UNIVERSITY OF CALIFORNIA

Santa Barbara

On pseudo-Anosov maps, symplectic, Perron-Frobenius matrices, and compression bodies

A Dissertation submitted in partial satisfaction

of the requirements for the degree of

Doctor of Philosophy

in

Mathematics

by

Robert Ackermann

Committee in Charge:

Professor Darren Long, Chair

Professor Jon McCammond

Professor Martin Scharlemann

December 2014

The Dissertation of

Robert Ackermann is approved:

Professor Jon McCammond

Professor Martin Scharlemann

Professor Darren Long, Committee Chairperson

December 2014

On pseudo-Anosov maps, symplectic, Perron-Frobenius matrices, and compression bodies

by

Robert Ackermann

Acknowledgements

I am thankful for the entire community in the math department at UCSB, who collectively have made the department a terrific place to study and work. I would especially like to thank my advisor, Darren Long, for his tremendous help and encouragement throughout my studies at UCSB. Finally, I am grateful for my family, who have always supported and encouraged my pursuit of mathematics.

VITA OF ROBERT ACKERMANN

EDUCATION

Bachelor of Science in Mathematics, Virginia Polytechnic Institute and State University, May 2007

Bachelor of Science in Computer Science, Virginia Polytechnic Institute and State University, May 2007

Master of Arts in Mathematics, Virginia Polytechnic Institute and State University, Virginia, May 2008

Doctor of Philosophy in Mathematics, University of California, Santa Barbara, December 2014 (expected)

PUBLICATIONS

"Achievable spectral radii of symplectic Perron-Frobenius matrices" *New York J Math.* 17 (2011), 683 - 697

"An Alternative Approach to Extending pseudo-Anosovs Over Compression Bodies". To appear in *Algebraic Geometry and Topology.* Accepted Dec. 14, 2014

PROFESSIONAL EMPLOYMENT

2008-2012: Teaching Assistant, Department of Mathematics, University of California, Santa Barbara

FIELDS OF STUDY

Low dimensional topology, especially automorphisms of surfaces, advised by Professor Darren Long

Abstract

On pseudo-Anosov maps, symplectic, Perron-Frobenius matrices, and compression bodies

Robert Ackermann

In 1988, William Thurston announced the completion of a classification of surface automorphisms into three types up to isotopy: periodic, reducible, and pseudo-Anosov. The most common but also least understood maps in this classification are pseudo-Anosovs. We extend our understanding of pseudo-Anosov maps in two ways. First, we show that every Perron unit of appropriate degree has a power which appears as the spectral radius of a symplectic, Perron-Frobenius matrix. This is significant due to possible applications to understanding the spectrum of dilatations for a surface. Second, we present an alternative proof to an important result of Biringer, Johnson, and Minsky ([2]) showing roughly that a power of a pseudo-Anosov extends over a compression body if and only if the stable lamination bounds. Our alternative proof follows ideas of Casson and Long first presented in ([7]).

1 Introduction

A natural object of study in mathematics, arising both from our natural world and more abstract mathematical questions, are functions called surface automorphisms. More specifically, a surface automorphism is a continuous, bijective map from a surface to itself having continuous inverse. One can see a "real life" example of one by imagining a bucket containing a liquid with some number of stirring rods dipping into it. If the rods stir the liquid, return to their original position, and then allow the liquid's surface to come to rest, the result can be described as an automorphism of a punctured disk with a number of punctures equal to the number of stirring rods.

The collection of all automorphisms on a surface is in many cases an overly large object, so we usually restrict ourselves to orientation-preserving automorphisms and study them up to isotopy. Roughly speaking, two automorphisms are isotopic if one can be "deformed" into the other. Similarly, we consider surfaces up to homeomorphism, for which a classification exists. For example, closed orientable surfaces are classified by their genus (i.e., they are all doughnuts with some number of holes).

On a sphere, up to isotopy there is only one orientation-preserving automorphism: the identity map. The torus has three types of automorphisms: periodic, reducible, and Anosov. An automorphism is periodic if it is isotopic to the identity after composing it with itself some number of times. On the Torus, an example would be the map interchanging parallel and meridian. An automorphism is reducible if it leaves invariant some closed 1-submanifold, for instance, a Dehn twist.

Anosov maps are more interesting. An Anosov map of the Torus leaves invariant a pair of transverse foliations, called the stable and unstable foliation. Every leaf of both foliations is an embedded copy of $\mathbb{R}$. It "stretches" the stable foliation and "shrinks" the unstable, so that the any curve on the surface begins to look like a leaf of the stable foliation after iterations of the automorphism. Though these automorphisms may seem complex, they are actually well understood because the group of all orientation preserving automorphisms of the torus up to isotopy is isomorphic to $\mathrm{SL}(2, \mathbb{Z})$. In fact, their dynamics can be understood by examining the action of $\mathrm{SL}(2, \mathbb{Z})$ on the universal cover $\mathbb{R}^2$.

On higher genus closed surfaces (or punctured surfaces with negative Euler characteristic), a similar classification exists. Nielsen initiated a study of automorphisms of these surfaces and his work was later completed by William Thurston in 1988 ([18]). In particular, for any surface of genus 2 or greater every surface automorphism is either periodic, reducible, or pseudo-Anosov. Periodic and reducible automorphisms are defined in the same way as above, but pseudo-Anosovs are more complex. There are different definitions, but we will say that an automorphism is pseudo-Anosov if it leaves invariant a pair of transverse, geodesic laminations called the stable and unstable lamination. Like Anosov maps on the torus, every leaf of the two laminations is an embedded copy of $\mathbb{R}$ and the stable lamination is "stretched" while the unstable is "shrunk". Curves begin to look like the stable lamination under iteration of the automorphism.

Pseudo-Anosov automorphisms are less understood than their Anosov counterparts. The group structure of orientation-preserving automorphisms on a negative Euler characteristic surface is much more complex than on the torus (with the exception of the four-punctured sphere). Yet developing a better understanding is important both due to the inherent importance of pseudo-Anosovs and their connection to the study of hyperbolic 3-manifolds, Teichmuller space, dynamical systems, and more. Some recent research has focused on dilatations of pseudo-Anosovs and extensions to compression bodies.

The dilatation of a pseudo-Anosov is the stretch factor for the stable lamination. From it one can learn other properties of the map, including its topological entropy, the growth rate of the length of geodesic curves under iteration of the map, and growth rate of intersections under iteration. The set of all dilatations on a particular surface is a discrete set and has a least element, the log of which is the systole of the moduli space for the surface. Recently, Farb, Leininger, and

Margalit showed that the mapping tori of all small dilatation pseudo-Anosovs on a closed surface are obtained by Dehn filling on a finite collection of cusped hyperbolic 3-manifolds ([11]). Ian Agol has presented an alternative proof, inspired by ideas of Hammenstadt which using ideal triangulations and splitting sequences of train tracks ([1]).

In section 3, we present some work towards understanding the collection of all pseudo-Anosovs which can occur on a particular surface, called the *spectrum of dilatation*. In particular, it is well known that all dilatations must be Perron units which appear as the largest eigenvalue of symplectic, Perron-Frobenius matrices. We show that every non-rational Perron unit appears in this way, after raising it to a high enough power:

Theorem. *Let M be an integral matrix with a unique, real eigenvalue of largest modulus greater than 1. Suppose also that this eigenvalue has algebraic multiplicity 1, and that M preserves a symplectic form L.*

Then $\exists n \in \mathbb{N}$ and $B \in \mathrm{GL}(2g)$ such that $B^{-1}M^nB$ is an integral, Perron-Frobenius matrix which preserves L.

The question of when a pseudo-Anosov extends over some compression body has been of interest both out of general interest and towards understanding the cobodorism group. In [7], Casson and Long provide an algorithm for determining when a pseudo-Anosov extends over some compression body. More recently, Biringer, Johnson, and Minsky ([2]) show that a power of a pseudo-Anosov extends if and only if the stable lamination is the limit of meridians:

Theorem. *Let $\varphi : F \to F$ be a pseudo-Anosov with stable lamination $\mathcal{L}^+$ and unstable lamination $\mathcal{L}^-$. Say also that a lamination $K^+ \supseteq \mathcal{L}^+$ bounds in a compression body M and M is minimal with respect to this condition.*

Then there exists k such that φ^k extends over M.

Ulrich Oertel has investigated these questions from a different perspective, finding a classification of automorphisms of handlebodies similar to that of surface automorphisms ([16]).

In section 4, we present an alternative proof of Biringer, Johnson, and Minsky's result. This uses ideas originally presented by Casson and Long. An intuitive understanding of when pseudo-Anosovs extend is still out of reach, but we hope by studying various approaches to this question some greater understanding will eventually be gained.

2 Preliminaries

Throughout, let S be an orientable surface of negative Euler characteristic (without boundary). An *automorphism* of S is a self-homeomorphism of S, and we will say two automorphisms are in the same *mapping class* if they are isotopic to one another. In 1988, Thurston announced a classification theorem for mapping classes based on the work of Nielsen ([18]):

Theorem 1. *Let $\varphi : S \to S$ be a surface automorphism. Then φ is isotopic to an automorphism which is at least one of the following types:*

1. *Periodic*
2. *Reducible*
3. *pseudo-Anosov*

Further, if φ is isotopic to a pseudo-Anosov then it is not isotopic to a periodic or reducible automorphism.

An automorphism φ is *periodic* if φ^k is isotopic to the identity for some k, and an automorphism is reducible if it leaves some closed 1-submanifold of S invariant. Reducible automorphisms get their name because we can cut along the invariant

1-submanifold and study the "reduced" automorphisms on each component of the simpler surface.

The definition of a pseudo-Anosov is more complicated. A *geodesic lamination* on a surface S is a closed subset which can be written as a union of disjoint geodesics. Given such a lamination, a *traverse measure* for $\mathcal{L}$ is a measure on the collection of arcs in S transverse to the leaves of $\mathcal{L}$. Further, the measure of an arc α is preserved under isotopy which preserves the leaves of $\mathcal{L}$ which α crosses (see [4] for a more precise definition). A pairing of $\mathcal{L}$ with a transverse measure μ is called a *measured lamination* and is denoted $(\mathcal{L}, \mu)$.

A *pseudo-Anosov* is an automorphism φ which leaves invariant a pair of transverse, measured laminations $(\mathcal{L}^+, \mu_+)$ and $(\mathcal{L}^-, \mu_-)$ called the *stable* and *unstable* lamination respectively. Furthermore, $\varphi \cdot (\mathcal{L}^+, \mu_+) = (\mathcal{L}^+, \lambda\mu_+)$ and $\varphi \cdot (\mathcal{L}^-, \mu_-) = (\mathcal{L}^-, 1/\lambda\mu_-)$ for some $\lambda > 1$. The number λ is called the *dilatation* of φ.

Pseudo-Anosov automorphisms get their name because they are an analog of Anosov automorphisms on the Torus, and like Anosov automorphisms they exhibit source-sink dynamics. Specifically, if C is any essential simple closed curve on S and φ is a pseudo-Anosov, then $\varphi^k(C)$ approaches the stable lamination $\mathcal{L}^+$. Here and throughout we measure distance between closed sets with the *Hausdorff metric*, which is defined by $d_H(A, B) \leq \epsilon$ if there are ϵ-neighborhoods $N_\epsilon(A) \subseteq B$ and $N_\epsilon(B) \subseteq A$.

2.1 Symplectic and Perron-Frobenius matrices

A *symplectic form* is a non-degenerate, skew-symmetric bilinear form; that is, a biinear form $\omega : \mathbb{R}^{2g} \times \mathbb{R}^{2g} \to \mathbb{R}$ is symplectic if:

1. $\omega(v, w) = 0$ for all $w \in \mathbb{R}^{2g}$ then $v = 0$

2. $\omega(v, w) = -\omega(w, v)$ for all $v, w \in \mathbb{R}^{2g}$

The *symplectic group* $\mathrm{Sp}(2g, \mathbb{R})$ is the group of all linear transformations $T : \mathbb{R}^{2g} \to \mathbb{R}^{2g}$ such that $\omega(Tv, Tw) = \omega(v, w)$ for all $v, w \in \mathbb{R}^{2g}$. We similarly define $\mathrm{Sp}(2g, \mathbb{Q})$ and, although $\mathbb{Z}$ is not a field, the integral symplectic group $\mathrm{Sp}(2g, \mathbb{Z})$. There is a symplectic form in each even dimension, though there are none in odd dimensions.

After fixing a basis, any symplectic form ω has a matrix representation L. We will use the notation $\mathrm{Sp}(2g, \mathbb{Z}, L)$ to represent the group of all matrices with integral entries which preserve the symplectic form L (that is, $M \in \mathrm{Sp}(2g, \mathbb{Z}, L)$ if $M^T L M = L$). There are two standard symplectic forms we will work with. They are:

$$J = \begin{pmatrix} 0 & 1 & & & 0 \\ -1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & 1 \\ 0 & & & -1 & 0 \end{pmatrix}$$

and

$$K = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

where I represents the $g \times g$ identity matrix. We include K because it tends to be easier to work with when performing calculations, and we include J because it arises more easily from the study of surface automorphisms. Note that for every even dimension, all matrix symplectic forms are conjugate to one another.

Every symplectic transformation has a characteristic polynomial whose coefficients are palindromic, i.e., a polynomial of the form:

$$p(t) = 1 - a_2 t - a_3 t^2 - ... - a_{g+1} t^g - a_g t^{g+1} - ... - a_2 t^{2g-1} + t^{2g} \tag{0.1}$$

We will call such polynomials *self-reciprocal*. An equivalent definition is that $p(t)$ is self-reciprocal if and only if $p(t) = t^{2g} p(t^{-1})$. Note that if α is the root of a self-reciprocal polynomial $p(\alpha)$, then $\alpha^{(2g)} p(1/\alpha) = p(\alpha) = 0$ and $1/\alpha$ is a root as well. If $p(t)$ has integral coefficients, then every root of $p(t)$ is an *algebraic unit*, that is, the root of a monic polynomial having constant coefficient ± 1.

If both λ and $1/\lambda$ are roots of a monic, irreducible polynomial $f(t)$, then $f(t)$ is self-reciprocal. This is because in this case λ and $1/\lambda$ are also both roots of $t^{2g} f(t^{-1})$ and by uniqueness of minimal polynomials $f(t) = t^{2g} f(t^{-1})$. It is also useful to note that if $f(t)$ is any degree $2g$ monic polynomial with constant coefficient ± 1 then $t^{2g} f(t) f(t^{-1})$ is a self-reciprocal polynomial.

We say a real matrix M is *Perron-Frobenius* if it has all nonnegative entries and M^k has strictly positive entries for some $k \in \mathbb{N}$. Such matrices have important applications in dynamical systems, graph theory, and in studying pseudo-Anosov surface automorphisms. A key result about such matrices was proved in the early 20th century:

Perron-Frobenius Theorem. *Let M be Perron-Frobenius. Then M has a unique eigenvalue of largest modulus λ. Furthermore, λ is real, positive, and has an associated real eigenvector with all positive entries.*

The eigenvalue λ is called the *spectral radius* or *growth rate* of M.

2.2 Dilatations

The dilatation λ of a pseudo-Anosov φ gives a great deal of information about the map. First, $\log(\lambda)$ is the topological entropy of φ (see [12]). Also of interest is that it gives the growth rate of the length of geodesics under iteration of φ and the rate at which geometric intersection between a fixed curve and images of another curve grows:

Theorem 2. *Let $\varphi : S \to S$ be a pseudo-Anosov with dilatation λ. Let C and D be essential simple closed curves on S, and let $i\,(\ ,\ )$ denote geometric intersection. Then:*

$$\lim_{k\to\infty} \frac{i\left(C, \varphi^k(D)\right)}{\lambda^k} = P$$

where $P > 0$.

Further, if S is compact and $l(C)$ denotes the length of the geodesic representative of the isotopy class of C, then:

$$\lim_{k\to\infty} \sqrt[k]{l\left(\varphi^k C\right)} = \lambda$$

See [9] for a proof.

The collection of all dilatations that can occur on a surface S is known to be a discrete subset of $\mathbb{R}$, and in particular has a minimum λ_S. It turns out that λ_S is the systole of the moduli space associated to S. It is in principle possible to calculate λ_S for every surface, however, very little is known about the spectrum of dilatations in general.

Recall that a number is an algebraic unit if it is the root of a monic polynomial with integer coefficients having constant term ± 1. A number is a *Perron unit* if it is an algebraic unit and greater in modulus than all of its algebraic conjugates.

Theorem 3. *Let S be a closed surface of genus g and $\varphi : S \to S$ a pseudo-Anosov with dilatation λ.*

Then λ is a Perron unit of degree at most $6g - 6$.

We first summarize how the bound on degree is proven (see [9] for more details). Here we will use an alternative definition of pseudo-Anosov, in which it preserves a pair of singular, transverse, measured foliations rather than laminations. If φ has orientable foliations, then they come from a closed 1-form and the dilatation appears as an eigenvalue of $\varphi^* : H^1(S) \to H^1(S)$. Since $\dim(H^1) = 2g$, the degree of the dilatation λ is at most $2g$ in this case.

If φ has non-orientable foliations, then we examine the double branched cover $\tilde{S}$, with branch points corresponding to odd pronged singularities of the stable foliation of φ. The cover $\tilde{\varphi} : \tilde{S} \to \tilde{S}$ has orientable foliations, and so the dilatation of φ appears as an eigenvalue of $\tilde{\varphi}^* : H^1\left(\tilde{S}\right) \to H^1\left(\tilde{S}\right)$. An Euler characteristic calculation shows that $\dim H^1\left(\tilde{S}\right) \leq 8g-6$. The deck transformation τ of $\tilde{S}$ gives a decomposition $H^1\left(\tilde{S}\right) = V_+ \oplus V_-$, and furthermore one can show that V_- is invariant under $\tilde{\varphi}^*$ and that the dilatation of φ is an eigenvalue of the restriction of the action of $\tilde{\varphi}^*$ to V_-. The deck transformation τ gives an isomorphism of V_+ onto $H^1(S)$, and so we conclude that V_- has dimension $6g-6$.

That dilatations are in fact Perron units can be seen using Markov partitions or train tracks, two different ways of describing the dynamics of pseudo-Anosovs. In both of these cases, the dilatation appears as the spectral radius of a Perron-Frobenius matrix and thus is a Perron unit. The theory of train tracks, in fact, further implies that this matrix also preserves a symplectic form.

There is a tighter degree bound in the case that the dilatation of φ is of odd degree. First we need the following lemma:

Lemma 4. *Say $f(x)$ is a reciprocal polynomial of degree $2n$ and that $f(x) = p(x)\,q(x)$ with $p(x)$, $q(x)$ both polynomials of odd degree.*

Then if $p(x)$ has degree greater than n, we have that $p(x)$ is not irreducible.

Proof. Say $p(x)$ is irreducible. Then ± 1 is not a root of $p(x)$, and since $p(x)$ is of odd degree we see that $p(x)$ cannot be reciprocal. But $f(x)$ is reciprocal, so there must be a root λ of $p(x)$ such that $1/\lambda$ is a root of $q(x)$. Now, λ and $1/\lambda$ both have the same degree, which by irreducibility of $p(x)$ is greater than n. Therefore the degree of $q(x)$ is greater than n, and the degree of $f(x)$ is greater than $2n$, a contradiction. □

From this lemma and discussion above, one obtains the following theorem:

Theorem 5. *Let S be a closed surface of genus g and $\varphi : S \to S$ be a pseudo-Anosov with dilatation λ. Say that λ has odd degree.*

Then the degree of λ is at most $3g-3$.

Proof. First note that if the foliations of φ are orientable then λ is an eigenvalue of $\phi^* : H^1(S) \to H^1(S)$ and thus has degree at most $2g$. Therefore assume that the foliations of φ are non-orientable.

Let $\tilde{S}$ be the orientation double cover of S and recall from above the decomposition $H^1\left(\tilde{S}\right) = V_+ \oplus V_-$. The map $\tilde{\varphi}* : H^1(\tilde{S}) \to H^1(\tilde{S})$ has a restriction to V_- which is also symplectic. Therefore the dilatation λ appears as the root of a $6g-6$ reciprocal polynomial $f(x)$. By lemma 4, the irreducible factor of $f(x)$ having λ as a root is of degree at most $3g-3$. □

3 Symplectic, Perron-Frobenius Matrices and Perron Units

A possible obstruction to a Perron unit being the dilatation of some pseudo-Anosov is that it does not appear as a the spectral radius of any symplectic, Perron-Frobenius matrix. For example, any Perron unit with negative trace cannot appear in this way. We show in this section, however, that for any Perron unit λ there is some k so that λ^k is the spectral radius of a symplectic, Perron-Frobenius matrix.

We show this in two main steps. First, we construct a canonical form for a matrix preserving one of the standard symplectic forms and having a prescribed self-reciprocal polynomial as its characteristic polynomial. Since all symplectic forms are conjugate, this allows us to construct a matrix M with spectral radius λ which preserves any integral symplectic form L. Next, we show that some power of M has a conjugate which is Perron-Frobenius and still preserves L. More specifically:

Theorem 6. *Let M be an integral matrix with a unique, real eigenvalue of largest modulus greater than 1. Suppose also that this eigenvalue has algebraic multiplicity 1, and that M preserves a symplectic form L.*

Then $\exists n \in \mathbb{N}$ *and* $B \in \mathrm{GL}(2g)$ *such that* $B^{-1}M^{n}B$ *is an integral, Perron-Frobenius matrix which preserves* L*.*

3.1 A canonical form for self-reciprocal polynomials

In this section, we establish a canonical form for integral matrices with self-reciprocal characteristic polynomial. These matrices preserve a symplectic form which is standard in the sense that it arises naturally from the study of surface automorphisms.

Recall that a polynomial $p(t)$ over the integers is *self-reciprocal* if its coefficients are palindromic, i.e, $p(t)$ has the form

$$p(t) = 1 - a_2 t - a_3 t^2 - ... - a_{g+1} t^g - a_g t^{g+1} - ... - a_2 t^{2g-1} + t^{2g} \tag{0.2}$$

and the two standard symplectic forms:

$$J = \begin{pmatrix} 0 & 1 & & & 0 \\ -1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & 1 \\ 0 & & & -1 & 0 \end{pmatrix}$$

and

$$K = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

We now define two canonical forms for a matrix which has the self-reciprocal polynomial $p(t)$ as its characteristic polynomial. We will also show that each preserves one of the standard symplectic forms above. The first canonical form, denoted A below, preserves J (that is, $A^T J A = J$).

$$A = \begin{pmatrix}
0 & \dots & & & & & \dots & 0 & -1 \\
0 & a_2 & 0 & a_3 & \dots & 0 & a_g & 1 & a_{g+1} \\
1 & 0 & & & & & & & a_2 \\
0 & 1 & & & & & & & 0 \\
\vdots & & \ddots & & & & & & a_3 \\
 & & & & & & & & \vdots \\
 & & & & \ddots & & & & 0 \\
 & & & & & 1 & 0 & & a_g \\
0 & \dots & & & \dots & 0 & 1 & 0 & 0
\end{pmatrix}$$

By performing the change of basis which carries J to K, we obtain a second canonical form, denoted B, which preserves K.

$$B = \begin{pmatrix} 0 & \dots & & & & & \dots & -1 \\ 1 & & & & & & & a_2 \\ & \ddots & & & & & & a_3 \\ & & \ddots & & & & & \vdots \\ & & & 1 & a_2 & a_3 & \dots & a_{g+1} \\ & & & & \ddots & & & 0 \\ & & & & & \ddots & & \vdots \\ 0 & & & & & & 1 & 0 \end{pmatrix}$$

The proofs of this section could be considered tedious, and the uninterested reader should have no problems skipping to section 3.2 after first reading theorem 9.

Lemma 7. *A preserves the symplectic form J and B preserves the symplectic form K.*

Proof. It suffices to show that B preserves K. Let $\{e_1, ..., e_{2g}\}$ denote the standard basis vectors for $\mathbb{R}^{2g}$. We note that the action of B on e_i is:

$$Be_i = e_{i+1} \qquad \text{if } 1 \leq i \leq g$$

$$Be_i = a_{i-g+1}e_{g+1} + e_{i+1} \qquad \text{if } g+1 \leq i \leq 2g-1$$

$$Be_{2g} = -e_1 + \sum_{i=2}^{g+1} a_i e_i$$

We now show that if $< \ , \ >$ is the bilinear form coming from K, $< Be_i, Be_k > = < e_i, e_k >$. Since this is all computational, we will do only a few cases here. A key observation to simplify calculations is that for $1 \leq i \leq g$ we have $< e_i, e_k > \neq 0$ if and only if $k = g + i$. In particular, $< e_i, e_{g+1} > \neq 0$ if and only if $i = 1$.

Assume first that $1 \leq i \leq g$. Then:

$$< Be_i, Be_k > = < e_{i+1}, Be_k > = \begin{cases} < e_{i+1}, e_{k+1} > & \text{if } 1 \leq k \leq g \\ < e_{i+1}, a_{k-g+1}e_{g+1} > + < e_{i+1}, e_{k+1} > & \text{if } g+1 \leq k \leq 2g-1 \\ < e_{i+1}, -e_1 > + < e_{i+1}, \sum_{j=2}^{g+1} a_j e_j > & \text{if } k = 2g \end{cases}$$

But checking our form K, we see that

$$< Be_i, Be_k > = \begin{cases} 0 & \text{if } 1 \leq k \leq g \\ 0+1 & \text{if } k = g+i \text{ and } g+1 \leq k \leq 2g-1 \\ 0+0 & \text{if } k \neq g+i \text{ and } g+1 \leq k \leq 2g-1 \\ 1+0 & \text{if } i = g \text{ and } k = 2g \\ 0+0 & \text{if } i \neq g \text{ and } k = 2g \end{cases}$$

A slightly more complicated case occurs if we let $g+1 \leq i \leq 2g-1$ and $k = 2g$. Then:

$$
\begin{aligned}
< Be_i, Be_k > &= a_{i-g+1} < e_{g+1}, Be_{2g} > + < e_{i+1}, Be_{2g} > \\
&= a_{i-g+1} + 0 + 0 + \sum_{j=2}^{g+1} a_j < e_{i+1}, e_j > \\
&= a_{i-g+1} - a_{i-g+1} \\
&= 0
\end{aligned}
$$

The other cases are not more difficult than the two above. □

Now we will show that A and B both have characteristic polynomials of the form (0.2).

Lemma 8. *The characteristic polynomials of A and B are both $p(t) = 1 - a_2 t - a_3 t^2 - ... - a_{g+1} t^g - a_g t^{g+1} - ... - a_2 t^{2g-1} + t^{2g}$*

Proof. As with the proof of lemma 7, we prove our result for B and the result immediately follows for A.

Let $B_0 = B - tI$, and let B_{k+1} be the matrix obtained from B_k by blocking off the first row and first column. Then the $(0, 2g-k)$ minor of B_k is 1 for $0 \leq k < g$.

Thus we see that

$$\det\left(B - tI\right) = 1 + a_2\left(-t\right) + \left(-a_3\right)\left(-t\right)^2 + ... + \left(-1\right)^g a_g \left(-t\right)^{g-1} + \left(-t\right)^g \det B_g \tag{0.3}$$

where B_g has form:

$$B_g = \begin{pmatrix} a_2 - t & a_3 & \dots & \dots & a_{g+1} \\ 1 & -t & & & 0 \\ & \ddots & \ddots & & \vdots \\ & & \ddots & \ddots & \vdots \\ 0 & & & 1 & -t \end{pmatrix}$$

Let $D_g = B_g$ and for $l \geq g$ let D_{l-1} be the matrix obtained from D_l by blocking off the last row and last column. Then for $g \geq l > 2$, the $(0, l)$ minor of D_l is 1. Thus we have:

$$\begin{aligned} \det B_g &= \left(-1\right)^{g+1} a_{g+1} + ... + \left(-t\right)^i \left(-1\right)^{g+1-i} a_{g+1-i} + ... + \left(-t\right)^{g-3}\left(-1\right)^4 a_4 + \left(-t\right)^{g-2} \det D_2 \\ &= \left(-1\right)^{g+1} a_{g+1} + ... + \left(-1\right)^{g+1} a_{g+1-i} t^i + ... + \left(-1\right)^{g+1} a_4 t^{g-3} + \left(-t\right)^{g-2} \det D_2 \end{aligned} \tag{0.4}$$

Notice that in the equation above that if g is even, then every coefficient is negative. If g is odd, every coefficient is positive. Now,

$$\det D_2 = \det \begin{pmatrix} a_2 - t & a_3 \\ 1 & -t \end{pmatrix} = t^2 - a_2 t - a_3 \tag{0.5}$$

Now by substituting (0.5) into (0.4) into (0.3), we obtain our result. □

Putting lemmas 7 and 8 together, we have the following theorem:

Theorem 9. *Every algebraic unit is an eigenvalue of some symplectic matrix.*

Proof. Let λ be an algebraic unit with minimum polynomial $q(t) = 1 + b_2 t + b_3 t^2 + ... + b_g t^{g-1} + t^g$. Then $t^g q(t) q(t^{-1})$ is a self-reciprocal polynomial. Applying lemmas 7 and 8 we obtain our result. □

3.2 Changing basis to be Perron-Frobenius

The main purpose of this section is to find integral matrices which can be conjugated to be Perron-Frobenius. We'd also like to do this in a way which preserves a fixed symplectic form (for example, the symplectic form J from section 3.1). In particular, we prove the following:

Theorem 10. *Let* $M \in \mathrm{Sp}(2g, \mathbb{Z}, L)$ *such that* M *has a unique, real eigenvalue of largest modulus greater than 1. Suppose also that this eigenvalue has algebraic multiplicity 1.*

Then $\exists n \in \mathbb{N}$ *and* $B \in \mathrm{GL}(2g)$ *such that* $B^{-1}M^nB$ *is a Perron-Frobenius matrix in* $\mathrm{Sp}(2g, \mathbb{Z}, L)$.

We will also obtain a similar result for integral, nonsingular matrices (see corollary 18).

Given a matrix M with a unique real eigenvalue of largest modulus greater than 1, we will denote this eigenvalue λ_M and an associated eigenvector v_M. We will refer to λ_M and v_M as the dominating eigenvalue and dominating eigenvector, respectively.

The idea behind the proof is to find an integral basis $\{b_1, ..., b_{2g}\}$ for $\mathbb{R}^{2g}$ such that v_M is contained in the cone determined by $b_1, ..., b_{2g}$. We also need that if W is the co-dimension 1 invariant subspace of M such that $v_M \notin W$, then $b_1, ..., b_{2g}$ all lie on the same side of W as v_M. To make the notion of side precise, denote by W^+ as the set of all vectors in $\mathbb{R}^{2g}$ that can be written as $av_M + w$ where $a \in \mathbb{R}^+$ and $w \in W$.

Lemma 11. *Let M be a matrix with a dominating real eigenvalue λ_M and an associated real eigenvector v_M. Say $\{b_1, ..., b_{2g}\}$ is a basis for $\mathbb{R}^{2g}$ such that $b_1, ..., b_{2g} \in W^+$ and v_M is contained in the interior of the cone determined by $b_1, ..., b_{2g}$.*

Then for some $n \in \mathbb{N}$, M^n has all positive entries after changing to the basis above.

Proof. Since we can replace M by M^2 if necessary, we may assume λ_M is positive. Let $\lambda_2, ..., \lambda_n$ be the other eigenvalues of M and let $v_M, v_2, ..., v_{2g}$ be a Jordan basis for M (i.e, a basis in which the linear transformation represented by M is in Jordan canonical form). Note that $v_2, ..., v_{2g}$ span W.

Consider a Jordan block associated to some eigenvalue λ_i of M:

$$J_i = \begin{pmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{pmatrix}$$

The definition of matrix multiplication guarantees that each entry of J_i^k will be a polynomial in λ_i. Each diagonal entry will equal λ_i^k and every other entry of J_i^k

will have degree strictly less than k. Thus we see that if v_j is a Jordan basis vector corresponding to the eigenvalue λ_i we get $\frac{J_i^k v_j}{\lambda_M^k} \to 0$ as $k \to \infty$, which implies:

$$\frac{M^k v_j}{\lambda_M^k} \to 0 \text{ as } k \to \infty \tag{0.6}$$

Since v_M is in the interior of the cone determined by $b_1, ..., b_{2g}$, for some positive real scalars $a_1, ..., a_{2g}$ we have $v_M = a_1 b_1 + ... + a_{2g} b_{2g}$. Furthermore, since $b_i \in W^+$, for some positive real scalar c_i and $w \in W$ we have $b_i = c_i v_M + w$. Since w may be expressed as a linear combination of $v_2, ..., v_{2g}$, we see that $\frac{M^k b_i}{\lambda_M^k} \to c_i v_M$ as $k \to \infty$ by (0.6). Rewriting v_M and w as (real) linear combinations of $b_1, ..., b_{2g}$, we see that for k large enough $M^k b_i$ is a positive linear combination of $b_1, ..., b_{2g}$. Hence, M^k has all positive entries in the basis $b_1, ..., b_{2g}$. □

The last paragraph of the proof above also gives us a quick but important corollary. We will use $|| \cdot ||$ to denote the standard Euclidean norm.

Corollary 12. *Let M as in lemma 11 and $v \in W^+$. Then $\frac{M^k v}{||M^k v||}$ approaches $\frac{v_M}{||v_M||}$ as $k \to \infty$.*

Our goal is now to construct a matrix $B \in \mathrm{Sp}(2g, \mathbb{Z}, L)$ such that the columns of B form a basis satisfying the hypotheses of lemma 11. The idea will be to construct a set of symplectic basis vectors which define a very narrow cone, and then apply a slightly perturbed symplectic isometry of the sphere $\mathbf{S}^{2g-1}$ to move that cone into the correct position.

A symplectic linear transformation τ is a (symplectic) transvection if $\tau \neq 1$, τ is the identity map on a codimension 1 subspace U, and $\tau v - v \in U$ for all $v \in \mathbb{R}^{2g}$. Geometrically, a tranvection is a shear fixing the hyperplane U. A symplectic transvection preserving the symplectic form J can be written

$$\tau_{u,a} v = v + aJ\left(v, u\right) u$$

for some scalar a and vector $u \in \mathbb{R}^{2g}$. Note that the fixed subspace is $< u >^{\perp}$ and that it contains u. $\mathrm{Sp}\left(2g\right)$ is generated by transvections (see [8]). If we wish to preserve a symplectic form L different from J, simply replace J with L in the formula.

Let $u \in \mathbb{R}^{2g}$ be the vector $(-1, 1, ..., -1, 1)$ and set $a = 1$. Let $e_1, ..., e_{2g}$ be the standard basis for $\mathbb{R}^{2g}$. Notice $J\left(e_i, u\right) = 1$, so $\tau_{u,1} e_i = e_i + u$. Thus, in matrix form:

$$\tau_{u,1} = \begin{pmatrix} 0 & -1 & & -1 & -1 \\ 1 & 2 & & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & & 0 & -1 \\ 1 & 1 & & 1 & 2 \end{pmatrix}$$

Composing this with transvections $\tau_{e_k,2}$ with k even, we get the symplectic matrix

$$A = \begin{pmatrix} 2 & 3 & & 1 & 1 \\ 1 & 2 & & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & & 2 & 3 \\ 1 & 1 & & 1 & 2 \end{pmatrix}$$

This matrix preserves the symplectic form J, and is also Perron-Frobenius. In fact, we can find such a matrix for any integral symplectic form:

Lemma 13. *There is a Perron-Frobenius matrix in* $\mathrm{Sp}\,(2g, \mathbb{Z}, L)$ *for any integral symplectic form* L*.*

Proof. Non-degeneracy of L guarantees that there is $u \in \mathbb{Q}^{2g}$ such that $L\,(e_i, u) = 1$ for every basis vector e_i. Let $w = (1, 1, ..., 1) \in \mathbb{Q}^{2g}$, and notice that $L\,(u, w) = -2g$. Then $\tau_{u,a} e_i = e_i + au$ for a very large we have that $\tau_{u,a} e_i$ is close to cu for some $c \in \mathbb{N}$. Now by continuity, $L(\tau_{u,a}e_i, w) = l < 0$ and for $b \in \mathbb{N}$ we have $\tau_{w,-b}\tau_{u,a}e_i = \tau_{u,a}e_i - blw$. Thus for b large enough, $\tau_{w,-b}\tau_{u,a}e_i$ is a rational vector with positive entries for all i. This transformation has Perron-Frobenius matrix representation. If it is not integral, we can adjust the values of a and b to clear denominators. $\square$

Let $\mathrm{U}(g)$ denote the group of unitary linear transformations of $\mathbb{C}^g$. Equivalently, we can think of the unitary group as a group of matrices: $\mathrm{U}(g) = \{M | M \in \mathrm{GL}(g, \mathbb{C}), M^*M = I\}$ where M^* denotes the conjugate transpose of M.

We identify $\mathrm{U}(g)$ with a subgroup of $\mathrm{GL}(2g, \mathbb{R})$ as follows: Let $M \in \mathrm{U}(g)$. Replace every entry $m = re^{(i\theta)} \in \mathbb{C}$ in M by the scaled 2×2 rotation matrix $R = \begin{pmatrix} r\cos(\theta) & -r\sin(\theta) \\ r\sin(\theta) & r\cos(\theta) \end{pmatrix}$ We now can consider $\mathrm{U}(g)$ as a group of real matrices acting on $\mathbb{R}^{2g}$. Notice that if $m \mapsto R$, then $\bar{m} \mapsto R^T$. Thus, if $M = [m_{i,j}] \in \mathrm{U}(g)$ is identified with $N = [R_{i,j}]$, we have $M^*M = [\bar{m}_{i,j}]^T [m_{i,j}] \mapsto \left[R^T_{i,j}\right]^T [R_{i,j}] = N^T N = I$. Hence with this identification $\mathrm{U}(g)$ is a subgroup of the real orthogonal group $\mathrm{O}(2g)$ (in fact it is a subgroup of $\mathrm{SO}(2\mathrm{g})$).

Notice that the symplectic form J gets identified with the complex matrix

$$\begin{pmatrix} -i & & \\ & \ddots & \\ & & -i \end{pmatrix}$$

which is in the center of $\mathrm{U}(g)$. Then if $M \in \mathrm{U}(g)$ we have $M^* J M = J$, and thus $\mathrm{U}(g)$ is a subgroup of $\mathrm{Sp}(2g)$. Below is a more powerful result which is proved in [15] as lemma 2.17.

Lemma 14. $\mathrm{Sp}(2g) \cap \mathrm{O}(2g) = \mathrm{U}(g)$

We also need the following fact:

Lemma 15. *The unitary group* $\mathrm{U}(g)$ *acts transitively on* $\mathbf{S}^{2g-1} \subseteq \mathbb{R}^{2g}$.

Proof. The $\mathbf{S}^{2g-1}$ sphere can be thought of as all vectors in $\mathbb{C}^g$ having unit length. Let $v \in \mathbf{S}^{2g-1}$ and $\{e_1, ..., e_g\}$ be the standard basis for $\mathbb{C}^g$. Using the Gram-Schmidt process, we can extend v to an orthonormal basis $\{v, v_2, ..., v_g\}$ for $\mathbb{C}^g$. Then the change of basis matrix is in $\mathrm{U}(g)$ and sends e_1 to v. □

At one point during the proof of our main theorem, it will become important to know that $\mathrm{Sp}(2g, \mathbb{Q})$ is dense in $\mathrm{Sp}(2g)$. This follows quickly from the Borel Density Theorem, but we include an elementary proof.

Lemma 16. $\mathrm{Sp}\,(2g, \mathbb{Q})$ *is dense in* $\mathrm{Sp}\,(2g)$.

Proof. Let $M' \in \mathrm{Sp}\,(2g, \mathbb{R}, J)$. Perturb the entries of M' by a small amount to obtain a matrix M with rational entries. We will systematically modify the columns $a_1, b_1, ..., a_g, b_g$ of M to form a new M which preserves J and still differs from M' by a small amount. Here for convenience we let $< \ , \ >$ denote the symplectic form given by J.

We iterate the following procedure for each pair of columns a_i, b_i, starting with a_1, b_1. First, say $< a_i, b_i >= 1+\eta_i$ where η_i is a small, rational number (its magnitude depends on the size of the perturbation of M'). Replace a_i with $\dfrac{a_i}{1+\eta_i}$, so that now $< a_i, b_i >= 1$. Now we modify each pair of columns a_j, b_j with $j > i$. Set $\epsilon_{i,j} =< a_i, a_j >$ and $\delta_{i,j} =< b_i, a_j >$. Replace a_j with $a_j - \epsilon_{i,j} b_i - \delta_{i,j} a_i$, so that now $< a_i, a_j >=< b_i, a_j >= 0$. Note that $\epsilon_{i,j}$ and $\delta_{i,j}$ are also small rational numbers. Now modify b_j by a similar procedure, so that $< a_i, b_j >=< b_i, b_j >= 0$.

Now repeat the procedure with the columns a_{i+1}, b_{i+1}. After modifying every column we obtain a new M which is in $\mathrm{Sp}\,(2g, \mathbb{Q}, J)$. Furthermore, since at each stage the modifications to the columns are small, M is still close to M'. □

We're now ready to prove theorem 10. Throughout we will use the notation that if $v \in \mathbb{R}^{2g} \setminus \{0\}$ then $\hat{v}$ denotes the normalization $v/||v|| \in \mathbf{S}^{2g-1}$. If M is a matrix with no zero columns, then $\hat{M}$ will denote the matrix obtained by normalizing each of the columns.

proof of theorem 10. Let $M \in \mathrm{Sp}(2g, \mathbb{Z}, L)$ with dominating real eigenvalue λ and associated eigenvector v_M. Let W be the co-dimension 1 invariant subspace of M with $v_M \notin W$, and W^+ the component of $\mathbb{R}^{2g} \setminus W$ containing v_M. Set ϵ to be the minimal distance in $\mathbf{S}^{2g-1}$ from $\hat{v}_M$ to $W \cap \mathbf{S}^{2g-1}$. Then by lemma 13 and corollary 12, there exists $n \in \mathbb{N}$ and $A \in \mathrm{Sp}(2g, \mathbb{Z}, L)$ such that A is Perron-Frobenius and the convex hull H of the columns of $\widehat{A^n}$ has diameter less than ϵ (here we take $H \subseteq \mathbf{S}^{2g-1}$ and measure distance in $\mathbf{S}^{2g-1}$).

Let ν be in the interior of H. Since $\mathrm{U}(g)$ acts transitively on $\mathbf{S}^{2g-1}$ (lemma 15), there is $S \in \mathrm{U}(g)$ such that $S\nu = \hat{v_M}$. As a real linear transformation, S is orthogonal and hence $\mathrm{diam}(H) = \mathrm{diam}(S(H))$. Thus the columns of $S\widehat{A^n}$ are contained in W^+. $\mathrm{U}(g)$ is a subgroup of $\mathrm{Sp}(2g)$ (lemma 14), so $S \in \mathrm{Sp}(2g)$. Furthermore, by lemma 16 we may perturb S slightly so that now $S \in \mathrm{Sp}(2g, \mathbb{Q}, L)$. Set $B' = SA^n$, note $B' \in \mathrm{Sp}(2g, \mathbb{Q}, L)$. Scale B' by an integer α so that $B = \alpha B'$ is a nonsingular, integral matrix.

Set $d = \det B$. Then $B^{-1} = \frac{1}{d}C$, where C is the adjugate of B. In particular, C is integral.

Consider the projection map $\mathrm{SL}(2g,\mathbb{Z}) \to \mathrm{SL}(2g,\mathbb{Z}/d\mathbb{Z})$. Since $\mathrm{SL}(2g,\mathbb{Z}/d\mathbb{Z})$ is finite, for some $m \in \mathbb{N}$ we have M^m in the kernel of this map. Hence, we can write $M^m = I + d\Lambda$ for some integral matrix Λ. Putting this together, we have:

$$\begin{aligned} B^{-1}M^mB &= \frac{1}{d}C\left(I + d\Lambda\right)B \\ &= I + C\Lambda B \end{aligned}$$

In particular, $B^{-1}M^mB$ is integral. By construction, the columns of B give a basis satisfying the conditions of lemma 11, so for large enough $k \in \mathbb{N}$ we have $B^{-1}M^{mk}B$ is Perron-Frobenius and integral. Furthermore $B^{-1}M^{mk}B$ is symplectic since B is a scaled symplectic matrix. □

Using theorems 9 and 10, we can prove our main result, which we restate here:

Theorem 17. *Let λ be a Perron unit, and let L be any integral symplectic form.*

Then for some $n \in \mathbb{N}$, λ^n is the spectral radius of an integral Perron-Frobenius matrix which preserves the symplectic form L.

Proof. Using the canonical form of section 3.1, we can build a matrix $M \in \mathrm{Sp}(2g,\mathbb{Z},J)$ with λ its spectral radius. For some $B' \in \mathrm{GL}(2g,\mathbb{Q})$ we have $(B')^T JB' = L$. Scale B' by an integer α so that $B = \alpha B'$ is integral. Now proceeding with the argument at the end of the proof for theorem 10, we get that $B^{-1}M^{r}B \in \mathrm{Sp}\left(2g, \mathbb{Z}, L\right)$. Now we can apply theorm 10 to obtain our result. $\square$

We end this section by noting that if the matrix M is not symplectic, we can modify the hypotheses slightly to achieve a result similar to theorm 10. The proof uses similar ideas, but is actually significantly easier.

Corollary 18. *Let M be an integral, nonsingular matrix with a unique, real eigenvalue of largest modulus greater than 1. Suppose also that this eigenvalue has algebraic multiplicity 1.*

Then $\exists n \in \mathbb{N}$ such that M^n is conjugate to an integral Perron-Frobenius matrix.

Proof. Let $\delta = \det M$, and pick a $B' \in \mathrm{SL}\left(r, \mathbb{Q}\right)$ such that the columns of B' satisfy the conditions of lemma 11. Choose $\alpha \in \mathbb{Z}$ such that $\widetilde{B} = \alpha B'$ has integer entries and δ divides every entry of $\widetilde{B}$. Assuming we also chose α to be large, we may set $B = \widetilde{B} + I$ and the columns of B will still satisfy lemma 11.

Consider $d = \det B$. Calculating the determinant by cofactor expansion, we see that $d = (\textit{sum of terms divisible by } \delta) + 1$. In particular, δ is relatively prime to d, so M projects to an element of $\mathrm{GL}\left(r, \mathbb{Z}/d\mathbb{Z}\right)$. It follows that $M^m = I + d\Lambda$ for some integral matrix Λ and we may proceed as in the proof of theorem 10. $\square$

3.3 Commuting symplectic Perron-Frobenius matrices and pseudo-Anosovs

In this section, we will explore an application of theorem 10 which has possible consequences to the study of the spectrum of dilatations for a particular surface. In particular, we are interested in exploring the following question:

Question 19. *Let λ be a Perron unit which is also the largest of the reciprocals of its algebraic conjugates. Is λ^k the dilatation of some pseudo-Anosov?*

More generally, if F is a number field containing λ, does the group of units of O_F contain a dilatation with degree equal to that of λ of some pseudo-Anosov?

Theorem 10 suggests the first question since it shows λ^k will arise as the dilatation of some symplectic, Perron-Frobenius matrix. The second question is broader and perhaps easier to answer.

Recall that a number field F is any finite field extension $\mathbb{Q}$. The algebraic integers of F, denoted O_F, are the elements of F which appear as roots of monic polynomials with coefficients in $\mathbb{Z}$. These form a ring, and the unit group U of O_F contains all elements of F which appear as roots of monic polynomials with constant coefficient ± 1. Dirichlet's unit theorem tells us that U is always a finitely generated abelian group of rank $r_1 + r_2 - 1$ where r_1 is the number of real embeddings of F

and r_2 is the number of conjugate pairs of complex embeddings of F. In practice, this means that if $F = Q(\alpha)$ then r_1 is the number of real conjugates of α and r_2 is the number of pairs of complex conjugates of α.

Since U is abelian, theorem 10 guarantees the existence of commuting, symplectic, Perron-Frobenius matrices. Such matrices are of interest because any non-commensurable pair of commuting, Perron-Frobenius matrices cannot both arise in the same manner from a pseudo-Anosov on a fixed surface:

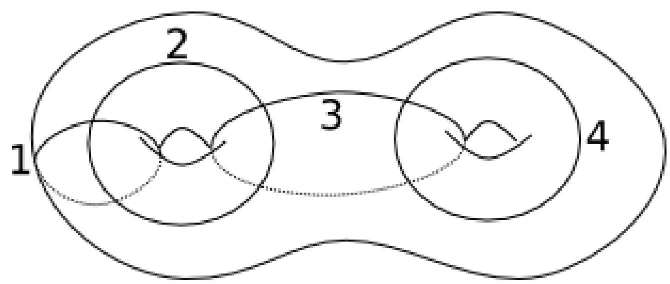
Theorem 20. *Say φ and ψ are pseudo-Anosovs on S with identical laminations $\mathcal{L}^+, \mathcal{L}^-$.*

Then $\varphi^k = \psi^l$ for some $k \in \mathbb{Z}$.

Proof. Let λ be the dilatation of φ and ν be the dilatation of ψ. Now, because φ and ψ have the same laminations, any map $\varphi^n \psi^m$ is pseudo-Anosov with dilatation $\lambda^n \nu^m$ (assuming it is not the identity). But the group action on $\mathbb{R}_+$ of the multiplicative group $< \lambda,\ \nu >$ is either isomorphic to $\mathbb{Z}$ or it has dense orbits. The latter case is not possible, for then there would be pseudo-Anosovs on S with dilatations arbitrarily close to 1. □

Note that the above theorem is equivalent to noting that non-commensurable pseudo-Anosov maps cannot commute.

We will now explore a couple of examples of commuting, symplectic Perron-Frobenius matrices. Let $p(t) = t^4 - 9t^3 + 18t^2 - 9t + 1$. Note that this is a self-reciprocal polynomial with four real roots, and in fact the largest root $\lambda \approx 6.405$ is the dilatation of a pseudo-Anosov on the closed genus 2 surface. More specifically, λ is the dilatation of the Dehn twists $T_2T_1T_3T_2T_4$ where T_i is a Dehn twist about the curve as labeled below.



Now, Dirichlet's unit theorem tells us that the group of units U for the number field $\mathbb{Q}(\lambda)$ is an abelian group of rank 3. Using a computer algebra system, we can compute generators for the torsion-free subgroup U:

$$u_1 = \lambda \tag{0.7}$$

$$u_2 = -u_1^3 + 9u_1^2 - 18u_1 + 7 \tag{0.8}$$

$$u_3 = -u_1 + 2 \tag{0.9}$$

Take M_1 to be the 4 x 4 matrix in rational canonical form for $u_1 = \lambda$. Formally substituting M_1 for u_1 in each of the equations above yields integral matrices M_2, M_3 whose characteristic polynomials have u_2, u_3 as roots respectively. Furthermore, since U is abelian, this guarantees that M_1, M_2, and M_3 commute. Note this also means that their eigenvectors are identical (up to scaling).

Unfortunately, neither M_2 nor M_3 have reciprocal characteristic polynomials and hence M_2, M_3 do not preserve any symplectic form. However, a quick search aided by a computer finds that $N = M_1^3 M_2 M_3$ has a self-reciprocal characteristic polynomial (in fact, there are many relatively short words in M_1, M_2, M_3 with this property). Also important is that N has the same dominating eigenvector as M_1, as one might expect given the M_1^3 part of the word.

Now, we will find a basis in which the transformations represented by $M = M_1$ and N are Perron-Frobenius after raising them both to a high enough power. The dominating eigenvector for both matrices is $v \approx (-0.156, 1.381, -2.595, 1)$. We choose as a change of basis matrix:

$$B = \begin{pmatrix} -5,656 & -11,549 & 0 & 0 \\ 50,021 & 102,138 & 0 & 0 \\ 0 & 0 & -9,5851 & -963,188 \\ 0 & 0 & 36,943 & 371,233 \end{pmatrix}$$

The entries were chosen by talking convergents for $\approx -0.156/1.381$ and for $\approx -2.595/1$. This technique does not guarantee that $B^{-1}M^kB$ is Perron-Frobenius for some k, but it does at least ensure that $B^{-1}v$ has positive entries and that the determinant of B is 1 (in fact, B will preserve the symplectic form J from section 3.1).

A computation (which should absolutely not be attempted by hand), shows that $P_1 = B^{-1}M^{23}B$ and $P_2 = B^{-1}N^6B$ are both Perron-Frobenius. These matrices commute, have self-reciprocal characteristic polynomials, and each preserves some symplectic form (though not necessarily the same one). Theorem 20 shows that P_1 and P_2 cannot both arise from a Pseudo-Anosov in the same manner.

Another example of commuting, symplectic Perron-Frobenius matrices arises from $\mathbb{Q}(\lambda)$ by using train tracks. The pseudo-Anosov $T_2T_1T_3T_2T_4$ is carried by a train track τ obtained by "smoothing out" the intersections of the four curves above. It's transition matrix is:

$$X_1 = \begin{pmatrix} 2 & 1 & 1 & 0 \\ 4 & 3 & 5 & 1 \\ 1 & 1 & 3 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

and X_1 preserves the symplectic form for τ:

$$L = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

(For more on Train Tracks and symplectic forms associated with them, see [17])

Following the process above, we obtain $X_2 = 6I - 18X_1 + 9X_1^2 - X_1^3$, $X_3 = 2I - X_1$, and $X_4 = X_1^3 X_2^{-1} X_3$. In particular,

$$X_4 = \begin{pmatrix} 118 & 87 & 177 & 48 \\ 372 & 271 & 552 & 153 \\ 153 & 111 & 226 & 63 \\ 27 & 21 & 42 & 10 \end{pmatrix}$$

That both X_1 and X_4 commute is guaranteed because they are both in U. It furthermore turns out that they both preserve L. This gives a $\mathbb{Z}_+ \times \mathbb{Z}_+$ monoid of Perron-Frobenius matrices which preserve L, but only those which can be written X_1^k for some k are transition matrices for a pseudo-Anosov by Theorem 20.

3.4 An application to subshifts of finite type

We will now apply previous results to symbolic dynamics, in particular to subshifts of finite type.

Let M be an $n \times n$ matrix of 0's and 1's. Let $A_n = \{1, 2, ..., n\}$, and form $\Sigma_n = A_n \times \mathbb{Z}$. We can think of Σ_n as the set of all bi-infinite sequences in symbols from A_n, and we endow it with the product topology. Now we form a subset $\Lambda_M \subseteq \Sigma_n$ by saying $(s_i) \in \Lambda_M$ if the s_i, s_{i+1} entry of M is equal to 1 for all i. We can think of the i, j entry of M as telling us whether it is possible to transition from state i to state j. Now let σ be the automorphism of Λ_M obtained by shifting every sequence one place to the left. The dynamical system (Λ_M, σ) is called a *subshift of finite type*, and can be thought of as a zero-dimensional dynamical system. These dynamical systems have relatively easy to understand dynamics and are often used to model more complicated systems (for example, pseudo-Anosov automorphisms).

Let $M = [m_{i,j}]$ be a square matrix with nonnegative, integer entries. We form a directed graph G from M as follows. G has one vertex for each row of M. Then connect the i-th vertex to the j-th vertex by $m_{i,j}$ edges, each directed from vertex i to vertex j. We call M the *transition matrix* for G. If M is Perron-Frobenius, then the graph G will be strongly connected and the i, j-th entry of M^k represents the number of paths of length k from vertex i to vertex j. The spectral radius λ

of M can be interpreted as the growth rate of the number of paths of length k in G, i.e. $\lim_{k\to\infty} \frac{M^k}{\lambda^k} = P \neq 0$.

We now show how to go from an integral Perron-Frobenius matrix M to another matrix with the same spectral radius whose entries are all 0 or 1. This construction can also be found in [10]. Given a directed graph G with Perron-Frobenius transition matrix M, label the edges of G as $e_1, ..., e_n$ and the vertices $v_1, ..., v_m$. From G, we form a directed graph H as follows: the vertex set $w_1, ..., w_n$ of H is in 1 - 1 correspondence with the edge set of G ($w_i \leftrightarrow e_i$). If the edge e_i terminates at the vertex from which e_j emanates, then we place an edge in H from w_i to w_j. Let N be the transition matrix of H. Note that by construction, every entry of N is either a 0 or a 1.

A subgraph of a graph G is a *cycle* if it is connected and every vertex has in and out valence 1. If M is a transition matrix for G, it is possible to reformulate the calculation of the characteristic polynomial $p(t) = \det(tI - M)$ in terms of cycles in G (see [3]):

Lemma 21. *Let G be a graph with transition matrix M. Denote by $\mathbf{C_i}$ the collection of all subgraphs which have i vertices and are the disjoint union of cycles. For $C \in \mathbf{C_i}$, denote by $\#(C)$ the number of cycles in C. Then the characteristic polynomial $p(t) = \det(tI - M)$ is*

$$p(t) = t^m + \sum_{i=1}^{m} c_i t^{m-i}$$

where m is the number of vertices in G and

$$c_i = \sum_{C \in \mathbf{C_i}} (-1)^{\#(C)}$$

Using this formula, we can prove that the characteristic polynomial of N (as above) has a nice form, and in particular that the spectral radius of N is the same as the spectral radius of M.

Theorem 22. *Let M be the transition matrix for a graph with m vertices and let N be an $n \times n$ matrix of 0's and 1's built from M by the construction above.*

Then if $p(t) = \det(tI - M)$ is the characteristic polynomial of M, the characteristic polynomial of N is $q(t) = t^{n-m} p(t)$

Proof. Let G be the graph associated to M, and H the graph associated with N. Order the vertices of G, and for each vertex v fix a lexicographic order of (*in-edge*, *out-edge*) pairs of edges incident to v. Let $\mathbf{D_i}$ be the collection of subgraphs of H which can be written as a union of disjoint cycles with i total vertices. For $D \in \mathbf{D_i}$, there is a canonical projection of D to a collection of paths in G (using the fact that vertices in H come from edges in G). Let $\mathbf{D_i^*}$ be the subset of $\mathbf{D_i}$ containing those disjoint unions of cycles in H which do not project to a disjoint union of cycles in G. We will show that there is a bijection between elements of $\mathbf{D_i^*}$ having an odd number of components and elements of $\mathbf{D_i^*}$ having an even number of components.

Let $D \in \mathbf{D_i^*}$ and say D has an odd number of components. Call C its projection to a collection of paths in G. Since C is not a disjoint union of cycles, there must be vertices of G that are either visited by two different paths in C and/or are visited twice by the same path. Choose v to be the minimal such vertex in the ordering of vertices of G, and note that v must have in-valence and out-valence both of at least 2. Choose two in/out-edge pairs, (e, f) and (e', f'), such that each pair occurs in some path in C and so that they are minimal among such pairs in the ordering of edges incident to v. Note that D contains vertices in H corresponding to e, e', f, f' and must contain edges from e to f and from e' to f'. Build $D' \in \mathbf{D_i^*}$ by letting D' have the same vertex collection as D, but instead of containing edges from e to f and from e' to f' it contains edges from e to f' and e' to f (call this operation an *edge swap*).

If the pairs (e, f) and (e', f') are both part of the same cycle in D, then D' will have one more component than D. If they are part of two different cycles, then D' will have one less component. In either case, D' has an even number of components and we have constructed a well-defined map from elements of $\mathbf{D_i^*}$ having odd components to elements having even components. Note also that the projection C' of D' still visits v twice, and contains in/out-edge pairs (e, f') and (e', f). Thus we can define the inverse of this map in exactly the same way, and hence we have a bijection.

Because of the bijection we built above, we see that disjoint unions of cycles in $\mathbf{D_i^*}$ cancel out when $q(t)$ is computed using lemma 21. Elements of $\mathbf{D_i} \setminus \mathbf{D_i^*}$ are in bijective correspondence with cycles in $\mathbf{C_i}$, so we get our conclusion. □

Finally, we have:

Theorem 23. *Let λ be a Perron unit. Then there is $k \in \mathbb{N}$ such that $\log\left(\lambda^k\right)$ is the topological entropy of some subshift of finite type.*

This follows directly from theorems 10, 9, 22, and comments of Fathi, Laudenbach, and Poénaru on subshifts of finite type (see [12]).

4 Extension of pseudo-Anosovs over Compression Bodies

Throughout this section, let F be a closed, orientable surface of genus at least two. A *compression body* is any 3-manifold formed by taking $F \times I$, attaching disjoint 2-handles to the boundary surface $F \times \{1\}$, and filling in any resulting 2-spheres with 3-handles. The boundary surface $F \times \{0\}$ is called the *exterior surface* of M. Call $F \times I$ the *trivial compression body.* We say that an automorphism φ of F *extends over a compression body* M if there is an automorphism $\psi : M \to M$ such that $\psi|_F = \varphi$ where F is the exterior surface of M.

In this section, we will present an alternate proof to a result of Biringer, Johnson, and Minsky first proved in [2]:

Theorem 24. *Let $\varphi : F \to F$ be a pseudo-Anosov with stable lamination $\mathcal{L}^+$ and unstable lamination $\mathcal{L}^-$. Say also that a lamination $K^+ \supseteq \mathcal{L}^+$ bounds in a compression body M and M is minimal with respect to this condition.*

Then there exists k such that φ^k extends over M.

Here we say that a lamination bounds if it is the Hausdorff limit of curves bounding disks in the compression body, and that a compression body M is minimal with respect to the condition that K^+ bounds if there is no inequivalent $N \subset M$ in which K^+ bounds. Their proof makes use of relatively recent ideas including δ-hyperbolic geometry, the curve complex, and Ahlfors-Bers theory. They also give examples which show that their theorem is false if φ^k is replaced with φ in the conclusion.

Here we present an alternative proof of this result using older ideas first introduced by Casson and Long. More specifically, in [7] Casson and Long provide an algorithm for determining whether a particular pseudo-Anosov extends over some compression body and in [13] Long goes on to show that a pair of minimal, transverse laminations can bound in only finitely many compression bodies.

This alternative proof is achieved by generalizing lemmas of Casson and Long. The basic idea is to show that disks of a particular type must exist in any compression body in which the stable lamination of φ bounds and some curve approximating the unstable lamination bounds as well. Using these disks, we build a non-empty but finite collection of compression bodies over which φ could potentially extend. Within this collection there is a (possibly smaller) collection which is invariant under the action of φ, implying that a power of φ extends.

4.1 Definitions and basic facts

A compression body M with exterior surface F has associated to it a normal subgroup $N = \ker(i_* : \pi_1(F) \to \pi_1(M))$ where $i : F \to M$ is inclusion. N is equal to the image of the fundamental group of a regular covering of F which is planar, so we call N the *planar kernel* of M (see [13]). The planar kernel can also be defined as the normal closure of the curves in F 2-handles were attached along to form M.

Given a fixed surface F, let $M_1 = (F \times I) \cup P_1$ and $M_2 = (F \times I) \cup P_2$ be two compression bodies formed by attaching collections P_1, P_2 of handles to $F \times \{1\}$. Say M_1 and M_2 are *equivalent* if the planar kernels $N_1 = \ker(i_* : \pi_1(F) \to \pi_1(M_1))$ and $N_2 = \ker(i_* : \pi_1(F) \to \pi_1(M_2))$ are equal. If M_1 and M_2 are equivalent, then any time a curve in F bounds a disk in M_1 there is an isotopic curve in F which bounds a disk in M_2.

Recall that a *geodesic lamination* $\mathcal{L}$ on F is a closed subset which can be written as the union of disjoint geodesic leaves. A geodesic lamination is *minimal* if the closure of any leaf is the whole lamination, and a geodesic lamination *fills* if each component of $F \setminus \mathcal{L}$ is simply connected. Call the closure of these components the *complementary regions* of $\mathcal{L}$, and say $\mathcal{L}$ is *maximal* if it has no isolated leaves and every complementary region is an ideal trigon. If $\mathcal{L}$ is one of the invariant laminations of a pseudo-Anosov, then $\mathcal{L}$ is always minimal and filling (but not necessarily maximal).

Crucial to our discussion is a definition from [7]:

Definition 25. *Let $\mathcal{L}$ be a geodesic lamination in the exterior surface of a compression body M. Then $\mathcal{L}$* bounds *in M if there is a sequence of simple closed curves $\{C_i\}$ all of which bound disks in M such that $C_i \to \mathcal{L}$ as $i \to \infty$.*

Here convergence is meant to be in the Hausdorff metric. However, if $\mathcal{L}^+$ and $\mathcal{L}^-$ are transverse, minimal, and maximal measured laminations with full support then $\mathcal{L}^+$ bounds after isotopy if and only if there is a sequence of essential simple closed curves $\{C_i\}$ all of which bound disks such that $\mu_+(C_i) \to 0$ as $i \to \infty$ (recall that we will always denote by μ_+, μ_- the transverse measures associated with $\mathcal{L}^+$, $\mathcal{L}^-$ respectively). In [2], a similar notion of bounding is expressed in terms of limit sets. One motivation for this definition is that if a pseudo-Anosov extends then both its stable and unstable lamination must bound by source-sink dynamics.

If $\mathcal{L}$ is a lamination, then we say that M *is minimal with respect to $\mathcal{L}$ bounding* if $\mathcal{L}$ bounds in M and if $N \subset M$ is a compression body inequivalent to M then $\mathcal{L}$ does not bound in N (we assume here that N and M have the same exterior surface F). Similarly, we say that M is minimal with respect to a collection of laminations $\mathcal{A}$ bounding if M is the smallest compression body in which every element of $\mathcal{A}$ bounds.

4.2 Disks in compression bodies

Throughout take $(\mathcal{L}^+, \mu_+)$ and $(\mathcal{L}^-, \mu_-)$ to be transverse, minimal, and maximal measured laminations. In this section, we construct a collection of bounding curves built from "short" arcs of $\mathcal{L}^-$ and "long" arcs of $\mathcal{L}^+$ with controlled μ_--measure. Necessary for the existence of these curves is that $\mathcal{L}^+$ bounds and some curve C approximating $\mathcal{L}^-$ bounds as well.

We begin by stating a lemma first proved in [13].

Lemma 26. *Let $\epsilon > 0$ be given. Then there are numbers $M(\epsilon)$, $m(\epsilon)$ such that:*

1. *If $\alpha^+ \subseteq \mathcal{L}^+$, $\alpha^- \subseteq \mathcal{L}^-$ are arcs with $\mu_-(\alpha^+) > M$ and $\mu_+(\alpha^-) > \epsilon$, then $\text{int}\ \alpha^+ \cap \text{int}\ \alpha^- \neq \emptyset$.*

2. *If $\alpha^+ \subseteq \mathcal{L}^+$, $\alpha^- \subseteq \mathcal{L}^-$ are arcs with $\mu_+(\alpha^-) < \epsilon$ and $|\text{int}\ \alpha^+ \cap \text{int}\ \alpha^-| \geq 2$, then $\mu_-(\alpha^+) > m$*

Furthermore, $M(\epsilon), m(\epsilon) \to \infty$ as $\epsilon \to 0$.

Let $N_\delta(\mathcal{L}^-)$ be a closed δ-neighborhood of $\mathcal{L}^-$. Such a neighborhood can be foliated by intervals so that it has the structure of a product, and each leaf t of the foliation (call these *ties*) can be thought of as an arc in F transverse to $\mathcal{L}^-$. Let $r = r(\delta) = \max\{\mu_-(t) \mid t \text{ is a tie of } N_\delta(\mathcal{L}^-)\}$.

Lemma 27. *Let C be a geodesic simple closed curve such that $d_H(C, \mathcal{L}^-) < \delta$ for some small $\delta > 0$ and say $\{A_n\}$ is a sequence of simple closed geodesics converging to $\mathcal{L}^+$. Then for any $\epsilon > 0$, there is N such that for all $n \geq N$ we have:*

1. *If $\alpha^+ \subseteq A_n$, $\alpha^- \subseteq C$ are arcs with $\mu_-(\alpha^+) > 2M$ and $\mu_+(\alpha^-) > \epsilon$, then* $\text{int } \alpha^+ \cap \text{int } \alpha^- \neq \emptyset$

2. *If $\alpha^+ \subseteq A_n$, $\alpha^- \subseteq C$ are arcs with $\mu_+(\alpha^-) < \epsilon$ and $|\text{int } \alpha^+ \cap \text{int } \alpha^-| \geq 2$, then $\mu_-(\alpha^+) > m - 2r$, where $r = r(\delta)$ as above.*

Proof. Given an arc $\alpha^- \subseteq C$, shrink it slightly and assume its endpoints are on leaves of $\mathcal{L}^+$ (without changing $\mu_+(\alpha^-)$). Then since $\delta > 0$ is small, we can slide its endpoints along leaves of $\mathcal{L}^+$ to obtain a nearby arc $\beta \subseteq \mathcal{L}^-$ with $\mu_+(\beta) = \mu_+(\alpha^-)$.

Choose N so that for all $n \geq N$ the curve A_n satisfies:

1. Lemma 26 holds with arcs of A_n in place of arcs in $\mathcal{L}^+$.

2. If $\alpha^- \subseteq C$ is an arc and $\beta \subseteq \mathcal{L}^-$ is chosen as above, then for any $p \in \text{int}\, \beta \cap A_n$ there is an arc $\phi \subseteq A_n$ with endpoints on β and α^- (one of these is p) with $\mu_-(\phi) < r = r(\delta)$.

These conditions can always be satisfied, after possibly a small isotopy of β, because the angles between nearby geodesics are close in a lamination (see [5]), and because the measure of an arc is preserved under homotopy respecting the leaves of $\mathcal{L}^-$.

To prove the first conclusion, let $n \geq N$ and take any arc $\alpha^+ \subseteq A_n$ with $\mu_-(\alpha^+) > 2M$. Say for contradiction that there is an arc $\alpha^- \subseteq C$ with $\mu_+(\alpha^-) > \epsilon$ such that $\text{int}\, \alpha^+ \cap \text{int}\, \alpha^- = \emptyset$. Take $\beta \subseteq \mathcal{L}^-$ as above and note that by condition (1) α^+ must intersect β at least twice. Thus by condition (2) both endpoints of α^+ must lie on short arcs of A_n with endpoints on int β and int α^-. But then by shrinking α^+ and allowing $\mu_-(\alpha^+)$ to change by at most $2r < M$ we obtain an arc which does not intersect β, a contradiction. Thus $\text{int}\, \alpha^+ \cap \text{int}\, \alpha^- \neq \emptyset$.

For the second conclusion, again fix $n \geq N$ and say $\alpha^- \subseteq C$, $\alpha^+ \subseteq A_n$ are arcs with $\mu_+(\alpha^-) < \epsilon$ and $|\text{int}\, \alpha^+ \cap \text{int}\, \alpha^-| \geq 2$. Again take $\beta \subseteq \mathcal{L}^-$ as above. Then by condition (2) α^+ can be extended to an arc that intersects int β at least twice with μ_--measure at most $2r$ more than $\mu_-(\alpha^+)$. Thus by lemma 26, $\mu_-(\alpha^+) > m - 2r$.

□

The next lemma mirrors the proof of lemma 2.4 in [13] and relies on lemma 27 to control the lengths of arcs.

Lemma 28. *Let $\delta > 0$ be small and say C be a geodesic simple closed curve with $d_H(C, \mathcal{L}^-) < \delta$. Say in addition that C and $\mathcal{L}^+$ both bound in a compression body M.*

Then for any small $\epsilon > 0$ there are arcs $\alpha^+ \subseteq \mathcal{L}^+$, $\alpha^- \subseteq \mathcal{L}^-$ such that $\alpha^+ \cup \alpha^-$ is the boundary of a disk and:

1. $\mu_+(\alpha^-) \leq \epsilon$
2. $m(2\epsilon) - 2r \leq \mu_-(\alpha^+) \leq 2M(\epsilon) + 2r$, *where* $r = r(\delta)$

Proof. Since $\mathcal{L}^+$ bounds, there is a sequence of closed geodesics $\{A_n\}$ all bounding disks in M and converging to $\mathcal{L}^+$. Choose N as in lemma 27 and let $A = A_n$ for some $n \geq N$. Let D^- be the disk with boundary C, and say D^+ is the disk with boundary A. We assume ϵ is very small compared to the μ_+-measure of C.

After isotopy, $D^+ \cap D^-$ is a collection of arcs with endpoints on $A \cap C$. Say an arc γ on C or A contains a *complete set* if whenever one endpoint of an arc in $D^+ \cap D^-$ is on γ the other endpoint is on γ as well. Choose an arc $\gamma^+ \subseteq A$ such that γ^+ is complete, $\mu_-(\gamma^+) \geq 2M$, and γ^+ is minimal with respect to these conditions. By lemma 27, γ^+ exists and intersects C many times.

Choose an arc $\phi \subseteq D^+ \cap D^-$ such that ϕ has endpoints on int γ^+ and ϕ is an outermost such arc in D^-. Then choose $\gamma^- \subseteq C = \partial D^-$ to be the arc with endpoints equal to the endpoints of ϕ and with the property that int $\gamma^- \cap$int $\gamma^+ = \emptyset$. By lemma 27, $\mu_+ (\gamma^-) < \epsilon$.

Now set $\beta^+ \subseteq A$ to be the sub-arc of γ^+ having endpoints $\partial\gamma^-$ (which equals $\partial\phi$), and note int $\beta^+ \cap$ int $\gamma^- = \emptyset$. We can stretch β^+ and γ^- a small amount so that $|\text{int } \beta^+ \cap \text{ int } \gamma^-| \geq 2$ and hence $\mu_- (\beta^+) \geq m (2\epsilon) - 2r$.

Since $\partial\beta^+ = \partial\phi \subseteq$ int γ^+, the arc β^+ is a complete, proper sub-arc of γ^+. Thus $\mu_- (\beta^+) \leq 2M$ by minimality of γ^+.

Now slide γ^- along the leaves of $\mathcal{L}^+$ to an arc α^- with $\mu_+ (\alpha^-) = \mu_+ (\gamma^-) < \epsilon$ and endpoints on leaves of $\mathcal{L}^+$. Isotopic to γ^+ is an arc $\alpha^+ \subseteq \mathcal{L}^+$ with $\partial\alpha^+ = \partial\alpha^-$ and $|\mu_- (\gamma^+) - \mu_- (\alpha^+)| \leq r$. The curve $\alpha^+ \cup \alpha^-$ is essential because it is the union of geodesic arcs, and is isotopic to the boundary of the disk formed by gluing pieces of D^+ and D^- cut out by ϕ.

□

4.3 Finitely many minimal compression bodies

Recall that a compression body M is *minimal* with respect to a collection of laminations $\mathcal{A}$ if every element of $\mathcal{A}$ bounds in M, and whenever $N \subseteq M$ is a compression body (with the same exterior surface) for which every element of $\mathcal{A}$ bounds, then the compression bodies M and N are equivalent. Also note that a single geodesic simple closed curve is a lamination.

The following lemma was first proved in [14]:

Lemma 29. *Let $\mathcal{C}$ be any finite collection of simple closed curves. Then there are at most finitely many compression bodies which are minimal with respect to $\mathcal{C}$.*

The next lemma gives even more control over what compression bodies can contain a specified collection of disks (this is referred to as a "folklore lemma" in [13]):

Lemma 30. *Let $\{M_1, ..., M_k\}$ be a collection of pairwise inequivalent compression bodies, all with the same exterior surface F. Say also that $M_1 \subset M_2 \subset ... \subset M_k$ (inclusions are strict).*

Then there is an integer P, depending only on the genus of F, so that $k \leq P$.

The following is a generalization of lemmas first proved in [5] and [13] and is one of the key ingredients in proving our main result.

Lemma 31. *Let* $\{\mathcal{C}_i\}$ *be a sequence of finite collections of essential simple closed curves such that any sequence* $\{C_i \mid C_i \in \mathcal{C}_i\}$ *converges to a lamination* $\mathcal{L}$*. Let* $\mathcal{M}$ *be the collection of all pairwise inequivalent compression bodies minimal with respect to* $\mathcal{L}$ *bounding and in which a sequence* $\{C_i \mid C_i \in \mathcal{C}_i\}$ *bounds.*

Then $\mathcal{M}$ *is finite.*

Proof. Let $\mathcal{P}$ be the collection of all pairwise inequivalent compression bodies M which are minimal with respect to some finite (or empty) collection $\{C_1, ..., C_n \mid C_i \in \mathcal{C}_i\}$ and either $\mathcal{L}$ does not bound in M or if it does then M is minimal. We consider $\mathcal{P}$ as a partially ordered set with $N \leq M$ if $N \subseteq M$. Note that the trivial compression body is the unique least element of P. To save on notation, let Δ_n formally denote a collection $\{C_1, ..., C_n \mid C_i \in \mathcal{C}_i\}$.

Any $M \in \mathcal{M}$ must be minimal with respect to some $\{C_i \mid C_i \in \mathcal{C}_i\}$, for any compression body in which such a sequence bounds has $\mathcal{L}$ bounding as well. Thus, by lemma 30, any $M \in \mathcal{M}$ must be minimal with respect to some finite collection Δ_n and hence $\mathcal{M} \subseteq \mathcal{P}$.

We show that $\mathcal{P}$ is finite. By lemma 30, any chain $M_1 \subset M_2 \subset ...$ in $\mathcal{P}$ is finite, so it only remains to show that for every M there are finitely many N such that whenever $M \subseteq X \subseteq N$ we have $X = M$ or $X = N$. Call such an N a *direct descendant* of M.

Say $M \in \mathcal{P}$ is minimal for some Δ_n and that $\mathcal{L}$ does not bound in M. Then there is a minimal $R \in \mathbb{N}$ such that no collection $\Delta_R \supset \Delta_n$ bounds in M. By lemma 29, there are only finitely many compression bodies minimal with respect to a collection Δ_l with $l \leq R$. Any direct descendant of M must be minimal with respect to one of these collections, and thus M has only finitely many direct descendants.

Now say $M \in \mathcal{P}$ such that an infinite sequence $\{C_i \mid C_i \in \mathcal{C}_i\}$ bounds. Then $\mathcal{L}$ bounds in M, and by minimality M has no direct descendants in this case. Thus $\mathcal{P}$ is finite, and $\mathcal{M}$ is finite as well. □

Given a compression body M, let $C_1, ..., C_n$ be disjoint curves to which 2-handles are attached to form M from $F \times I$. For any automorphism φ, define φM to be the compression body formed by attaching 2-handles along $\varphi C_1, ..., \varphi C_n$. Note that φ extends over M if and only if φM is equivalent to M.

Lemma 32. *Let φ be a pseudo-Anosov with maximal stable, unstable laminations $\mathcal{L}^+$, $\mathcal{L}^-$ and say that $\mathcal{L}^+$ bounds in some compression body. Let $\delta > 0$ and let $\mathcal{N}$*

be the collection of all compression bodies minimal with respect to $\mathcal{L}^+$ and which have a disk D with $d_H(\partial D, \mathcal{L}^-) < \delta$.

Then $\mathcal{N}$ is non-empty.

Proof. Let M be any compression body minimal for $\mathcal{L}^+$ and say D is a disk in M. By the source-sink dynamics of pseudo-Anosovs, for some k the curve $\varphi^{-k}(\partial D)$ is, after isotopy, a geodesic simple closed curve with $d_H\left(\varphi^{-k}(\partial D), \mathcal{L}^-\right) < \delta$.

Now, let $\{C_i\}$ be a sequence of curves bounding disks in M, such that $\{C_i\}$ approaches $\mathcal{L}^+$ in the Hausdorff topology. Then $\{\varphi^{-k}(C_i)\}$ has the same properties in the compression body $\varphi^{-k}M$. Furthermore, $\varphi^{-k}M$ is still minimal for if $N \subseteq \varphi^{-k}M$ such that $\mathcal{L}^+$ bounds, then $\varphi^k N \subseteq M$ and in fact $\varphi^k N$ is equivalent to M by minimality. □

Finally, we prove the main theorem.

Theorem 33. *Let $\varphi : F \to F$ be a pseudo-Anosov with stable lamination $\mathcal{L}^+$. Assume that $\mathcal{L}^+$ is maximal, that $\mathcal{L}^+$ bounds in a compression body N, and that N is minimal with respect to this condition.*

Then there exists k such that φ^k extends over N.

Proof. Let $\delta > 0$ be small and choose a decreasing sequence $\{\epsilon_i\}$ with $\epsilon_1 > 0$ and $\epsilon_i \to 0$ as $i \to \infty$. Define $\mathcal{C}_i$ to be the collection of all simple closed curves $\alpha^+ \cup \alpha^-$ formed from arcs $\alpha^+ \subseteq \mathcal{L}^+$ and $\alpha^- \subseteq \mathcal{L}^-$ where $m\,(2\epsilon_i) - 2r \leq \mu_-\,(\alpha^+) \leq 2M\,(\epsilon_i) + 2r$ and $\mu_+\,(\alpha^-) \leq \epsilon_i$ (here $\mathcal{L}^-$ is the unstable lamination of φ and M, m, and $r = r\,(\delta)$ are as in lemma 27). After identifying isotopic curves, each $\mathcal{C}_i$ is finite and any sequence $\{C_i \mid C_i \in \mathcal{C}_i\}$ converges to $\mathcal{L}^+$.

Now let $\mathcal{N}$ be the collection of all compression bodies which are minimal for $\mathcal{L}^+$ and also have a sequence of curves $\{C_i \mid C_i \in \mathcal{C}_i\}$ all of which bound disks. By lemma 31, $\mathcal{N}$ is finite.

Let $\mathcal{N}'$ be the collection of compression bodies which are minimal for $\mathcal{L}^+$ and also have a disk D with $d_H\,(\partial D, \mathcal{L}^-) < \delta$. By lemma 32, the set $\mathcal{N}'$ is nonempty and by lemma 28 it is contained in $\mathcal{N}$. Applying the techniques of lemma 32 once again shows that $\varphi^{-t}\mathcal{N}' \subseteq \mathcal{N}'$ for some t, and thus there is a subset of $\mathcal{N}$ invariant under the action of φ^{-1}. Call this collection $\mathcal{N}^*$.

Now, $\varphi^{-s}N$ lies in $\mathcal{N}^*$ and thus for some k we have $\varphi^{-s-k}N = \varphi^{-s}N$. Composing with φ^{s+k} we have $N = \varphi^k N$ and so φ^k extends over N. □

The above proof implies the following corollary, though it also follows from results of [7].

Corollary 34. *Let φ be a pseudo-Anosov with maximal invariant laminations.*

Then φ extends over at most finitely many compression bodies minimal with respect to the condition that the stable lamination of φ bounds.

Remark: The observant reader will note that the theorem above is not quite the same as the theorem of Biringer, Johnson, and Minsky as we have added the hypothesis that the invariant laminations of φ are maximal. If they are not, it is necessary to consider a finite collection of laminations which are formed from $\mathcal{L}^+$ and $\mathcal{L}^-$ by adding isolated leaves which "cut across" the diagonals of principal regions (see [5]). Lemmas 27, 28, and 32 can be modified to take into account this situation, however it makes their statements and proofs far more clumsy so we do not do so here.

Bibliography

[1] Agol, Ian. Ideal triangulations of pseudo-Anosov mapping tori. *Topology and Geometry in Dimension Three*, vol. 560 of *Contemp. Math.. Amer. Mathematical Society* (2012)

[2] Biringer, Ian; Johnson, Jesse; Minsky, Yair. Extending pseudo-Anosov maps into compression bodies. *J. Topol.* **6** (2013), no. 4, 1019 - 1042

[3] Brouwer, Andries H.; Haemers, Willem H. Spectra of graphs. Course notes, 2003. `http://homepages.cwi.nl/~aeb/math/ipm.pdf`

[4] Casson, A. J. Automorphisms of Surfaces after Nielsen and Thurston, vol. 9 of *London Mathematical Society Student Texts. Cambridge University Press*, 1988

[5] Casson, Andrew J.; Bleiler, Steven A. Automorphisms of Sur-

faces after Nielsen and Thurston, vol. 9 of *London Mathematical Society Student Texts. Cambridge University Press, Cambridge*, 1988

[6] Casson, A. J.; Gordon, C. McA. A loop theorem for duality spaces and fibered ribbon knots. *Inventiones. mathematicae*, **74** (1983), no. 1, 119 - 137

[7] Casson, A.J.; Long, D.D. Algorithmic compression of surface automorphisms. *Inventiones mathematicae*, **81** (1985) 295 - 303

[8] Grove, Larry C. Classical Groups and Geometric Algebra, vol. 39 of *Graduate Studies in Mathematics. American Mathematical Society, Providence, RI*, 2002

[9] Farb, Benson; Margalit, Dan. A Primer on mapping class groups, vol. 49 of *Princeton Mathematical Series. Princeton Univ. Press*, 2011.

[10] Franks, John M. Homology and Dynamical Systems. vol. 49 of *Regional Conference Series in Mathematics. American Mathematical Society, Providence, RI*, 1982

[11] Farb, Benson; Leininger, Chris; Margalit, Dan Small dilatation pseudo-Anosovs and 3-manifolds. *Adv. Math.* **228** (2011), no. 3, 1466 - 1502

[12] Fathi, Albert; Laudenbach, Francois; Poénaru, Valentin. Travaux de Thurston sur les surfaces. *Astérisque* **66**. *Société Mathématique de France, Paris* (1979) Séminaire Orsay, With an English summary. MR0568308 (82m:57003)

[13] Long, D.D. Bounding laminations. *Duke Mathematical Journal*, **56** (1) (1988)

[14] Long, D.D. Planar kernels in surface groups. *Oxford Quart. J. Math.*, **35** (2) (1984) 305 - 310

[15] McDuff, Dusa and Salamon, Dietmar. Introduction to Symplectic Topology. *Oxford University Press, Oxford, UK*, 1995

[16] Oertel, Ulrich. Automorphisms of three-dimensional handlebodies. *Topology*, **41** (2002)

[17] Penner, R.C. and Harer, John L. Combinatorics of Train Tracks. vol. 125 of *Annals of Mathematics Studies. Princeton University Press*, 1991

[18] Thurston, William P. On the geometry and dynamics of diffeomorphisms of surfaces. *Bull. Amer. Math. Soc. (N.S.)*, **19** (2), 1988