), \quad \text{where} \quad G \in \mathbb{R}^3 \otimes \mathbb{R}^3,\ v \in \mathbb{R}^3,$$

we define

$$A(\nabla)U = (\nabla v, \nabla \cdot G) \quad \text{and} \quad BU = (0, v).$$

We can then rewrite (3.1.6a) and (3.1.6b) as:

$$LU \equiv \partial_t U - A(\nabla)U - \nu B \Delta U = N(U, \nabla U) + (0, -\nabla p), \tag{3.1.7a}$$

where the nonlinearity is of the form

$$N(U, \nabla U) = (N^1(U, \nabla U), N^2(U, \nabla U)) \tag{3.1.7b}$$

with

$$N^1(U, \nabla U) = \nabla v G - v \cdot \nabla G,$$
$$N^2(U, \nabla U) = \nabla \cdot (GG^T) - v \cdot \nabla v. \tag{3.1.7c}$$

## 3.2 Notation

The vector fields that we will use are:

$$\nabla = (\partial_1, \partial_2, \partial_3), \quad \Omega = x \wedge \nabla, \quad S = t\partial_t + r\partial_r, \quad S_0 = r\partial_r.$$

Since we are working with vector-valued functions, the rotational operators $\Omega$ are correspondingly modified so that they are consistent with the rotational invariance of the linear system. The definition of the modified rotational operators depends on whether they are applied to a matrix-valued, vector-valued, or a scalar function. With a slight abuse of notation, we define $\widetilde{\Omega}$ as follows:

$$\widetilde{\Omega}_i G = \Omega_i G + [V_i, G] \quad \text{for} \quad G \in \mathbb{R}^3 \otimes \mathbb{R}^3$$
$$\widetilde{\Omega}_i v = \Omega_i v + V_i v \quad \text{for} \quad v \in \mathbb{R}^3$$
$$\widetilde{\Omega}_i f = \Omega_i f \quad \text{for} \quad f \in \mathbb{R},$$

where [ , ] denotes the commutator of two matrices and

$$V_1 = e_3 \otimes e_2 - e_2 \otimes e_3, \; V_2 = e_1 \otimes e_3 - e_3 \otimes e_1, \; V_3 = e_2 \otimes e_1 - e_1 \otimes e_2,$$

with $(e_1, e_2, e_3)$ representing the standard basis in $\mathbb{R}^3$. Furthermore, we have:

$$\widetilde{\Omega}_i U = (\Omega_i G + [V_i, G], \Omega_i v + V_i v) \quad \text{for} \quad U = (G, v).$$

The specific use of $\widetilde{\Omega}$ will be clear from context.

We will again rely on the decomposition:

$$\nabla = \omega\partial_r - \frac{\omega}{r} \wedge \Omega. \tag{3.2.1}$$

For a more concise notation we define:

$$\Gamma = \{\nabla, \widetilde{\Omega}\}.$$

The scaling operators $S$ and $S_0$ are not included in $\Gamma$ because they do not commute with linear part of (3.1.6a) and (3.1.6b). Their occurrence will be tracked individually as evident in the following definition of the solution space:

$$X^{p,q} = \big\{U = (G, v) : \mathbb{R}^3 \to (\mathbb{R}^3 \otimes \mathbb{R}^3) \times \mathbb{R}^3 \mid \\ \|S_0^k \Gamma^a U\|_{L^2} < \infty \text{ for all } |a| + k \leq p, k \leq q\big\},$$

for integers $0 \leq q \leq p$. This is a Hilbert space with inner product:

$$\langle U_1, U_2 \rangle_{X^{p,q}} = \sum_{\substack{|a|+k\leq p \\ k\leq q}} \langle S_0^k \Gamma^a U_1, S_0^k \Gamma^a U_2 \rangle_{L^2}.$$

Once again, $p$ indicates the total number of derivatives taken, while $q$ indicates the number of occurrences of $S_0$.

The energy associated with a solution $U = (G, v)$ of the PDEs (3.1.6a) and (3.1.6b) with constrains (3.1.6c) - (3.1.6e) is given by

$$\mathcal{E}_{p,q}[U](t) = \sum_{\substack{|a|+k\leq p \\ k\leq q}} \left[\frac{1}{2}\|S^k \Gamma^a U(t)\|_{L^2}^2 + \nu \int_0^t \|\nabla S^k \Gamma^a v(s)\|_{L^2}^2 \, ds\right].$$

If $U(0) = U_0$, then the energy at time $t = 0$ will be denoted as:

$$\mathcal{E}_{p,q}[U_0] \equiv \mathcal{E}_{p,q}[U](0) = \frac{1}{2}\|U_0\|^2_{X^{p,q}}.$$

As in the damped wave equation case, we use cut-off functions to define two time-space regions, referred to as interior and exterior. For completeness, we list again those functions and some of their properties. Denote

$$\zeta(t,x) = \psi\left(\frac{|x|}{\sigma\langle t\rangle}\right) \quad \text{and} \quad \eta(t,x) = 1 - \psi\left(\frac{2|x|}{\sigma\langle t\rangle}\right), \tag{3.2.2a}$$

where

$$\psi \in C^\infty(\mathbb{R}), \quad \psi(s) = \begin{cases} 1, & s \le 1/2 \\ \\ 0, & s \ge 1 \end{cases}, \quad \psi' \le 0.$$

The parameter $\sigma$ in the definition of $\zeta$ and $\eta$ will be later chosen to be sufficiently small. The cut-off functions satisfy the following relations:

$$1 \le \zeta + \eta \quad \text{and} \quad 1 - \eta \le \zeta^2 \tag{3.2.2b}$$

and

$$\langle r + t\rangle \Big[|\partial\zeta(t,x)| + |\partial\eta(t,x)|\Big] \lesssim 1.$$

In the interior region, we will provide estimates for

$$\mathcal{Y}^{\text{int}}_{p,q}[U](t) = \sum_{\substack{|a|+k\le p-1 \\ k\le q}} \|\zeta\nabla S^k\Gamma^a U(t)\|^2_{L^2}$$

and

$$\mathcal{Z}^{\text{int}}_{p,q}[U](t) = \sum_{\substack{|a|+k\le p-1 \\ k\le q}} \|\zeta\Delta S^k\Gamma^a v(t)\|^2_{L^2},$$

for $q < p$.

In the exterior region, however, no such weighted $L^2$-estimates are needed. The special Hookean structure of the quadratic nonlinearity (see (3.1.4)) and the incompressible constraints (3.1.6c) and (3.1.6d) provide the decay needed through the Sobolev inequality (3.5.2).

## 3.3 Main Results

**Theorem 3.3.1** (Global existence)**.** *Choose* $(p, q)$ *such that* $p \geq 11$*, and* $p \geq q > p^*$*, where* $p^* = \left[\frac{p+5}{2}\right]$.

*There are positive constants* $C_0,\ C_1 > 1$ *with the property that if the initial data* $U_0$ *satisfies*

$$C_0 \mathcal{E}_{p^*,p^*}[U_0] \left(1 + \mathcal{E}_{p,q}^{1/2}[U_0]\right) < \varepsilon^2,$$

*for some* $\varepsilon^2 \ll 1$ *then* (3.1.6a)-(3.1.6e) *has a unique global solution*

$$U \in C(\mathbb{R}^+; X^{p,q})$$

*with*

$$\sup_{0 \leq t < \infty} \mathcal{E}_{p,q}[U](t) \leq C_1 \mathcal{E}_{p,q}[U_0] \langle t \rangle^{C_1 \varepsilon}$$

*and*

$$\sup_{0 \leq t < \infty} \mathcal{E}_{p^*,p^*}[U](t) < \varepsilon^2.$$

*Outline of Proof.* We follow the same strategy as in the proof of Theorem 2.3.1. □

## 3.4 Commutation

The linear operator defined in (3.1.7a):

$$L = \partial_t - A(\nabla) - \nu B \Delta$$

commutes with $\Gamma$, i.e. we have that

$$L \Gamma^a U = \Gamma^a L U$$

for any multi index $a$. Here we emphasize that each $\widetilde{\Omega}_i$ satisfies the following commutation properties:

$$\nabla(\Omega_i f) = \widetilde{\Omega}_i(\nabla f) \qquad \nabla \cdot (\widetilde{\Omega}_i v) = \Omega_i(\nabla \cdot v)$$

$$\nabla(\widetilde{\Omega}_i v) = \widetilde{\Omega}_i(\nabla v) \qquad \nabla \cdot (\widetilde{\Omega}_i G) = \widetilde{\Omega}_i(\nabla \cdot G)$$

for functions $f \in \mathbb{R}^3$, $v \in \mathbb{R}^3$, and $G \in \mathbb{R}^3 \otimes \mathbb{R}^3$.

The scaling operator $S$, however, does not commute with $L$. From the following commutation properties of $S$:

$$\begin{aligned} \partial S^k f &= (S+1)^k \partial f \\ \Delta S^k f &= (S+2)^k \Delta f, \end{aligned} \tag{3.4.1}$$

for $k \geq 0$ any integer, we can show that

$$L S^k U = (S+1)^k L U - \sum_{j=0}^{k-1} (-1)^{k-j} \binom{k}{j} \nu B \Delta S^j U.$$

Altogether, the linear operator satisfies:

$$LS^k\Gamma^a U = (S+1)^k\Gamma^a LU - \sum_{j=0}^{k-1}(-1)^{k-j}\binom{k}{j}\nu B\Delta S^j\Gamma^a U \tag{3.4.2a}$$

with the following vector field version of the constraints (3.1.6c), (3.1.6d), and (3.1.6e):

$$\nabla\cdot S^k\Gamma^a v = 0 \tag{3.4.2b}$$

$$\nabla\cdot (S^k\Gamma^a G)^T = 0 \tag{3.4.2c}$$

$$\begin{aligned}\partial_k(S^k\Gamma^a G)^{ij} - \partial_j(S^k\Gamma^a G)^{ik} &= \sum_{\substack{a_1+a_2=a\\ k_1+k_2=k}} \frac{a!}{a_1!\;a_2!}\frac{k!}{k_1!\;k_2!}Q_k^{ij}(S^{k_1}\Gamma^{a_1}G, \nabla S^{k_2}\Gamma^{a_2}G)\\ &\equiv \widetilde{Q}_k^{ij}(G,\nabla G).\end{aligned} \tag{3.4.2d}$$

The vector field constraints (3.4.2b) and (3.4.2c) are essentially a consequence of the commutation properties of the vector fields together with:

$$(\widetilde{\Omega}_i G)^T = \widetilde{\Omega}_i G^T.$$

As for (3.4.2d), we emphasize that $\widetilde{\Omega}_i$'s satisfy:

$$\begin{aligned}\partial_k(\widetilde{\Omega}_i G)^{ij} - \partial_j(\widetilde{\Omega}_i G)^{ik} \;&=\; (\widetilde{\Omega}_i G)^{\ell j}\partial_\ell G^{ik} + G^{\ell j}\partial_\ell(\widetilde{\Omega}_i G)^{ik}\\ &\quad - (\widetilde{\Omega}_i G)^{\ell k}\partial_\ell G^{ij} - G^{\ell k}\partial_\ell(\widetilde{\Omega}_i G)^{ij}.\end{aligned}$$

Also note that the rotational operators $\widetilde{\Omega}_i$ distribute across the nonlinear terms as follows:

$$\widetilde{\Omega}_i N^1(U,\nabla U) \;=\; \nabla(\widetilde{\Omega}_i v)G + \nabla v(\widetilde{\Omega}_i G) - (\widetilde{\Omega}_i v)\cdot\nabla G - v\cdot\nabla(\widetilde{\Omega}_i G)$$

$$\widetilde{\Omega}_i N^2(U,\nabla U) \;=\; \nabla\cdot\Big((\widetilde{\Omega}_i G)G^T + G(\widetilde{\Omega}_i G)^T\Big) - (\widetilde{\Omega}_i v)\cdot\nabla v - v\cdot\nabla(\widetilde{\Omega}_i v).$$

Together with (3.4.1), we have that the vector fields distribute across the nonlinear terms according to the following Leibnitz-type formula:

$$(S+1)^k\Gamma^a N(U,\nabla U) = \sum_{\substack{a_1+a_2=a\\ k_1+k_2=k}} \frac{a!}{a_1!\,a_2!}\frac{k!}{k_1!\,k_2!} N(S^{k_1}\Gamma^{a_1}U, \nabla S^{k_2}\Gamma^{a_2}U).$$

## 3.5 Sobolev Inequalities

**Lemma 3.5.1.** *Suppose that $U\in X^{2,0}$. Set $r=|x|$. Then*

$$\begin{aligned}
\|U\|_{L^\infty} &\lesssim \sum_{|a|\le 2}\|\nabla^a U\|_{L^2}\\
\|r^{-1}U\|_{L^2} &\lesssim \|\partial_r U\|_{L^2}\\
\|r^{1/2}U\|_{L^\infty} &\lesssim \sum_{|a|\le 1}\|\nabla\widetilde{\Omega}^a U\|_{L^2}\\
\|rU\|_{L^\infty} &\lesssim \left(\sum_{|a|\le 1}\|\partial_r\widetilde{\Omega}^a U\|_{L^2(|y|\ge r)}\sum_{|a|\le 2}\|\widetilde{\Omega}^a U\|_{L^2(|y|\ge r)}\right)^{1/2} \qquad (3.5.1a)
\end{aligned}$$

*Proof.* Those inequalities are equivalent to the ones in Lemma 2.5.1. □

**Proposition 3.5.2.** *Suppose that $U:[0,T)\times\mathbb{R}^3\to(\mathbb{R}^3\otimes\mathbb{R}^3)\times\mathbb{R}^3$ satisfies*

$$\mathcal{Y}^{int}_{2,0}[U](t) + \mathcal{E}_{2,0}[U](t) < \infty.$$

*Then using the weights* (3.2.2a), *we have*

$$\|\,\zeta\,U(t)\,\|_{L^\infty} \;\lesssim\; \big(\mathcal{Y}^{int}_{2,0}[U](t)\big)^{1/2} + \langle t\rangle^{-1}\mathcal{E}^{1/2}_{1,0}[U](t)$$

$$\| \, r\zeta \, \nabla U(t) \, \|_{L^\infty} \lesssim \left(\mathcal{Y}^{int}_{3,0}[U](t)\right)^{1/2} + \langle t\rangle^{-1}\mathcal{E}^{1/2}_{2,0}[U](t)$$

$$\| \, r^{-1}\zeta \, U(t) \, \|_{L^2} \lesssim \left(\mathcal{Y}^{int}_{1,0}[U](t)\right)^{1/2} + \langle t\rangle^{-1}\mathcal{E}^{1/2}_{0,0}[U](t)$$

$$\| \, \eta \, U(t)\|_{L^\infty} \lesssim \langle t\rangle^{-1}\mathcal{E}^{1/2}_{2,0}[U](t).$$

*Also, for $u = v$ or $u = G^{*j}$, i.e the $j^{th}$ column of $G$, $j = 1, 2, 3$, and $\omega = \frac{x}{|x|}$, we have*

$$\| \, \eta \, \omega \cdot u(t) \, \|_{L^\infty} \lesssim \langle t\rangle^{-3/2}\mathcal{E}^{1/2}_{2,0}[U](t). \tag{3.5.2}$$

*Proof.* We only need to show (3.5.2). We first prove the following generalized version of (3.5.1a):

$$\|u\|_{L^\infty} \lesssim \left(\sum_{|a|\le 1} \| \, |y|^{-\lambda}\partial_r\widetilde{\Omega}^a u\|_{L^2(|y|\ge r)} \times \sum_{|a|\le 2} \| \, |y|^{\lambda-2}\widetilde{\Omega}^a u\|_{L^2(|y|\ge r)}\right)^{1/2}. \tag{3.5.3}$$

We start with:

$$\begin{aligned} \|u(r\omega)\|^4_{L^4(S^2)} &= \int_{S^2} |u(r\omega)|^4 d\omega \\ &\lesssim \int_r^\infty \int_{S^2} |\partial_r u(\rho\omega)| \, |u(\rho\omega)|^3 d\rho d\omega \\ &\lesssim \int_{|y|\ge r} |y|^{-2}|\partial_r u(y)| \, |u(y)|^3 dy \\ &\lesssim \left(\int_{|y|\ge r} |y|^{-2\lambda}|\partial_r u(y)|^2 dy\right)^{1/2} \left(\int_{|y|\ge r} |y|^{2(\lambda-2)}|u(y)|^6 dy\right)^{1/2}. \end{aligned} \tag{3.5.4}$$

We can bound the second term on the right of (3.5.4) by:

$$\begin{aligned} \int_{|y|\ge r} |y|^{2(\lambda-2)}|u(y)|^6 dy &= \int_r^\infty \int_{S^2} \rho^{2(\lambda-2)+2}|u(\rho\omega)|^6 d\omega d\rho \\ &\lesssim \int_r^\infty \rho^{2(\lambda-2)+2}\|u(\rho\omega)\|^6_{L^6(S^2)} d\rho. \end{aligned} \tag{3.5.5}$$

An application of Gagliardo-Nirenberg

$$\|u(r\omega)\|_{L^6(S^2)} \lesssim \sum_{|\alpha|\le 1} \|\widetilde{\Omega}^\alpha u(r\omega)\|_{L^2(S^2)}^{1/3} \|u(r\omega)\|_{L^4(S^2)}^{2/3}$$

to the last term in (3.5.5) gives:

$$\begin{aligned}
&\int_{|y|\ge r} |y|^{2(\lambda-2)} |u(y)|^6 dy \qquad (3.5.6)\\
&\lesssim \int_r^\infty \rho^{2(\lambda-2)+2} \left( \sum_{|\alpha|\le 1} \|\widetilde{\Omega}^\alpha u(\rho\omega)\|_{L^2(S^2)}^2 \right) \|u(\rho\omega)\|_{L^4(S^2)}^4 d\rho\\
&\lesssim \sup_{\rho\ge r} \|u(\rho\omega)\|_{L^4(S^2)}^4 \sum_{|\alpha|\le 1} \int_r^\infty \int_{S^2} \rho^{2(\lambda-2)+2} |\widetilde{\Omega}^\alpha u(\rho\omega)|^2 d\omega d\rho\\
&\lesssim \sup_{\rho\ge r} \|u(\rho\omega)\|_{L^4(S^2)}^4 \sum_{|\alpha|\le 1} \int_{|y|\ge r} |y|^{2(\lambda-2)} |\widetilde{\Omega}^\alpha u(y)|^2 dy.
\end{aligned}$$

Putting (3.5.4) and (3.5.6) together, we get:

$$\begin{aligned}
\|u(r\omega)\|_{L^4(S^2)}^2 \lesssim &\left( \int_{|y|\ge r} |y|^{-2\lambda} |\partial_r u(y)|^2 dy \right)^{1/2}\\
&\times \left( \sum_{|\alpha|\le 1} \int_{|y|\ge r} |y|^{2(\lambda-2)} |\widetilde{\Omega}^\alpha u(y)|^2 dy \right)^{1/2}. \qquad (3.5.7)
\end{aligned}$$

Combining (3.5.7) with the isoperimetric Sobolev inequality

$$|u(x)| \lesssim \sum_{|\alpha|\le 1} \|\widetilde{\Omega}^\alpha u(r\omega)\|_{L^4(S^2)}$$

gives (3.5.3).

To prove (3.5.2), we observe that on the support of the cut-off function $\eta$, we have

$$\langle t\rangle^{3/2} \|\eta\, \omega \cdot u(t)\|_{L^\infty} \lesssim \|r^{3/2} \eta\, \omega \cdot u(t)\|_{L^\infty} \lesssim \|r^{3/2} \omega \cdot u(t)\|_{L^\infty}. \qquad (3.5.8)$$

Then, from (3.5.3) with $\lambda = \frac{1}{2}$ we get:

$$\|r^{3/2}\omega \cdot u(t)\|_{L^\infty} \tag{3.5.9}$$
$$\lesssim \left( \sum_{|a|\le 1} \| \, |y|^{-1/2} \partial_r \widetilde{\Omega}^a \left( |y|^{3/2} \omega \cdot u(t) \right) \|_{L^2} \sum_{|a|\le 2} \| \, |y|^{-3/2} \widetilde{\Omega}^a \left( |y|^{3/2} \omega \cdot u(t) \right) \|_{L^2} \right)^{1/2}$$

$$\lesssim \sum_{|a|\le 1} \| \, |y|^{-1/2} \partial_r \left( |y|^{3/2} \omega \cdot \widetilde{\Omega}^a u(t) \right) \|_{L^2} \; + \sum_{|a|\le 2} \|\omega \cdot \widetilde{\Omega}^a u(t)\|_{L^2}$$
$$\lesssim \sum_{|a|\le 1} \| \, |y| \; \omega \cdot \partial_r \widetilde{\Omega}^a u(t)\|_{L^2} \; + \sum_{|a|\le 2} \|\widetilde{\Omega}^a u(t)\|_{L^2}.$$

Using the the gradient decomposition (3.2.1) and the constraint (3.1.6c), we have the following bound for the first term on the right of (3.5.9):

$$\sum_{|a|\le 1} \| \, |y| \; \omega \cdot \partial_r \widetilde{\Omega}^a u(t)\|_{L^2} \tag{3.5.10}$$
$$\lesssim \sum_{|a|\le 1} \| \, |y| \; \nabla \cdot \widetilde{\Omega}^a u(t)\|_{L^2} \; + \sum_{|a|\le 1} \|(\omega \wedge \Omega) \cdot \widetilde{\Omega}^a u(t)\|_{L^2}$$
$$\lesssim \sum_{|a|\le 2} \|\widetilde{\Omega}^a u(t)\|_{L^2}.$$

The result (3.5.2) follows from (3.5.8), (3.5.9), and (3.5.10).

□

## 3.6 Calculus Inequalities

**Lemma 3.6.1.** *Suppose that* $U : [0, T) \times \mathbb{R}^3 \to (\mathbb{R}^3 \otimes \mathbb{R}^3) \times \mathbb{R}^3$. *If*

$$k_1 + k_2 + |a_1| + |a_2| \le \bar{p} \quad \textit{and} \quad k_1 + k_2 \le \bar{q},$$

*then we have*

$$\| \zeta |S^{k_1}\Gamma^{a_1}U(t)|\ |S^{k_2}\Gamma^{a_2+1}U(t)|\ \|_{L^2} \lesssim \left( \left( \mathcal{Y}^{int}_{[\frac{\bar{p}+5}{2}],[\frac{\bar{p}}{2}]}[U](t) \right)^{1/2} + \langle t \rangle^{-1} \mathcal{E}^{1/2}_{[\frac{\bar{p}+3}{2}],[\frac{\bar{p}}{2}]}[U](t) \right) \mathcal{E}^{1/2}_{\bar{p}+1,\bar{q}}[U](t),$$

*provided the right-hand side is finite.*

*In the special case when $k_2 + |a_2| < \bar{p}$, we have*

$$\| \zeta |S^{k_1}\Gamma^{a_1}U(t)|\ |S^{k_2}\Gamma^{a_2+1}U(t)|\ \|_{L^2} \lesssim \left( \left( \mathcal{Y}^{int}_{[\frac{\bar{p}+5}{2}],[\frac{\bar{p}}{2}]}[U](t) \right)^{1/2} + \langle t \rangle^{-1} \mathcal{E}^{1/2}_{[\frac{\bar{p}+3}{2}],[\frac{\bar{p}}{2}]}[U](t) \right) \mathcal{E}^{1/2}_{\bar{p},\bar{q}}[U](t),$$

*provided the right-hand side is finite.*

**Lemma 3.6.2.** *Suppose that $U : [0, T) \times \mathbb{R}^3 \to (\mathbb{R}^3 \otimes \mathbb{R}^3) \times \mathbb{R}^3$. If*

$$k_1 + k_2 + |a_1| + |a_2| \leq \bar{p} \quad \text{and} \quad k_1 + k_2 \leq \bar{q},$$

*then we have*

$$\| \eta |S^{k_1}\Gamma^{a_1}U(t)|\ |S^{k_2}\Gamma^{a_2+1}U(t)|\ \|_{L^2} \lesssim \langle t \rangle^{-1} \mathcal{E}^{1/2}_{[\frac{\bar{p}+5}{2}],[\frac{\bar{p}}{2}]}[U](t) \mathcal{E}^{1/2}_{\bar{p}+1,\bar{q}}[U](t),$$

*provided the right-hand side is finite.*

*In the special case when $k_2 + |a_2| < \bar{p}$, we have*

$$\| \eta |S^{k_1}\Gamma^{a_1}U(t)|\ |S^{k_2}\Gamma^{a_2+1}U(t)|\ \|_{L^2} \lesssim \langle t \rangle^{-1} \mathcal{E}^{1/2}_{[\frac{\bar{p}+5}{2}],[\frac{\bar{p}}{2}]}[U](t) \mathcal{E}^{1/2}_{\bar{p},\bar{q}}[U](t),$$

*provided the right-hand side is finite.*

The above two lemmas have similar proofs to those of Lemma 2.6.1 and Lemma 2.6.2.

## 3.7 Estimates of the Linear Equation

Consider the linear version of the system (3.1.6a) and (3.1.6b):

$$\partial_t G - \nabla v = H \tag{3.7.1a}$$

$$\partial_t v - \nabla \cdot G - \nu \Delta v = h \tag{3.7.1b}$$

with constraint (3.1.6e):

$$\partial_k G^{ij} - \partial_j G^{ik} = Q^{ij}_k. \tag{3.7.1c}$$

In this section we provide estimates of the linear system (3.7.1a)-(3.7.1c).

**Lemma 3.7.1.** *Assume that $\sigma$ in (3.2.2a) is sufficiently small and that $\nu \le 1$. Let $H$, $\nabla \cdot H$, $Q$, and $h \in L^2([0,T]; L^2(\mathbb{R}^3))$, for some $0 < T < \infty$. If $U = (G, v)$ is a solution of (3.7.1a)-(3.7.1c) such that*

$$\sup_{0 \le t \le T} \mathcal{E}_{1,1}[U](t) < \infty,$$

*then for any $0 \le \theta \le 1$,*

$$\begin{aligned}
&\int_0^T \langle t \rangle^\theta \left[ \|\zeta \nabla U\|_{L^2}^2 + \nu^2 \|\zeta \Delta v\|_{L^2}^2 \right] dt \\
&\qquad \lesssim \nu \langle T \rangle^{\theta - 2} \mathcal{E}_{1,0}[U](T) + \int_0^T \langle t \rangle^{\theta-2} \mathcal{E}_{1,1}[U](t) dt \\
&\qquad\quad + \int_0^T \langle t \rangle^\theta \left[ 2\nu \langle \zeta \nabla \cdot G, \zeta \nabla \cdot H \rangle_{L^2} \right. \\
&\qquad\qquad \left. + \|\zeta Q\|_{L^2}^2 + \|\zeta H\|_{L^2}^2 + \|\zeta h\|_{L^2}^2 \right] dt.
\end{aligned}$$

*Proof.* Multiplying (3.7.1a) and (3.7.1b) by $t$ and using that $S = t\partial_t + r\partial_r$, we have that

$$t\nabla v = -r\partial_r G + SG - tH$$

$$t\nabla \cdot G + t\nu\Delta v = -r\partial_r v + Sv - th.$$

Next, we multiply each equation by $\zeta$ and take the $L^2$-inner product:

$$\|\zeta t\nabla v\|_{L^2}^2 \leq \|\zeta r\partial_r G\|_{L^2}^2 + \|\zeta SG\|_{L^2}^2 + \|\zeta tH\|_{L^2}^2$$

$$\|\zeta t\nabla \cdot G\|_{L^2}^2 + \nu^2\|\zeta t\Delta v\|_{L^2}^2 + 2\langle \zeta t\nabla \cdot G, \zeta t\nu\Delta v\rangle_{L^2}$$

$$\leq \|\zeta r\partial_r v\|_{L^2}^2 + \|\zeta Sv\|_{L^2}^2 + \|\zeta th\|_{L^2}^2.$$

Adding the two inequalities, we obtain:

$$t^2\left[\|\zeta\nabla \cdot G\|_{L^2}^2 + 2\nu\langle \zeta\nabla \cdot G, \zeta\Delta v\rangle_{L^2} + \nu^2\|\zeta\Delta v\|_{L^2}^2 + \|\zeta\nabla v\|_{L^2}^2\right]$$

$$\leq \|\zeta r\partial_r U\|_{L^2}^2 + \mathcal{E}_{1,1}[U](t) + t^2(\|\zeta H\|_{L^2}^2 + \|\zeta h\|_{L^2}^2), \quad (3.7.2)$$

where $U = (G, v)$.

Taking the divergence of (3.7.1a), we have:

$$\Delta v = \partial_t\nabla \cdot G - \nabla \cdot H$$

and so we can write the inner product as:

$$2\nu\langle \zeta\nabla \cdot G, \zeta\Delta v\rangle_{L^2} \;=\; 2\nu\langle \zeta\nabla \cdot G, \zeta\partial_t\nabla \cdot G\rangle_{L^2} - 2\nu\langle \zeta\nabla \cdot G, \zeta\nabla \cdot H\rangle_{L^2}. \quad (3.7.3)$$

The first term can be bounded as follows:

$$2\nu\langle \zeta\nabla \cdot G, \zeta\partial_t\nabla \cdot G\rangle_{L^2} \quad (3.7.4)$$

$$
\begin{aligned}
&= \nu \int_{\mathbb{R}^3} \zeta^2 \partial_t |\nabla \cdot G|^2 dx \\
&\geq \nu \partial_t \|\zeta \nabla \cdot G\|_{L^2}^2 - C\nu \int_{\mathbb{R}^3} \zeta \langle t \rangle^{-1} |\nabla \cdot G|^2 dx \\
&\geq \nu \partial_t \|\zeta \nabla \cdot G\|_{L^2}^2 - \frac{1}{2} \|\zeta \nabla \cdot G\|_{L^2}^2 - 2C^2 \nu^2 \langle t \rangle^{-2} \|\nabla \cdot G\|_{L^2}^2,
\end{aligned}
$$

where we have used Young's inequality and the fact that $\partial_t \zeta^2 \leq C\zeta \langle t \rangle^{-1}$, for some constant $C$. Inserting (3.7.3) and (3.7.4) into (3.7.2), we have:

$$
\begin{aligned}
t^2 &\left[ \nu \partial_t \|\zeta \nabla \cdot G\|_{L^2}^2 + \frac{1}{2} \|\zeta \nabla \cdot G\|_{L^2}^2 + \nu^2 \|\zeta \Delta v\|_{L^2}^2 + \|\zeta \nabla v\|_{L^2}^2 \right] \\
&\lesssim \|\zeta r \partial_r U\|_{L^2}^2 + \mathcal{E}_{1,1}[U](t) + 2\nu t^2 \langle \zeta \nabla \cdot G, \zeta \nabla \cdot H \rangle_{L^2} \\
&\quad + t^2 (\|\zeta H\|_{L^2}^2 + \|\zeta h\|_{L^2}^2). \quad (3.7.5)
\end{aligned}
$$

Choosing $0 \leq \theta \leq 1$, we multiply (3.7.5) by $\langle t \rangle^{\theta - 2}$, and then we integrate:

$$
\begin{aligned}
\int_0^T & t^2 \langle t \rangle^{\theta-2} \left[ \frac{1}{2} \|\zeta \nabla \cdot G\|_{L^2}^2 + \nu^2 \|\zeta \Delta v\|_{L^2}^2 + \|\zeta \nabla v\|_{L^2}^2 \right] dt \\
&\lesssim \int_0^T \langle t \rangle^{\theta-2} \left[ \|\zeta r \partial_r U\|_{L^2}^2 + \mathcal{E}_{1,1}[U](t) + 2\nu t^2 \langle \zeta \nabla \cdot G, \zeta \nabla \cdot H \rangle_{L^2} \right. \\
&\quad \left. + t^2 (\|\zeta H\|_{L^2}^2 + \|\zeta h\|_{L^2}^2) \right] dt \\
&\quad - \int_0^T t^2 \langle t \rangle^{\theta-2} \nu \partial_t \|\zeta \nabla \cdot G\|_{L^2}^2 dt. \quad (3.7.6)
\end{aligned}
$$

Next, we estimate the time-derivative term on the right hand side. We have shown in (2.7.3) that

$$
\partial_t (\nu t^2 \langle t \rangle^{\theta-2}) \leq \frac{1}{4} t^2 \langle t \rangle^{\theta-2} + C\nu^2 \langle t \rangle^{\theta-2},
$$

so, by integration by parts, we have:

$$
- \int_0^T t^2 \langle t \rangle^{\theta-2} \nu \partial_t \|\zeta \nabla \cdot G\|_{L^2}^2 dt
$$

$$\leq \int_0^T \left(\frac{1}{4}t^2\langle t\rangle^{\theta-2} + C\nu^2\langle t\rangle^{\theta-2}\right)\|\zeta\nabla\cdot G\|_{L^2}^2 dt$$
$$\leq \frac{1}{4}\int_0^T t^2\langle t\rangle^{\theta-2}\|\zeta\nabla\cdot G\|_{L^2}^2 dt + C\int_0^T \langle t\rangle^{\theta-2}\mathcal{E}_{1,0}[U](t)dt.$$

Substitution in (3.7.6) gives:

$$\int_0^T t^2\langle t\rangle^{\theta-2}\left[\frac{1}{4}\|\zeta\nabla\cdot G\|_{L^2}^2 + \nu^2\|\zeta\Delta v\|_{L^2}^2 + \|\zeta\nabla v\|_{L^2}^2\right]dt$$
$$\lesssim \int_0^T \langle t\rangle^{\theta-2}\left[\|\zeta r\partial_r U\|_{L^2}^2 + \mathcal{E}_{1,1}[U](t) + 2\nu t^2\langle\zeta\nabla\cdot G, \zeta\nabla\cdot H\rangle_{L^2}\right.$$
$$\left. + t^2(\|\zeta H\|_{L^2}^2 + \|\zeta h\|_{L^2}^2)\right]dt.$$

By Lemma 3.7.2, we can control the divergence term on the left-hand side by the full gradient:

$$\int_0^T t^2\langle t\rangle^{\theta-2}\left[\|\zeta\nabla U\|_{L^2}^2 + \nu^2\|\zeta\Delta v\|_{L^2}^2\right]dt$$
$$\lesssim \int_0^T \langle t\rangle^{\theta-2}\left[\|\zeta r\partial_r U\|_{L^2}^2 + \mathcal{E}_{1,1}[U](t) + 2\nu t^2\langle\zeta\nabla\cdot G, \zeta\nabla\cdot H\rangle_{L^2}\right.$$
$$\left. + t^2(\|\zeta Q\|_{L^2}^2 + \|\zeta H\|_{L^2}^2 + \|\zeta h\|_{L^2}^2)\right]dt.$$

Using that $t^2 = \langle t\rangle^2 - 1$, we can write:

$$\int_0^T \langle t\rangle^{\theta}\left[\|\zeta\nabla U\|_{L^2}^2 + \nu^2\|\zeta\Delta v\|_{L^2}^2\right]dt$$
$$\lesssim \int_0^T \langle t\rangle^{\theta-2}\left[\|\zeta r\partial_r U\|_{L^2}^2 + \mathcal{E}_{1,1}[U](t) + \nu^2\|\zeta\Delta v\|_{L^2}^2\right.$$
$$+ 2\nu t^2\langle\zeta\nabla\cdot G, \zeta\nabla\cdot H\rangle_{L^2}$$
$$\left. + t^2(\|\zeta Q\|_{L^2}^2 + \|\zeta H\|_{L^2}^2 + \|\zeta h\|_{L^2}^2)\right]dt.$$

We have that $r \leq \sigma\langle t\rangle$ on the support of $\zeta$. Thus, we have the following estimate:

$$\int_0^T \langle t\rangle^{\theta-2} \|\zeta r \partial_r U\|_{L^2}^2 dt \;\leq\; \int_0^T \sigma^2 \langle t\rangle^{\theta} \|\zeta \partial_r U\|_{L^2}^2 dt.$$

For small enough $\sigma$ the last term can be absorbed on the left-hand side of the main inequality and thus, we obtain:

$$\begin{aligned}
&\int_0^T \langle t\rangle^{\theta} \left[\|\zeta\nabla U\|_{L^2}^2 + \nu^2\|\zeta\Delta v\|_{L^2}^2\right] dt \\
&\qquad\lesssim \int_0^T \langle t\rangle^{\theta-2} \left[\mathcal{E}_{1,1}[U](t) + \nu^2\|\zeta\Delta v\|_{L^2}^2 + 2\nu t^2\langle \zeta\nabla\cdot G, \zeta\nabla\cdot H\rangle_{L^2}\right. \\
&\qquad\qquad\left. + \, t^2(\|\zeta Q\|_{L^2}^2 + \|\zeta H\|_{L^2}^2 + \|\zeta h\|_{L^2}^2)\right] dt. \qquad (3.7.7)
\end{aligned}$$

The Laplacian term on the right has the following bound:

$$\begin{aligned}
\int_0^T \langle t\rangle^{\theta-2}\nu^2\|\zeta\Delta v\|_{L^2}^2 dt \;&=\; \int_0^T \langle t\rangle^{\theta-2}\nu^2 \frac{d}{dt}\int_0^t \|\zeta\Delta v\|_{L^2}^2 ds dt \qquad (3.7.8)\\
&= \langle T\rangle^{\theta-2}\nu^2 \int_0^T \|\zeta\Delta v\|_{L^2}^2 dt \;+\; \int_0^T (2-\theta)t\langle t\rangle^{\theta-4}\nu^2 \int_0^t \|\zeta\Delta v\|_{L^2}^2 ds dt \\
&\lesssim \nu\langle T\rangle^{\theta-2}\mathcal{E}_{1,0}[U](T) \;+\; \nu\int_0^T \langle t\rangle^{\theta-2}\mathcal{E}_{1,0}[U](t) dt.
\end{aligned}$$

Substituting into (3.7.7) and using that $\nu \leq 1$, we arrive at:

$$\begin{aligned}
&\int_0^T \langle t\rangle^{\theta} \left[\|\zeta\nabla U\|_{L^2}^2 + \nu^2\|\zeta\Delta v\|_{L^2}^2\right] dt \\
&\qquad\lesssim \nu\langle T\rangle^{\theta-2}\mathcal{E}_{1,0}[U](T) + \int_0^T \langle t\rangle^{\theta-2}\mathcal{E}_{1,1}[U](t) dt \\
&\qquad\quad + \int_0^T \langle t\rangle^{\theta}\left[2\nu\langle \zeta\nabla\cdot G, \zeta\nabla\cdot H\rangle_{L^2}\right. \\
&\qquad\qquad\left. + \|\zeta Q\|_{L^2}^2 + \|\zeta H\|_{L^2}^2 + \|\zeta h\|_{L^2}^2\right] dt.
\end{aligned}$$

□

In the proof of Lemma (3.7.1), we used the following estimate:

**Lemma 3.7.2.** *If* $G \in H^1(\mathbb{R}^3, \mathbb{R}^3 \otimes \mathbb{R}^3)$ *and*

$$\partial_k G^{ij} - \partial_j G^{ik} = Q_k^{ij} \tag{3.7.9}$$

*for* $i, j, k = 1, 2, 3$ *and* $\|Q\|_{L^2} < \infty$*, then*

$$\frac{1}{2}\|\zeta \nabla G\|_{L^2}^2 - \|\zeta \nabla \cdot G\|_{L^2}^2 \lesssim \langle t \rangle^{-2} \|G\|_{L^2}^2 + \|\zeta Q\|_{L^2}^2.$$

*Proof.* The constraint (3.7.9) implies that

$$\begin{aligned}
-|\nabla \cdot G|^2 &= -\partial_j G^{ij} \partial_k G^{ik} \\
&= -\partial_k(\partial_j G^{ij} G^{ik}) + \partial_j \partial_k G^{ij} G^{ik} \\
&= -\partial_k(\partial_j G^{ij} G^{ik}) + \partial_j(\partial_j G^{ik} + Q_k^{ij}) G^{ik} \\
&= -\partial_k(\partial_j G^{ij} G^{ik}) + \partial_j \partial_j G^{ik} G^{ik} + \partial_j Q_k^{ij} G^{ik} \\
&= -\partial_k(\partial_j G^{ij} G^{ik}) + \partial_j(\partial_j G^{ik} G^{ik}) - \partial_j G^{ik} \partial_j G^{ik} + \partial_j Q_k^{ij} G^{ik} \\
&= -\partial_k(\partial_j G^{ij} G^{ik}) + \partial_j(\partial_j G^{ik} G^{ik}) - |\nabla G|^2 + \partial_j Q_k^{ij} G^{ik}.
\end{aligned}$$

Therefore, we have:

$$|\nabla G|^2 - |\nabla \cdot G|^2 \ = \ \partial_j(\partial_j G^{ik} G^{ik}) - \partial_k(\partial_j G^{ij} G^{ik}) + \partial_j Q_k^{ij} G^{ik}.$$

We next multiply by $\zeta^2$ and integrate:

$$\begin{aligned}
&\|\zeta \nabla G\|_{L^2}^2 - \|\zeta \nabla \cdot G\|_{L^2}^2 \\
&= \int_{\mathbb{R}^3} \zeta^2 \left[ \partial_j(\partial_j G^{ik} G^{ik}) - \partial_k(\partial_j G^{ij} G^{ik}) \right] dx \ + \ \int_{\mathbb{R}^3} \zeta^2 \partial_j Q_k^{ij} G^{ik} dx
\end{aligned}$$

$$\lesssim \int_{\mathbb{R}^3} \zeta \langle t\rangle^{-1} |\nabla G||G| dx \;+\; \int_{\mathbb{R}^3} \zeta \langle t\rangle^{-1} |Q_k^{ij}||G^{ik}| dx \;-\; \int_{\mathbb{R}^3} \zeta^2 Q_k^{ij} \partial_j G^{ik} dx$$
$$\leq \frac{1}{2}\|\zeta \nabla G\|_{L^2}^2 + C\langle t\rangle^{-2}\|G\|_{L^2}^2 + C\|\zeta Q\|_{L^2}^2,$$

where we have used Young's inequality and $|\nabla \zeta^2| \lesssim \zeta\langle t\rangle^{-1}$. The statement of the theorem follows immediately from the above inequality. □

We now establish a higher order version of Lemma 3.7.1. Applying the vector fields $S^k\Gamma^a$ to (3.7.1a) - (3.7.1c) and using the commutation properties (3.4.2a) - (3.4.2d), we obtain the PDEs:

$$\partial_t \widetilde{G} - \nabla \widetilde{v} = \widetilde{H}$$
$$\partial_t \widetilde{v} - \nabla \cdot \widetilde{G} - \nu \Delta \widetilde{v} = \widetilde{h_0},$$

subject to the constraint

$$\partial_k \widetilde{G}^{ij} - \partial_j \widetilde{G}^{ik} = \widetilde{Q}_k^{ij}.$$

We have used the notation

$$\begin{aligned}
&\widetilde{G} = S^k\Gamma^a G && \widetilde{v} = S^k\Gamma^a v \\
&\widetilde{H} = (S+1)^k\Gamma^a H && \widetilde{h} = (S+1)^k\Gamma^a h \\
&\widetilde{Q}_k^{ij} = (S+1)^k\Gamma^a Q_k^{ij} && \\
&\widetilde{h_0} = \widetilde{h} - \sum_{j=0}^{k-1}(-1)^{k-j}\binom{k}{j}\nu B\Delta S^j\Gamma^a U &&
\end{aligned} \tag{3.7.11}$$

Fixing $0 \leq q < p$, summing over $k \leq q$, $|a| \leq p-1$, and using induction as in Proposition 2.7.3, we establish:

**Proposition 3.7.3.** *Assume that $\sigma$ in (3.2.2a) is sufficiently small and that $\nu \le 1$. Fix $0 \le q < p$. Suppose that*

$$S^k H,\ \nabla \cdot S^k H,\ S^k Q,\ S^k h \in L^2([0,T]; X^{p-k-1,0}), \quad k = 0, \dots, q,$$

*for some $0 < T < \infty$. If $U = (G, v)$ is a solution of (3.7.1a) - (3.7.1c) such that*

$$\sup_{0 \le t \le T} \mathcal{E}_{p,q+1}[U](t) < \infty,$$

*then for any $0 \le \theta \le 1$,*

$$\int_0^T \langle t\rangle^\theta \left[\mathcal{Y}^{int}_{p,q}[U](t) + \nu^2 \mathcal{Z}^{int}_{p,q}[U](t)\right] dt$$
$$\lesssim \nu \langle T\rangle^{\theta-2} \mathcal{E}_{p,q}[u](T) + \int_0^T \langle t\rangle^{\theta-2} \mathcal{E}_{p,q+1}[U](t) dt$$
$$+ \sum_{\substack{|a|+k \le p-1 \\ k \le q}} \left\{ \int_0^T \langle t\rangle^\theta [2\nu \langle \zeta \nabla \cdot S^k \Gamma^a G, \zeta \nabla \cdot (S+1)^k \Gamma^a H\rangle_{L^2} \right.$$
$$\left. + \|\zeta (S+1)^k \Gamma^a Q\|^2_{L^2} + \|\zeta (S+1)^k \Gamma^a H\|^2_{L^2} + \|\zeta (S+1)^k \Gamma^a h\|^2_{L^2}] dt \right\}.$$

## 3.8 Decay Estimates

In this section we establish the dispersive estimates for the nonlinear equation using a bootstrap argument and an application of Propositions 3.7.3.

**Theorem 3.8.1.** *Choose $(p, q)$ so that $p^* = \left[\frac{p+5}{2}\right] < q \le p$. Suppose that $U = (G, v) \in C([0,T); X^{p,q})$ is a solution of (3.1.6a) - (3.1.6e) with*

$$\sup_{0 \le t \le T} \mathcal{E}_{p,q}[U](t) < \infty,$$

*and*

$$\sup_{0\le t\le T} \mathcal{E}_{p^*,p^*}[U](t) \le \varepsilon^2 \ll 1.$$

*Then*

$$\int_0^T \langle t\rangle^\theta \left[\mathcal{Y}^{int}_{p^*,p^*-1}[U](t) + \nu^2 \mathcal{Z}^{int}_{p^*,p^*-1}[U](t)\right] dt$$

$$\lesssim \begin{cases} \sup\limits_{0\le t\le T} \langle t\rangle^{-\gamma} \mathcal{E}_{p^*,p^*}[U](t), & 0<\theta+\gamma<1 \\ \\ \log(e+T) \sup\limits_{0\le t\le T} \mathcal{E}_{p^*,p^*}[U](t), & \theta = 1 \end{cases},$$

*and*

$$\int_0^T \langle t\rangle^\theta \left[\mathcal{Y}^{int}_{p^*+1,p^*}[U](t) + \nu^2 \mathcal{Z}^{int}_{p^*+1,p^*}[U](t)\right] dt$$

$$\lesssim \sup_{0\le t\le T} \langle t\rangle^{-\gamma} \mathcal{E}_{p,q}[U](t), \quad 0<\theta+\gamma<1.$$

*Proof.* We start with an application of Propositions 3.7.3:

$$\begin{aligned} &\int_0^T \langle t\rangle^\theta \left[\mathcal{Y}^{\rm int}_{p,q}[U](t) + \nu^2 \mathcal{Z}^{\rm int}_{p,q}[U](t)\right] dt \\ &\qquad \lesssim \nu \langle T\rangle^{\theta-2} \mathcal{E}_{p,q}[u](T) + \int_0^T \langle t\rangle^{\theta-2} \mathcal{E}_{p,q+1}[U](t) dt \\ &\qquad + \sum_{\substack{|a|+k\le p-1 \\ k\le q}} \left\{ \int_0^T \langle t\rangle^\theta [2\nu \langle \zeta\nabla\cdot S^k\Gamma^a G, \zeta\nabla\cdot (S+1)^k\Gamma^a H\rangle_{L^2} \right. \\ &\left. +\|\zeta(S+1)^k\Gamma^a Q\|^2_{L^2} + \|\zeta(S+1)^k\Gamma^a H\|^2_{L^2} + \|\zeta(S+1)^k\Gamma^a h\|^2_{L^2}] dt \right\} \quad (3.8.2) \end{aligned}$$

with the following quadratic terms:

$$
\begin{aligned}
H &= \nabla v G - v \cdot \nabla G \\
h &= \nabla \cdot (GG^T) - v \cdot \nabla v - \nabla p \\
Q_k^{ij} &= G^{mj}\partial_m G^{ik} - G^{mk}\partial_m G^{ij}.
\end{aligned}
\tag{3.8.3}
$$

We will first show that the inner product term

$$
\int_0^T \langle t\rangle^\theta 2\nu \langle \zeta\nabla \cdot S^k\Gamma^a G, \zeta\nabla\cdot (S+1)^k\Gamma^a H\rangle_{L^2} dt \tag{3.8.4}
$$

is bounded by the sum of the four integrals (defined in (3.8.7a) and (3.8.7b)):

$$
|I_1| + |I_2| + |I_3| + |I_4|.
$$

Each of those four terms will be further estimated by

$$
\begin{aligned}
\frac{1}{8}\int_0^T \langle t\rangle^\theta \left[\mathcal{Y}_{p,q}^{\mathrm{int}}[U](t) + \nu^2 \mathcal{Z}_{p,q}^{\mathrm{int}}[U](t)\right] dt \\
+ C\nu\langle T\rangle^{\theta-2}\mathcal{E}_{p,q}[u](T) + C\int_0^T \langle t\rangle^{\theta-2}\mathcal{E}_{p,q+1}[U](t)dt \\
+ C\int_0^T \langle t\rangle^\theta \|\zeta R^{ka}\|_{L^2}^2 dt.
\end{aligned}
\tag{3.8.5}
$$

In the above expression, $R^{ka} \equiv R^{ka}(U, \nabla U)$ represents any of the following terms:

$$
\begin{aligned}
&|\nabla S^k\Gamma^a U|\,|\nabla U|, \quad \nabla\cdot[(\nabla S^{k_1}\Gamma^{a_1} v)(S^{k_2}\Gamma^{a_2} G)], \text{ and} \\
&\nabla\cdot(S^{k_1}\Gamma^{a_1} v\cdot\nabla S^{k_2}\Gamma^{a_2} G),
\end{aligned}
\tag{3.8.6}
$$

for $|a_1| + |a_2| \le |a|$ and $k_1 + k_2 \le k$ with $k_1 + a_1 \ne k + a$ in the second term and $k_2 + a_2 \ne k + a$ in the third term.

Upon distributing the derivatives $(S+1)^k\Gamma^a H$ in (3.8.4), we will treat separately the two instances in which all the vector fields fall on the gradient term, i.e.

$$(\nabla S^k\Gamma^a v)G \quad \text{and} \quad -v\cdot(\nabla S^k\Gamma^a G).$$

The remaining terms in $(S+1)^k\Gamma^a H$ are of the form:

$$(\nabla S^{k_1}\Gamma^{a_1}v)(S^{k_2}\Gamma^{a_2}G) \quad \text{and} \quad -(S^{k_1}\Gamma^{a_1}v)\cdot(\nabla S^{k_2}\Gamma^{a_2}G),$$

for $|a_1|+|a_2|\le|a|$ and $k_1+k_2\le k$ with $k_1+a_1\ne k+a$ in the first term and $k_2+a_2\ne k+a$ in the second term.

Therefore, in order to control (3.8.4), we need to estimate the following integrals:

$$\begin{aligned} I_1 &= \int_0^T \langle t\rangle^\theta 2\nu\langle\zeta\nabla\cdot\widetilde{G},\zeta\nabla\cdot(\nabla\widetilde{v}G)\rangle_{L^2}dt \\ I_2 &= \int_0^T \langle t\rangle^\theta 2\nu\langle\zeta\nabla\cdot\widetilde{G},\zeta\nabla\cdot(-v\cdot\nabla\widetilde{G})\rangle_{L^2}dt, \end{aligned} \tag{3.8.7a}$$

together with

$$\begin{aligned} I_3 &= \int_0^T \langle t\rangle^\theta 2\nu\langle\zeta\nabla\cdot\widetilde{G},\zeta\nabla\cdot(\nabla S^{k_1}\Gamma^{a_1}vS^{k_2}\Gamma^{a_2}G)\rangle_{L^2}dt \\ I_4 &= \int_0^T \langle t\rangle^\theta 2\nu\langle\zeta\nabla\cdot\widetilde{G},\zeta\nabla\cdot(-S^{k_1}\Gamma^{a_1}v\cdot\nabla S^{k_2}\Gamma^{a_2}G)\rangle_{L^2}dt, \end{aligned} \tag{3.8.7b}$$

where $k_1+a_1\ne k+a$ in $I_3$ and $k_2+a_2\ne k+a$ in $I_4$. Note that we have adopted the notation from (3.7.11).

We start first with $I_1$ by considering

$$2\nu\langle\zeta\nabla\cdot\widetilde{G},\zeta\nabla\cdot(\nabla\widetilde{v}G)\rangle_{L^2} \;=\; 2\nu\int_{\mathbb{R}^3}\zeta^2(\nabla\cdot\widetilde{G})^i[\nabla\cdot(\nabla\widetilde{v}G)]^i dx, \tag{3.8.8}$$

where

$$[\nabla \cdot (\nabla \widetilde{v} G)]^i \;=\; \partial_k (\nabla \widetilde{v} G)^{ik} \;=\; \partial_k (\partial_m \widetilde{v}^{\,i} G^{mk})$$

$$= (\partial_k \partial_m \widetilde{v}^i)(G^{mk}) + (\partial_m \widetilde{v}^i)(\partial_k G^{mk}). \quad (3.8.9)$$

Consider (3.8.8) with the first term of (3.8.9):

$$\begin{aligned} 2\nu \int_{\mathbb{R}^3} \zeta^2 (\nabla \cdot \widetilde{G})^i (\partial_k \partial_m \widetilde{v}^i)(G^{mk}) dx & \qquad (3.8.10) \\ = 2\nu \int_{\mathbb{R}^3} \zeta^2 (\partial_n \widetilde{G}^{in})(\partial_k \partial_m \widetilde{v}^i)(G^{mk}) dx & \\ \leq \frac{1}{M} \|\zeta \nabla \widetilde{U}\|_{L^2}^2 + C\nu^2 \|\zeta (\partial_k \partial_m \widetilde{v}^i)(G^{mk})\|_{L^2}^2 & \\ \leq \frac{1}{M} \|\zeta \nabla \widetilde{U}\|_{L^2}^2 + C\nu^2 \|G\|_\infty^2 \|\zeta \nabla^2 \widetilde{v}\|_{L^2}^2 & \\ \leq \frac{1}{M} \|\zeta \nabla \widetilde{U}\|_{L^2}^2 + C\nu^2 \varepsilon^2 \|\zeta \nabla^2 \widetilde{v}\|_{L^2}^2, & \end{aligned}$$

where $M$ is some big enough constant. Also, in the last step we have used that $\|G\|_\infty^2 \leq \mathcal{E}_{2,0}[U](t) \leq \varepsilon^2$.

The second term in (3.8.10) has the following bound:

$$\begin{aligned} C\nu^2\varepsilon^2 \|\zeta \nabla^2 \widetilde{v}\|_{L^2}^2 &= C\nu^2\varepsilon^2 \int_{\mathbb{R}^3} \zeta^2 (\partial_k \partial_m \widetilde{v}^i)(\partial_k \partial_m \widetilde{v}^i) dx \qquad (3.8.11) \\ &= -C\nu^2\varepsilon^2 \int_{\mathbb{R}^3} (\partial_k \zeta^2)(\partial_m \widetilde{v}^i)(\partial_k \partial_m \widetilde{v}^i) dx \\ &\quad - C\nu^2\varepsilon^2 \int_{\mathbb{R}^3} \zeta^2 (\partial_m \widetilde{v}^i)(\partial_k^2 \partial_m \widetilde{v}^i) dx \\ &= -C\nu^2\varepsilon^2 \int_{\mathbb{R}^3} (\partial_k \zeta^2)(\partial_m \widetilde{v}^i)(\partial_k \partial_m \widetilde{v}^i) dx \\ &\quad + C\nu^2\varepsilon^2 \int_{\mathbb{R}^3} (\partial_m \zeta^2)(\partial_m \widetilde{v}^i)(\partial_k^2 \widetilde{v}^i) dx \\ &\quad + C\nu^2\varepsilon^2 \int_{\mathbb{R}^3} \zeta^2 (\partial_m^2 \widetilde{v}^i)(\partial_k^2 \widetilde{v}^i) dx. \end{aligned}$$

Using that $|\nabla\zeta^2| \lesssim \zeta\langle t\rangle^{-1}$, (3.8.11) implies that

$$C\nu^2\varepsilon^2\|\zeta\nabla^2\widetilde{v}\|_{L^2}^2 \;\leq\; C\nu\int_{\mathbb{R}^3}\zeta\langle t\rangle^{-1}|\nabla\widetilde{v}|\;|\nabla^2\widetilde{v}|dx \;+\; C\varepsilon^2\nu^2\|\zeta\Delta\widetilde{v}\|_{L^2}^2$$
$$\leq \frac{1}{M}\|\zeta\nabla\widetilde{v}\|_{L^2}^2 + C\langle t\rangle^{-2}\nu^2\|\nabla^2\widetilde{v}\|_{L^2}^2 + C\varepsilon^2\nu^2\|\zeta\Delta\widetilde{v}\|_{L^2}^2. \quad (3.8.12)$$

In (3.8.12) we have also relied on $\nu \ll 1$, $\varepsilon^2 \ll 1$.

Next we estimate (3.8.8) with the second term in (3.8.9):

$$2\nu\int_{\mathbb{R}^3}\zeta^2(\nabla\cdot\widetilde{G})^i(\partial_m\widetilde{v}^i)(\partial_k G^{mk})dx$$
$$\leq \frac{1}{M}\|\zeta\nabla\widetilde{U}\|_{L^2}^2 + C\|\zeta|\nabla S^k\Gamma^a U|\;|\nabla U|\;\|_{L^2}^2. \quad (3.8.13)$$

Altogether, from (3.8.8) - (3.8.13), we conclude that:

$$I_1 \leq \frac{3}{M}\int_0^T\langle t\rangle^{\theta}\|\zeta\nabla\widetilde{U}\|_{L^2}^2dt$$
$$+ C\int_0^T\langle t\rangle^{\theta-2}\nu^2\|\nabla^2\widetilde{v}\|_{L^2}^2dt$$
$$+ C\varepsilon^2\nu^2\int_0^T\langle t\rangle^{\theta}\|\zeta\Delta\widetilde{v}\|_{L^2}^2dt$$
$$+ C\int_0^T\langle t\rangle^{\theta}\|\zeta|\nabla S^k\Gamma^a U|\;|\nabla U|\;\|_{L^2}^2dt.$$

Similarly to (3.7.8), we can show that

$$C\int_0^T\langle t\rangle^{\theta-2}\nu^2\|\nabla^2\widetilde{v}\|_{L^2}^2dt$$
$$\leq C\nu\langle T\rangle^{\theta-2}\mathcal{E}_{p,q}[u](T) + C\int_0^T\langle t\rangle^{\theta-2}\mathcal{E}_{p,q+1}[U](t)dt.$$

Imposing smallness conditions $\frac{3}{M} \leq \frac{1}{8}$ and $C\varepsilon^2 \leq \frac{1}{8}$, we show that $I_1$ satisfies the bound (3.8.5).

Next, we estimate $I_2$. We have:

$$2\nu\langle\zeta\nabla\cdot\widetilde{G},\zeta\nabla\cdot(-v\cdot\nabla\widetilde{G})\rangle_{L^2} \;=\; 2\nu\int_{\mathbb{R}^3}\zeta^2(\nabla\cdot\widetilde{G})^i[\nabla\cdot(-v\cdot\nabla\widetilde{G})]^i dx. \quad (3.8.14)$$

We can write:

$$[\nabla\cdot(-v\cdot\nabla\widetilde{G})]^i = \partial_k(-v\cdot\nabla\widetilde{G})^{ik}$$
$$= -\partial_k(v^m\partial_m\widetilde{G}^{ik}) \;=\; -(\partial_k v^m)(\partial_m\widetilde{G}^{ik}) - v^m(\partial_m\partial_k\widetilde{G}^{ik}). \quad (3.8.15)$$

Consider (3.8.14) with the second term of (3.8.15). Applying the constraint $\nabla\cdot v = 0$, we have:

$$-2\nu\int_{\mathbb{R}^3}\zeta^2(\nabla\cdot\widetilde{G})^i v^m\partial_m\partial_k\widetilde{G}^{ik}dx \quad (3.8.16)$$
$$= -2\nu\int_{\mathbb{R}^3}\zeta^2(\nabla\cdot\widetilde{G})^i v^m\partial_m(\nabla\cdot\widetilde{G})^i dx$$
$$= -\nu\int_{\mathbb{R}^3}\zeta^2 v^m\partial_m|\nabla\cdot\widetilde{G}|^2 dx$$
$$= \nu\int_{\mathbb{R}^3}(\partial_m\zeta^2)v^m|\nabla\cdot\widetilde{G}|^2 dx$$
$$\lesssim \int_{\mathbb{R}^3}\zeta\langle t\rangle^{-1}|v|\,|\nabla\cdot\widetilde{G}|^2 dx.$$

Application of Young's inequality gives:

$$\int_{\mathbb{R}^3}\zeta\langle t\rangle^{-1}|v|\,|\nabla\cdot\widetilde{G}|^2 dx \;\le\; \frac{1}{M}\|\zeta\nabla\widetilde{U}\|_{L^2}^2 + C\langle t\rangle^{-2}\|\,|v|\,|\nabla\cdot\widetilde{G}|\,\|_{L^2}^2$$
$$\le \frac{1}{M}\|\zeta\nabla\widetilde{U}\|_{L^2}^2 + C\varepsilon^2\langle t\rangle^{-2}\|\nabla\cdot\widetilde{G}\|_{L^2}^2, \quad (3.8.17)$$

where in the last line we used $\|v\|_\infty^2 \le \mathcal{E}_{2,0}[U](t) \le \varepsilon^2$.

From (3.8.16) and (3.8.17), we obtain:

$$-2\nu\int_{\mathbb{R}^3}\zeta^2(\nabla\cdot\widetilde{G})^i v^m\partial_m\partial_k\widetilde{G}^{ik}dx \;\le\; \frac{1}{M}\|\zeta\nabla\widetilde{U}\|_{L^2}^2 + C\langle t\rangle^{-2}\|\nabla\cdot\widetilde{G}\|_{L^2}^2. \quad (3.8.18)$$

We now estimate (3.8.14) with the first term of (3.8.15):

$$
\begin{aligned}
-2\nu \int_{\mathbb{R}^3} \zeta^2 (\nabla \cdot \widetilde{G})^i (\partial_k v^m)(\partial_m \widetilde{G}^{ik}) dx& \\
\le \frac{1}{M} \|\zeta \nabla \widetilde{U}\|_{L^2}^2 + C \|\zeta |\nabla S^k \Gamma^a U| \, |\nabla U| \, \|_{L^2}^2.& \quad (3.8.19)
\end{aligned}
$$

From (3.8.18) and (3.8.19) we arrive at:

$$
\begin{aligned}
I_2 \; \le \; & \frac{2}{M} \int_0^T \langle t \rangle^\theta \|\zeta \nabla \widetilde{U}\|_{L^2}^2 dt + C \int_0^T \langle t \rangle^{\theta - 2} \|\nabla \cdot \widetilde{G}\|_{L^2}^2 dt \\
& + C \int_0^T \langle t \rangle^\theta \|\zeta |\nabla S^k \Gamma^a U| \, |\nabla U| \, \|_{L^2}^2,
\end{aligned}
$$

which shows that $I_2$ satisfies (3.8.5) for $\frac{2}{M} \le \frac{1}{8}$.

Finally, we treat $I_3$ and $I_4$ . Application of Young's inequality gives:

$$
\begin{aligned}
I_3 + I_4 \; \le \; & \frac{2}{M} \int_0^T \langle t \rangle^\theta \|\zeta \nabla \widetilde{U}\|_{L^2}^2 dt \\
& + C \int_0^T \langle t \rangle^\theta \|\zeta \nabla \cdot [(\nabla S^{k_1} \Gamma^{a_1} v)(S^{k_2} \Gamma^{a_2} G)]\|_{L^2}^2 \\
& + C \int_0^T \langle t \rangle^\theta \|\zeta \nabla \cdot (S^{k_1} \Gamma^{a_1} v \cdot \nabla S^{k_2} \Gamma^{a_2} G)\|_{L^2}^2,
\end{aligned}
$$

which is again bounded by (3.8.5) for $\frac{2}{M} \le \frac{1}{8}$.

If we apply the bound (3.8.5) on the inner product term (3.8.4), we can write the estimate in the interior region (3.8.2) as:

$$
\begin{aligned}
\int_0^T & \langle t \rangle^\theta \left[ \mathcal{Y}_{p,q}^{\text{int}}[U](t) + \nu^2 \mathcal{Z}_{p,q}^{\text{int}}[U](t) \right] dt \\
& \lesssim \nu \langle T \rangle^{\theta - 2} \mathcal{E}_{p,q}[u](T) + \int_0^T \langle t \rangle^{\theta - 2} \mathcal{E}_{p,q+1}[U](t) dt \\
& + \sum_{\substack{|a|+k \le p-1 \\ k \le q}} \left\{ \int_0^T \langle t \rangle^\theta [\|\zeta (S+1)^k \Gamma^a Q\|_{L^2}^2 + \|\zeta (S+1)^k \Gamma^a H\|_{L^2}^2 \right.
\end{aligned}
$$

$$+\|\zeta(S+1)^k\Gamma^a h\|_{L^2}^2+\|\zeta R^{ka}\|_{L^2}^2]dt\Big\}. \quad (3.8.20)$$

Recall the definitions of $R^{ka}$ (see (3.8.6)), and $Q$, $H$, $h$ (see (3.8.3)). All terms are quadratic except for the linear portion of $h$ involving the pressure term. By Lemma 3.9.1, however, $\nabla p$ can be bounded by quadratic terms. Therefore, we have that all the $L^2$ norms appearing on the righthand side of (3.8.20) can be bounded by the sum of terms of the form:

$$\|\zeta|S^{k_1}\Gamma^{a_1}U|\,|S^{k_2}\Gamma^{a_2+1}U|\,\|_{L^2}^2,$$

where $|a_1|+|a_2|\le|a|$, $k_1+k_2\le k\le q$, and $k_2+|a_2|\ne k+a=p$.

At this point we can proceed as in the proof of (2.8.1). The main difference here is that because of the extra $\nabla$ vector field in the $R^{ka}$ term we have $k+|a|\le\bar{p}$ (vs. $k+|a|\le\bar{p}-1$ in the dissipative wave equation case). As a result, $\bar{p}'$ and $\bar{q}'$ would be defined slightly differently. For example, when $(\bar{p},\bar{q})=(p^*+1,p^*)$ we have:

$$\bar{p}'=\left[\frac{p^*+6}{2}\right] \quad \text{and} \quad \bar{q}'=\left[\frac{p^*+1}{2}\right].$$

However, for $p^*\ge 5$ we still have $\left[\frac{p^*+6}{2}\right]\le p^*$ and $\left[\frac{p^*+1}{2}\right]\le p^*-1$ and therefore we can still adopt the proof of (2.8.1). □

## 3.9 Bound for the Pressure

**Lemma 3.9.1.** *Suppose that $p$ satisfies* (3.1.6b) *with constraints* (3.1.6c) *and* (3.1.6d)*. Then for any multi-index $a$ and integer $k \geq 0$,*

$$\|\nabla S^k \Gamma^a p\|_{L^2}^2 \leq \|(S+1)^k \Gamma^a \left[\nabla \cdot (GG^T) - v \cdot \nabla v\right]\|_{L^2}^2.$$

*Proof.* From the PDE (3.1.6b) we have:

$$\nabla p = -\partial_t v + \nabla \cdot G + \nu \Delta v + \nabla \cdot (GG^T) - v \cdot \nabla v. \tag{3.9.1}$$

We apply the vector fields $(S+1)^k \Gamma^a$ to each side of (3.9.1) to get:

$$\nabla S^k \Gamma^a p \;=\; (S+1)^k \Gamma^a \left[-\partial_t v + \nabla \cdot G + \nu \Delta v + \nabla \cdot (GG^T) - v \cdot \nabla v\right]. \tag{3.9.2}$$

Next, we take the divergence of (3.9.2). The constraints (3.1.6c) and (3.1.6d) together with the identity

$$\nabla \cdot (\nabla \cdot G) = \nabla \cdot (\nabla \cdot G^T) = 0$$

give:

$$\Delta S^k \Gamma^a p = \nabla \cdot (S+1)^k \Gamma^a \left[\nabla \cdot (GG^T) - v \cdot \nabla v\right]. \tag{3.9.3}$$

With the notation

$$\widetilde{Q_0} = (S+1)^k \Gamma^a \left[\nabla \cdot (GG^T) - v \cdot \nabla v\right]$$

we can rewrite (3.9.3) as

$$\Delta S^k \Gamma^a p = \nabla \cdot \widetilde{Q_0}. \tag{3.9.4}$$

Using (3.9.4), we can estimate next the $L^2$-norm of the gradient of the pressure as follows:

$$\begin{aligned}\|\nabla S^k\Gamma^a p\|_{L^2}^2 &= -\langle \Delta S^k\Gamma^a p, S^k\Gamma^a p\rangle_{L^2}\\ &= -\langle \nabla\cdot\widetilde{Q_0}, S^k\Gamma^a p\rangle_{L^2}\\ &= \langle \widetilde{Q_0}, \nabla S^k\Gamma^a p\rangle_{L^2}\\ &\le \|\widetilde{Q_0}\|_{L^2}\|\nabla S^k\Gamma^a p\|_{L^2},\end{aligned}$$

from which the statement of the theorem follows. □

## 3.10 High Energy Estimates

**Proposition 3.10.1.** *Choose* $(p,q)$ *so that* $5 \le p^* = \left[\frac{p+5}{2}\right] \le q \le p$. *Suppose that* $U = (G,v) \in C([0,T_0), X^{p,q})$ *is a solution of* (3.1.6a) - (3.1.6e) *with*

$$\sup_{0\le t<T_0} \mathcal{E}_{p^*,p^*}[U](t) \le \varepsilon^2 \ll 1.$$

*Then there exists a constant* $C_1 > 1$ *such that*

$$\begin{aligned}\mathcal{E}_{p,q}[U](t) &\le C_1\mathcal{E}_{p,q}[U_0]\langle t\rangle^{C_1\varepsilon}\\ \mathcal{E}_{p^*,p^*}[U](t) &\le C_1\mathcal{E}_{p^*,p^*}[U_0]\langle t\rangle^{C_1\varepsilon},\end{aligned}$$

*for* $0 \le t < T_0$.

*Proof.* Taking the $L^2$ dot product of

$$LU = \partial_t U - A(\nabla)U - \nu B\Delta U$$

with $U = (G, v)$, we obtain:

$$\langle \partial_t U(t), U(t)\rangle_{L^2} - \langle A(\nabla)U(t), U(t)\rangle_{L^2} - \langle \nu B\Delta U(t), U(t)\rangle_{L^2} = \langle LU(t), U(t)\rangle_{L^2}. \quad (3.10.1)$$

The second term on the left of (3.10.1) vanishes:

$$\begin{aligned}\langle A(\nabla)U(t), U(t)\rangle_{L^2} &= -\langle \nabla v(t), G(t)\rangle_{L^2} - \langle \nabla\cdot G(t), v(t)\rangle_{L^2} \\ &= -\langle \nabla v(t), G(t)\rangle_{L^2} + \langle G(t), \nabla v(t)\rangle_{L^2} = 0.\end{aligned}$$

Using integration by parts, the third term on left of (3.10.1) can be written as:

$$\begin{aligned}\langle \nu B\Delta U(t), U(t)\rangle_{L^2} = \langle \nu\Delta v(t), v(t)\rangle_{L^2} &= \int_{\mathbb{R}^3} \nu\partial_k^2 v^i(t) v^i(t)dx \\ &= -\int_{\mathbb{R}^3} \nu\partial_k v^i(t)\partial_k v^i(t)dx = -\nu\|\nabla v(t)\|_{L^2}^2.\end{aligned}$$

Therefore, (3.10.1) becomes:

$$\frac{1}{2}\partial_t\|U(t)\|_{L^2}^2 + \nu\|\nabla v(t)\|_{L^2}^2 = \langle LU(t), U(t)\rangle_{L^2}.$$

Integration over time gives:

$$\frac{1}{2}\|U(T)\|_{L^2}^2 + \nu\int_0^T \|\nabla v(t)\|_{L^2}^2 dt = \frac{1}{2}\|U(0)\|_{L^2}^2 + \int_0^T \langle LU(t), U(t)\rangle_{L^2} dt,$$

which implies that for $0 \le T < T_0$

$$\mathcal{E}_{0,0}[U](T) = \mathcal{E}_{0,0}[U_0] + \int_0^T \langle LU(t), U(t)\rangle_{L^2} dt.$$

For $p \ge q \ge 0$, we apply the above estimate to higher order vector fields and together with the commutation property (3.4.2a) we obtain:

$$\mathcal{E}_{p,q}[U](T) = \mathcal{E}_{p,q}[U_0] + I + \sum_{\substack{|a|+k\le p\\ k\le q}} \int_0^T \langle (S+1)^k\Gamma^a LU(t), S^k\Gamma^a U(t)\rangle_{L^2} dt, \quad (3.10.2)$$

where

$$I = -\sum_{\substack{|a|+k\le p\\ k\le q}} \sum_{j=0}^{k-1} \int_0^T \left\langle (-1)^{k-j}\binom{k}{j} \nu B \Delta S^j \Gamma^a U(t), S^k \Gamma^a U(t) \right\rangle_{L_2} dt.$$

For $q > 0$ we show by integration by parts that:

$$\begin{aligned}
&\int_0^T \langle \nu B \Delta S^j \Gamma^a U(t), S^k \Gamma^a U(t)\rangle_{L^2} dt \\
&\qquad = \int_0^T \int_{\mathbb{R}^3} \nu \Delta (S^j \Gamma^a v(t))^i (S^k \Gamma^a v(t))^i dx dt \\
&\qquad = -\int_0^T \int_{\mathbb{R}^3} \nu \nabla (S^j \Gamma^a v(t))^i \cdot \nabla (S^k \Gamma^a v(t))^i dx dt \\
&\qquad \le \int_0^T \nu \|\nabla S^j \Gamma^a v(t)\|_{L^2} \|\nabla S^k \Gamma^a v(t)\|_{L^2} dt \\
&\qquad \le \left(\nu \int_0^T \|\nabla S^j \Gamma^a v(t)\|_{L^2}^2 dt\right)^{1/2} \left(\nu \int_0^T \|\nabla S^k \Gamma^a v(t)\|_{L^2}^2 dt\right)^{1/2}.
\end{aligned}$$

Therefore, we have:

$$I \lesssim \mathcal{E}_{p,q-1}^{1/2}[U](T) \mathcal{E}_{p,q}^{1/2}[U](T).$$

Applying Young's inequality to the above bound and substituting into (3.10.2) give:

$$\begin{aligned}
\mathcal{E}_{p,q}[U](T) \;\lesssim\; & \mathcal{E}_{p,q}[U_0] + \mathcal{E}_{p,q-1}[U](T) + \mu \mathcal{E}_{p,q}[U](T) \\
& + \sum_{\substack{|a|+k\le p\\ k\le q}} \int_0^T \langle (S+1)^k \Gamma^a L U(t), S^k \Gamma^a U(t)\rangle_{L^2} dt,
\end{aligned}$$

where $\mu$ is a small enough constant so that the corresponding energy term can be absorbed on the left. Induction on $q$ further gives:

$$\mathcal{E}_{p,q}[U](T) \;\lesssim\; \mathcal{E}_{p,q}[U_0] + \sum_{\substack{|a|+k\le p\\ k\le q}} \int_0^T \langle (S+1)^k \Gamma^a L U(t), S^k \Gamma^a U(t)\rangle_{L^2} dt. \quad (3.10.3)$$

Using (3.1.7a), (3.1.7b), and (3.1.7c), we can write the inner product inside the integral as:

$$\langle (S+1)^k \Gamma^a LU(t), S^k \Gamma^a U(t) \rangle_{L^2} = \left\langle (S+1)^k \Gamma^a \begin{pmatrix} \nabla v G - v \cdot \nabla G \\ \nabla \cdot (GG^T) - v \cdot \nabla v - \nabla p \end{pmatrix}, S^k \Gamma^a \begin{pmatrix} G \\ v \end{pmatrix} \right\rangle_{L_2}.$$

We will first address the special case when vector fields distribute onto the gradient term. Using the notation introduced in (3.7.11), we start with:

$$\langle -v \cdot \nabla \widetilde{G}, \widetilde{G} \rangle_{L^2} = -\int_{\mathbb{R}^3} v^k \partial_k \widetilde{G}^{ij} \widetilde{G}^{ij} dx = -\frac{1}{2} \int_{\mathbb{R}^3} v^k \partial_k |\widetilde{G}|^2 dx = 0,$$

where in the above expression we have used integration by parts and the constraint $\nabla \cdot v = 0$.

Similarly, we show that

$$\langle -v \cdot \nabla \widetilde{v}, \widetilde{v} \rangle_{L^2} = 0$$

$$\langle -\nabla \widetilde{p}, \widetilde{v} \rangle_{L^2} = 0.$$

We continue with:

$$\begin{aligned} \langle \nabla \cdot \widetilde{G} G^T, \widetilde{v} \rangle_{L^2} &= \int_{\mathbb{R}^3} (\nabla \cdot \widetilde{G} G^T)^i \widetilde{v}^i dx = \int_{\mathbb{R}^3} \partial_j (\widetilde{G} G^T)^{ij} \widetilde{v}^i dx \\ &= \int_{\mathbb{R}^3} \partial_j (\widetilde{G}^{ik} G^{jk}) \widetilde{v}^i dx = \int_{\mathbb{R}^3} \partial_j \widetilde{G}^{ik} G^{jk} \widetilde{v}^i dx, \end{aligned} \quad (3.10.4)$$

where in the last line we have applied the constraint $\nabla \cdot G^T = 0$. Similarly we show that:

$$\langle \nabla \cdot G \widetilde{G}^T, \widetilde{v} \rangle_{L^2} = \int_{\mathbb{R}^3} \partial_j G^{ik} \widetilde{G}^{jk} \widetilde{v}^i dx$$

and we note that this expression does not involve a term with $k + |a| + 1$ vector fields.

For the last one of the special cases, we once again use $\nabla \cdot G^T = 0$, together with integration by parts to show that:

$$\begin{aligned} \langle \nabla \widetilde{v} G, \widetilde{G} \rangle_{L^2} &= \int_{\mathbb{R}^3} (\nabla \widetilde{v} G)^{ik} \widetilde{G}^{ik} dx = \int_{\mathbb{R}^3} \partial_j \widetilde{v}^i G^{jk} \widetilde{G}^{ik} dx \\ &= - \int_{\mathbb{R}^3} \widetilde{v}^i G^{jk} \partial_j \widetilde{G}^{ik} dx. \quad (3.10.5) \end{aligned}$$

We notice at this point that (3.10.4) and (3.10.5) cancel out.

The treatment of the above special cases shows that no single term has more than $k + |a|$ vector field derivatives and therefore we can write (3.10.3) as:

$$\begin{aligned} \mathcal{E}_{p,q}[U](T) &\lesssim \mathcal{E}_{p,q}[U_0] \\ &+ \sum_{\substack{|a_1+a_2|+k_1+k_2 \le p \\ k_1+k_2 \le q \\ |a_2|+k_2 < p}} \int_0^T \| \, |S^{k_1} \Gamma^{a_1} U(t)| \, |S^{k_2} \Gamma^{a_2+1} U(t)| \, \|_{L^2} \mathcal{E}_{p,q}^{1/2}[U](t) dt. \end{aligned}$$

For the remaining of the proof we refer to the arguments that follow (2.9.10) in the damped wave equation case.

□

## 3.11 Low Energy Estimates

**Proposition 3.11.1.** *Choose* $(p, q)$ *such that* $p \geq 11$, *and* $p \geq q > p^*$, *where* $p^* = \left[\frac{p+5}{2}\right]$. *Suppose that* $U = (G, v) \in C([0, T_0), X^{p,q})$ *is a solution of* (3.1.6a) -

(3.1.6e) *with*

$$\sup_{0\leq t<T_0} \mathcal{E}_{p^*,p^*}[U](t) \leq \varepsilon^2 \ll 1.$$

*There exists a constant $C_0 > 1$ such that*

$$\sup_{0\leq t<T_0} \mathcal{E}_{p^*,p^*}[U](t) \leq C_0 \mathcal{E}_{p^*,p^*}[U_0] \left(1 + \mathcal{E}_{p,q}^{1/2}[U_0]\right).$$

*Proof.* We start with (3.10.3) applied to $(p,q) = (p^*, p^*)$:

$$\mathcal{E}_{p^*,p^*}[U](T) \ \lesssim \ \mathcal{E}_{p^*,p^*}[U_0] \ + \sum_{\substack{|a|+k\leq p^* \\ k\leq p^*}} \int_0^T \langle (S+1)^k \Gamma^a LU(t), S^k \Gamma^a U(t)\rangle_{L^2} dt$$

and we recall that

$$LU = N(U, \nabla U) + (0, -\nabla p).$$

Using integration by parts and the constraint $\nabla \cdot v = 0$, we show that the pressure term vanishes:

$$\begin{aligned} \langle -(S+1)^k \Gamma^a \nabla p, S^k \Gamma^a v\rangle_{L^2} \ &= \ \langle -\nabla S^k \Gamma^a p, S^k \Gamma^a v\rangle_{L^2} \\ &= -\int_{\mathbb{R}^3} \partial_i S^k \Gamma^a p (S^k \Gamma^a v)^i dx \ = \ \int_{\mathbb{R}^3} S^k \Gamma^a p \, \partial_i (S^k \Gamma^a v)^i dx \ = \ 0. \end{aligned}$$

Therefore, we can write the energy inequality as:

$$\begin{aligned} &\mathcal{E}_{p^*,p^*}[U](T) \lesssim \mathcal{E}_{p^*,p^*}[U_0] \\ &\qquad + \sum_{\substack{a_1+a_2=a \\ k_1+k_2=k \\ |a|+k\leq p^* \\ k\leq p^*}} \left| \int_0^T \langle N(S^{k_1}\Gamma^{a_1} U(t), \nabla S^{k_2} \Gamma^{a_2} U(t)), S^k \Gamma^a U(t)\rangle_{L^2} dt \right|. \end{aligned} \tag{3.11.1}$$

We have shown in section (3.10) that the summation indices satisfy $|a_2| + k_2 < p^*$, but we will not need this fact here.

Using the cut-off functions (3.2.2a) and the property (3.2.2b), we can bound the integral on the right-hand side by:

$$\begin{aligned}
&\left| \int_0^T \langle N(S^{k_1}\Gamma^{a_1}U(t), \nabla S^{k_2}\Gamma^{a_2}U(t)), S^k\Gamma^a U(t)\rangle_{L^2} dt \right| \\
&\qquad \lesssim \int_0^T \int_{\mathbb{R}^3} \zeta^2 |S^{k_1}\Gamma^{a_1}U(t)|\, |\nabla S^{k_2}\Gamma^{a_2}U(t)|\, |S^k\Gamma^a U(t)| dx dt \\
&\qquad\qquad + \left| \int_0^T \langle \eta N(S^{k_1}\Gamma^{a_1}U(t), \nabla S^{k_2}\Gamma^{a_2}U(t)), S^k\Gamma^a U(t)\rangle_{L^2} dt \right| \\
&\qquad \equiv I_1 + I_2,
\end{aligned} \tag{3.11.2}$$

where $I_1$ and $I_2$ denote correspondingly the two integrals on the right and $T$ is in the range $0 \leq T < T_0$.

### Interior Low Energy

Similarly to Interior Low Energy section of damped wave equation case we show that

$$I_1 \lesssim \mathcal{E}_{p^*,p^*}[U_0]\mathcal{E}_{p,q}^{1/2}[U_0]. \tag{3.11.3}$$

### Exterior Low Energy

By the definition of the exterior term:

$$I_2 = \left| \int_0^T \langle \eta N(S^{k_1}\Gamma^{a_1}U(t), \nabla S^{k_2}\Gamma^{a_2}U(t)), S^k\Gamma^a U(t)\rangle_{L^2} dt \right|,$$

where the indices satisfy $a_1 + a_2 = a$, $k_1 + k_2 = k$ with $|a| + k \leq p^*$ and $k \leq p^*$.

Referring to the definition of $N(U, \nabla U)$ (see (3.1.7b) and (3.1.7c)) and applying the constraint (3.1.6d), we note that, in components, the quadratic nonlinear terms are of the form:

$$\begin{aligned}
&\left[(\nabla S^{k_2}\Gamma^{a_2}v)(S^{k_1}\Gamma^{a_1}G)\right]^{ij} = (S^{k_1}\Gamma^{a_1}G)^{*j} \cdot \nabla(S^{k_2}\Gamma^{a_2}v)^i \\
&\left[(S^{k_1}\Gamma^{a_1}v) \cdot \nabla(S^{k_2}\Gamma^{a_2}G)\right]^{ij} = S^{k_1}\Gamma^{a_1}v \cdot \nabla(S^{k_2}\Gamma^{a_2}G)^{ij} \\
&\left[\nabla \cdot \left\{(S^{k_2}\Gamma^{a_2}G)(S^{k_1}\Gamma^{a_1}G)^T\right\}\right]^i = (S^{k_1}\Gamma^{a_1}G)^{*j} \cdot \nabla(S^{k_2}\Gamma^{a_2}G)^{ij} \\
&\left[S^{k_1}\Gamma^{a_1}v \cdot \nabla S^{k_2}\Gamma^{a_2}v\right]^i = S^{k_1}\Gamma^{a_1}v \cdot \nabla(S^{k_2}\Gamma^{a_2}v)^i,
\end{aligned} \tag{3.11.4}$$

where $G^{*j}$ denotes the vector which is the the $j^{\text{th}}$ column of the matrix $G$.

Using the gradient decomposition $\nabla = \omega\partial_r - \frac{1}{r}\omega \wedge \Omega$, we can write:

$$\begin{aligned}
I_2 &\lesssim \left| \int_0^T \langle \eta N(S^{k_1}\Gamma^{a_1}U(t), \omega\partial_r S^{k_2}\Gamma^{a_2}U(t)), S^k\Gamma^a U(t)\rangle_{L^2} dt \right| \\
&\quad + \int_0^T \int_{\mathbb{R}^3} \eta r^{-1} |S^{k_1}\Gamma^{a_1}U(t)|\, |\Omega S^{k_2}\Gamma^{a_2}U(t)|\, |S^k\Gamma^a U(t)| dx dt \\
&\equiv I_2' + I_2''.
\end{aligned} \tag{3.11.5}$$

By Lemma 3.6.2, we can bound the second term on the right as follows:

$$\begin{aligned}
I_2'' &\lesssim \int_0^T \langle t\rangle^{-1} \|\eta |S^{k_1}\Gamma^{a_1}U(t)|\, |\Omega S^{k_2}\Gamma^{a_2}U(t)|\, \|_{L^2} \|S^k\Gamma^a U(t)\|_{L^2} dt \\
&\lesssim \int_0^T \langle t\rangle^{-1}\langle t\rangle^{-1} \mathcal{E}^{1/2}_{p^*,p^*}[U](t)\mathcal{E}^{1/2}_{p^*+1,p^*}[U](t)\mathcal{E}^{1/2}_{p^*,p^*}[U](t) dt \\
&\lesssim \int_0^T \langle t\rangle^{-2} \mathcal{E}_{p^*,p^*}[U](t)\mathcal{E}^{1/2}_{p,q}[U](t) dt.
\end{aligned} \tag{3.11.6}$$

We proceed with estimating $I_2'$. Substituting $\nabla$ with $\omega\partial_r$ in (3.11.4), we see that $I_2'$ is bounded by terms of the form

$$\int_0^T \|\, |\eta\, \omega \cdot S^{k_1}\Gamma^{a_1}u(t)|\, |S^{k_2}\Gamma^{a_2+1}U(t)|\, \|_{L^2} \|S^k\Gamma^a U(t)\|_{L^2} dt, \tag{3.11.7}$$

where $S^{k_1}\Gamma^{a_1}u(t)$ stands for either of:

$$S^{k_1}\Gamma^{a_1}u(t) = \begin{cases} (S^{k_1}\Gamma^{a_1}G)^{*j} \\ S^{k_1}\Gamma^{a_1}v \end{cases}.$$

Furthermore, recalling the constraints $\nabla\cdot v = 0$ and $\nabla\cdot G^T = 0$, we have that $\nabla\cdot S^{k_1}\Gamma^{a_1}u(t) = 0$.

In the case $k_1+|a_1|\le\left[\frac{p^*}{2}\right]$ and $k_2+|a_2|\le p^*$, the Sobolev inequality (3.5.2) gives:

$$\begin{aligned}
&\|\,|\,\eta\,\omega\cdot S^{k_1}\Gamma^{a_1}u(t)|\,|S^{k_2}\Gamma^{a_2+1}U(t)|\,\|_{L^2}\\
&\qquad\lesssim \|\eta\,\omega\cdot S^{k_1}\Gamma^{a_1}u(t)\|_{L^\infty}\|S^{k_2}\Gamma^{a_2+1}U(t)\|_{L^2}\\
&\qquad\lesssim \langle t\rangle^{-3/2}\mathcal{E}^{1/2}_{\left[\frac{p^*+4}{2}\right],\left[\frac{p^*}{2}\right]}[U](t)\mathcal{E}^{1/2}_{p^*+1,p^*}[U](t)\\
&\qquad\lesssim \langle t\rangle^{-3/2}\mathcal{E}^{1/2}_{p^*,p^*}[U](t)\mathcal{E}^{1/2}_{p,q}[U](t).
\end{aligned}$$

And in the case $k_2+|a_2|\le\left[\frac{p^*-1}{2}\right]$ and $k_1+|a_1|\le p^*$, again by (3.5.2), we have:

$$\begin{aligned}
&\|\,|\,\eta\,\omega\cdot S^{k_1}\Gamma^{a_1}u(t)|\,|S^{k_2}\Gamma^{a_2+1}U(t)|\,\|_{L^2}\\
&\qquad\lesssim \|\eta\,\omega\cdot S^{k_1}\Gamma^{a_1}u(t)\|_{L^\infty}\|S^{k_2}\Gamma^{a_2+1}U(t)\|_{L^2}\\
&\qquad\lesssim \langle t\rangle^{-3/2}\mathcal{E}^{1/2}_{p^*+2,p^*}[U](t)\mathcal{E}^{1/2}_{\left[\frac{p^*+1}{2}\right],\left[\frac{p^*-1}{2}\right]}[U](t)\\
&\qquad\lesssim \langle t\rangle^{-3/2}\mathcal{E}^{1/2}_{p,q}[U](t)\mathcal{E}^{1/2}_{p^*,p^*}[U](t).
\end{aligned}$$

From the two cases above and (3.11.7) we conclude that

$$I_2' \lesssim \int_0^T \langle t\rangle^{-3/2}\mathcal{E}_{p^*,p^*}[U](t)\mathcal{E}^{1/2}_{p,q}[U](t)dt. \qquad (3.11.8)$$

Therefore, from (3.11.5),(3.11.6), and (3.11.8), we show that

$$I_2 \lesssim \int_0^T \langle t\rangle^{-3/2} \mathcal{E}_{p^*,p^*}[U](t)\mathcal{E}_{p,q}^{1/2}[U](t)dt.$$

Referring back to the corresponding estimates for the damped wave equation case (see (2.10.17) and (2.10.18)), we have

$$I_2 \lesssim \mathcal{E}_{p^*,p^*}[U_0]\mathcal{E}_{p,q}^{1/2}[U_0]. \tag{3.11.9}$$

Therefore, from the energy inequality (3.11.1), (3.11.2), and the estimates (3.11.3) and (3.11.9), we have:

$$\mathcal{E}_{p^*,p^*}[U](T) \leq C_0\mathcal{E}_{p^*,p^*}[U_0](1 + \mathcal{E}_{p,q}^{1/2}[U_0])$$

for some constant $C_0 > 1$ and for every $0 \leq T < T_0$. This gives us the desired result.

□

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