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Propagation of Regularity within Solutions to Korteweg-de Vries Type Equations

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by

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For Iris, Myla and Lisa.

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Curriculum Vitæ

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Abstract

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For many dispersive equations, decay of the initial data leads to increased regularity of the solution for positive times. The unidirectional dispersion of the k-generalized Korteweg-de Vries equation

$$\partial_t u + \partial_x^3 u + u^k \partial_x u = 0, \qquad x, t \in \mathbb{R}, k \in \mathbb{Z}^+, \tag{0.1}$$

produces the following propagation of regularity phenomena [12]: if for some $l \in \mathbb{Z}^+$

$$\|\partial_x^l u_0\|_{L^2(0,\infty)} < \infty,$$

then for positive times the corresponding local solution $u = u(x,t)$ satisfies

$$\|\partial_x^l u(\cdot,t)\|_{L^2(y,\infty)} < \infty \qquad \text{for every } y \in \mathbb{R}.$$

We show that similar results hold for fifth and higher order KdV equations, as well as for quasilinear KdV type equations.

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Chapter 1

Introduction

Nonlinear dispersive equations arise as models of several physical phenomena, for instance wave propagation in media such as liquids, gases and plasmas [33]. One of the most famous dispersive models is the Korteweg-de Vries (KdV) equation

$$\partial_t u + \partial_x^3 u + u\partial_x u = 0, \quad x, t \in \mathbb{R}, \tag{1.1}$$

derived in 1895 to describe unidirectional long waves propagating in a shallow channel [25]. It is the second in a sequence of completely integrable equations

$$\partial_t u + \partial_x^{2j+1} u + Q_j(u, \partial_x u, \dots, \partial_x^{2j-1} u) = 0, \quad x, t \in \mathbb{R}, j \in \mathbb{Z}^+, \tag{1.2}$$

known as the KdV hierarchy. The polynomials $Q_j : \mathbb{R}^{2j} \to \mathbb{R}$ are chosen so that the above equation has the Lax pair formulation

$$\partial_t u = [B_j; L]u \tag{1.3}$$

for $L = \frac{d^2}{dx^2} - u(x,\cdot)$ the stationary Schrödinger operator [28]. The first and third equations in the hierarchy are

$$\partial_t u - \partial_x u = 0, \quad x,t \in \mathbb{R}, \tag{1.4}$$

and

$$\partial_t u - \partial_x^5 u - 30u^2 \partial_x u + 20\partial_x u \partial_x^2 u + 10u\partial_x^3 u = 0, \quad x,t \in \mathbb{R}, \tag{1.5}$$

respectively. Utilizing the Lax pair formulation, the initial value problem (IVP) associated to each equation in the KdV hierarchy can be solved by the inverse scattering method in a space of "rapidly decaying functions" [9].

The study of qualitative properties of solutions to dispersive equations has attracted considerable attention in recent decades. Several remarkable results have been attained concerning: local and global well-posedness of the IVP associated to the KdV equation, for instance, under minimal regularity assumptions on the initial data, blowup profiles and global in time behavior of solutions, the stability of special solutions, among others.

It is the aim of this work to explore the propagation of regularity within solutions to dispersive equations of KdV-type, including those in the KdV hierarchy. As motivation, the following discussion will review known smoothing effects found in solutions to the IVP for the k-generalized Korteweg-de Vries (k-gKdV) equation

$$\begin{cases} \partial_t u + \partial_x^3 u + u^k \partial_x u = 0, \quad x,t \in \mathbb{R}, k \in \mathbb{Z}^+, \\ u(x,0) = u_0(x), \end{cases} \tag{1.6}$$

as well as the associated IVP for the linear Airy equation

$$\begin{cases} \partial_t v + \partial_x^3 v = 0, \quad x,t \in \mathbb{R}, \\ v(x,0) = v_0(x). \end{cases} \tag{1.7}$$

Consider initial data in the Sobolev space

$$H^s(\mathbb{R}) := \left\{ f \in L^2(\mathbb{R}) : (1+\xi^2)^{s/2}\hat{f}(\xi) \in L^2(\mathbb{R}) \right\} \qquad (s \geq 0) \tag{1.8}$$

with the norm

$$\|f\|_{s,2} := \|(1+\xi^2)^{s/2}\hat{f}\|_2. \tag{1.9}$$

Then the solution to the linear problem (1.7) may be written

$$v(x,t) = V(t)v_0(x) := S_t * v_0(x) \tag{1.10}$$

where

$$S_t(x) = \int_{-\infty}^{\infty} e^{2\pi i x\xi} e^{8\pi i t\xi^3}\, d\xi = \frac{1}{\sqrt[3]{t}} S_1\left(\frac{x}{\sqrt[3]{t}}\right). \tag{1.11}$$

Applying the Plancherel identity yields

$$\|v(\cdot,t)\|_{s,2} = \|v_0\|_{s,2} \qquad (t \in \mathbb{R}). \tag{1.12}$$

In other words, the family of operators $\{V(t)\}_{t=-\infty}^{\infty}$ forms a unitary group on $H^s(\mathbb{R})$ for every $s \in \mathbb{R}$. Moreover, for fixed $v_0 \in H^s(\mathbb{R})$, the map $v_0 \mapsto v(\cdot,t)$ traces a continuous curve in $H^s(\mathbb{R})$. This is abbreviated

$$v \in C(\mathbb{R}; H^s(\mathbb{R})). \tag{1.13}$$

As a consequence of (1.12) and the time reversible nature of the equation, there can be no global smoothing effect for solutions to problem (1.7). As remarked in [41] for solutions to the KdV equation, if $v_0 \in H^s(\mathbb{R}) \backslash H^{s'}(\mathbb{R})$ for $s < s'$, then for any $t \in \mathbb{R}$, $v(\cdot,t) \in H^s(\mathbb{R}) \backslash H^{s'}(\mathbb{R})$.

Next we study a smoothing property of the group $\{V(t)\}_{t=-\infty}^{\infty}$. First note that

$$\|V(t)v_0(x)\|_\infty \leq \|S_t\|_\infty \|v_0\|_1 \leq c|t|^{-1/3}\|v_0\|_1 \qquad (t \neq 0), \tag{1.14}$$

so that when $v_0 \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$, the solution to problem (1.7) is bounded for $t \neq 0$. Combined with L^2-conservation and Riesz-Thorin interpolation, this inequality is one ingredient in establishing the mixed-norm estimates

$$\left(\int_{-\infty}^{\infty} \|D^{\alpha\theta/2}V(t)v_0\|_p^q \, dt\right)^{1/q} \leq c\|v_0\|_2 \tag{1.15}$$

with $(q,p) = (6/\theta(\alpha+1), 2/(1-\theta))$, $0 \leq \theta \leq 1$ and $0 \leq \alpha \leq 1/2$. Choosing, for example, $\theta = 1$ and $\alpha = 0$, if $v_0 \in L^2(\mathbb{R})$, then the solution to problem (1.7) satisfies

$$\|V(t)v_0\|_\infty < \infty \qquad \text{for a.e. } t \in \mathbb{R}. \tag{1.16}$$

Moreover, $V(t)v_0(\cdot)$ is continuous a.e. for $v_0 \in L^2(\mathbb{R})$. Inequalities of the form (1.15) were first discovered by Strichartz [44] in the context of the wave equation, with the above version proved by Kenig, Ponce and Vega [19].

Inequality (1.14) suggests a relationship between decay of the initial data and regularity of the corresponding solution. Selecting v_0 to be the characteristic function of an interval, it is seen from (1.10) and the smoothness of $S_t(\cdot)$ that the solution to the linear Airy equation is smooth for positive times. For the same initial data, Murray [36] used the inverse scattering method to construct solutions which weakly recover the data and are of class $C^\infty(\{x,t : x \in \mathbb{R}, t > 0\})$. Kato [15] described this *quasiparabolic smoothing effect* as stemming from the unidirectional dispersion inherent in the equation. He studied solutions to the k-gKdV (1.6) in the asymmetric spaces

$$H^s(\mathbb{R}) \cap L^2(e^{2\beta x}\, dx) \qquad (s \geq 0,\ \beta > 0), \tag{1.17}$$

in which the operator $\partial_t + \partial_x^3$ is formally equivalent to $\partial_t + (\partial_x - \beta)^3$. The expansion

$$(\partial_x - \beta)^3 = \partial_x^3 - 3\beta\partial_x^2 + 3\beta^2\partial_x - \beta^3 \tag{1.18}$$

reveals the dissipation in this operator. Kato found that solutions to the k-gKdV corresponding to data of class (1.17) satisfy $u(\cdot, t) \in C^\infty(\mathbb{R})$ for $t > 0$; the result is not reversible in time due to the asymmetrical weight. Note that these results do not contradict the failure of a global $H^s(\mathbb{R})$ smoothing effect.

In the same paper, Kato demonstrated the existence of global weak solutions to the IVP associated to the KdV equation corresponding to initial data u_0 in $L^2(\mathbb{R})$. A key step in the proof is the following *local smoothing effect*.

Theorem A. (*Kato [15]*) *Let* $s > 3/2$ *and* $0 < T < \infty$. *If* $u \in C([0,T]; H^s(\mathbb{R}))$ *is the solution to the IVP associated to* (1.1) *with data* u_0, *then*

$$u \in L^2([0,T]; H^{s+1}(-R,R)) \quad \textit{for any } 0 < R < \infty, \tag{1.19}$$

with the associated norm depending only on $\|u_0\|_{H^s}$, R *and* T.

Roughly, the proof follows by observing that a smooth solution u to the KdV satisfies

$$\begin{aligned} &\frac{d}{dt}\int (\partial_x^l u)^2 \psi \, dx + 3\int (\partial_x^{l+1} u)^2 \psi' \, dx \\ &\qquad = \int (\partial_x^l u)^2 \psi''' \, dx + \int (\partial_x^l u)^2 \partial_x(\psi u) \, dx + \int \partial_x^l u [\partial_x^l; u] \partial_x u \psi \, dx \end{aligned} \tag{1.20}$$

for $l \in \mathbb{Z}^+ \cup \{0\}$. Selecting $l = 0$ and $\psi \in C^3(\mathbb{R})$ to be an appropriate nonnegative, monotonically increasing cutoff function with ψ' compactly supported, integration in time yields

$$\int_0^T \int_{-R}^R (\partial_x u)^2(x,t)\, dxdt \le c(R; T; \|u_0\|_2). \tag{1.21}$$

Kenig, Ponce and Vega [18] used Fourier analysis to establish the sharp smoothing effect

$$\int_{-\infty}^{\infty} (\partial_x V(t) v_0(x))^2\, dt = c\|v_0\|_2^2 \qquad \text{for all } x \in \mathbb{R}, \tag{1.22}$$

for solutions to the linear problem (1.7).

Kruzhkov and Faminskiĭ [26] connected polynomial decay of the initial data on the positive half-line to increased regularity of solutions to the KdV equation. Refining in some sense Kato's quasiparabolic smoothing effect, they showed if $x^n u_0 \in L^2(0, \infty)$, then for positive times the solution possesses $2n-1$ continuous spatial derivatives. Moreover, the decay persists in that $x^n u(\cdot, t) \in L^2(0, \infty)$ for $t > 0$. Their proof relies on decay properties of the fundamental linear solution (1.11) and its derivatives. However, this reliance is not essential. Craig, Kappeler and Strauss [8] reproduced the above correspondence between polynomial decay and regularity for solutions to a fully nonlinear KdV-type equation. Following Craig and Goodman [7], they incorporated a nonlinear multiplier into Kato's weighted energy method (1.20).

The local smoothing effect has also been established for a wide class of dispersive equations and systems using Fourier analysis. Consider linear equations of the form

$$\partial_t u + iP(\mathcal{D})u = 0, \quad x, t \in \mathbb{R}, \tag{1.23}$$

where $\mathcal{D} = \frac{1}{i}\partial_x$ and $P(\mathcal{D})f$ is defined via the real symbol $p(\xi)$ as

$$\widehat{P(\mathcal{D})f}(\xi) := p(\xi)\hat{f}(\xi). \tag{1.24}$$

Constantin and Saut [6] showed that if $p(\xi)$ behaves like $|\xi|^m$ for $\xi \gg 1, m > 1$ and $u_0 \in H^s(\mathbb{R})$, then the corresponding solution to the IVP associated to (1.23) satisfies

$$u \in L^2([0,T]; H^{s+d}_{\text{loc}}(\mathbb{R})) \tag{1.25}$$

with $d = (m-1)/2$. Vega [45] and Sjölin [43] used a similar technique to investigate pointwise convergence issues for the linear Schrödinger equation. We note that such local smoothing is a dispersive phenomenon; it cannot hold in hyperbolic systems.

Isaza, Linares and Ponce [12] recently discovered a *propagation of regularity* result for solutions to the k-generalized KdV equation. Suppose $u_0 \in H^{3/4^+}(\mathbb{R})$ and denote by $u \in C([0,T]; H^{3/4^+}(\mathbb{R}))$ the corresponding local solution to (1.6). Further assume for some $l \in \mathbb{Z}^+$ that

$$\|\partial_x^l u_0\|^2_{L^2(0,\infty)} = \int_0^\infty (\partial_x^l u_0)^2(x)\, dx < \infty. \tag{1.26}$$

Then for each $\epsilon > 0, \nu \geq 0$, the solution satisfies

$$\sup_{0 \leq t \leq T} \int_{\epsilon - \nu t}^\infty (\partial_x^l u)^2(x,t)\, dx < \infty \tag{1.27}$$

The regularity (1.26) has propagated leftward with infinite speed; for each $x_0 \in \mathbb{R}$ and $0 < t \leq T$, then $u(\cdot, t) \in H(x_0, \infty)$. In fact, for $\delta > 0$ and $0 < t \leq T$,

$$\int_{-\infty}^\infty \frac{1}{\langle x_- \rangle^{l+\delta}} (\partial_x^l u)^2(x,t)\, dx \leq \frac{c}{t}. \tag{1.28}$$

If, as an alternative to (1.26), $u_0 \in H^{3/4^+}(\mathbb{R})$ and for some $n \in \mathbb{Z}^+$

$$\|x^{n/2}u_0\|^2_{L^2(0,\infty)} = \int_0^\infty |x|^n u_0^2(x)\,dx < \infty, \tag{1.29}$$

then

$$\sup_{0\leq t\leq T} \int_0^\infty |x|^n u^2(x,t)\,dx < \infty. \tag{1.30}$$

That is, one-sided polynomial decay of the initial data persists in the solution to IVP (1.6) for positive times. The proof also reproduces the trade-off between decay and regularity as observed by Kruzhkov and Faminskiĭ [26]. The proof utilizes Kato's weighted energy method (1.20) with the primary technical hurdle being the use of a multiplier which is supported in $(x,t) \in [-R,\infty) \times [0,T]$ for some $R > 0$. Isaza, Linares and Ponce subsequently obtained the propagation of regularity result for the completely integrable Benjamin-Ono [14] and Kadomstev-Petviashvilli-II [13] equations.

The first goal of this work is to demonstrate both the propagation of regularity and persistence of decay for equations in the KdV hierarchy (1.2). This will be accomplished first for relatively rough solutions to the fifth order KdV equation (1.5). By increasing slightly the regularity and decay assumptions on the solutions, the technique will be seen to extend to a large class of fifth order equations. This class includes the following models from mathematical physics:

$$\partial_t u + \partial_x u + c_1 u\partial_x u + c_2\partial_x^3 u + c_3\partial_x u\partial_x^2 u + c_4 u\partial_x^3 u + c_5\partial_x^5 u = 0 \tag{1.31}$$

modelling the water wave problem for long, small amplitude waves over shallow bottom [37], a model describing short and long wave interaction [1]

$$\partial_t u - 2\partial_x u\partial_x^2 u - u\partial_x^3 u + \partial_x^5 u = 0, \tag{1.32}$$

and Lisher's model for motion of a lattice of anharmonic oscillators [32]

$$\partial_t u + (u + u^2)\partial_x u + (1 + u)(\partial_x u \partial_x^2 u + u\partial_x^3 u) + \partial_x^5 u = 0. \tag{1.33}$$

For further discussion of these models see [39] and references therein.

In fact, our results will apply to all equations of the form (1.2), not only the completely integrable members of the KdV hierarchy. We review next the existence theory for such equations in the Sobolev scale.

Following Kato [15], an IVP is said to be to be *locally well-posed* in the Banach space X if for every $u_0 \in X$ there exists $T > 0$ and a unique solution $u(t)$ satisfying

$$u \in C([0, T]; X) \cap Y_T, \tag{1.34}$$

where Y_T is an auxillary function space. Moreover, the solution map $u_0 \mapsto u$ is continuous from X into the class (1.34). If T can be taken arbitrarily large, the IVP is said to be *globally well-posed*. The persistence condition (1.34) states that the solution curve describes a dynamical system.

While studying the fifth order KdV (1.5) and equations (1.31)-(1.33), Ponce [39] remarked that the use of dispersive estimates appears essential to attain local well-posedness in Sobolev spaces. He proved that the IVP associated to each of these models is locally well-posed in $H^s(\mathbb{R}), s \geq 4$, by using the weighted energy method, sharp linear estimates and parabolic regularization.

Kenig, Ponce and Vega ([21], [22]) investigated the class

$$\partial_t u + \partial_x^{2j+1} u + P(u, \partial_x u, \ldots, \partial_x^{2j} u) = 0, \qquad x, t \in \mathbb{R}, \tag{1.35}$$

with $j \in \mathbb{Z}^+$ and $P : \mathbb{R}^{2j+1} \to \mathbb{R}$ (or $\mathbb{C}^{2j+1} \to \mathbb{C}$) a polynomial having no constant or linear terms. By incorporating a commuting vector field identity into the contraction principle argument used in [20], they established that to each equation in the above class there exists nonnegative integers m_0 and s_0 such that the corresponding IVP is locally well-posed in the weighted Sobolev space

$$X_{s,m} = H^s(\mathbb{R}) \cap L^2(|x|^m \, dx) \tag{1.36}$$

for any $m \geq m_0$ and $s \geq \max\{s_0, 2jm\}$. As a consequence of the implicit function theorem, the smoothness of the polynomial P yields the smoothness of the solution map (see [29] for further comments).

Following Molinet, Saut and Tzvetkov ([34], [35]), Pilod [38] showed that for certain equations in the class (1.35), the associated IVP is in some sense ill-posed in the Sobolev scale. In particular, if P contains the term $u\partial_x^l u$ for $l > j$, then the solution map $H^s(\mathbb{R}) \to C([0,T]; H^s(\mathbb{R}))$ is not C^2 at the origin for any choice of $s \in \mathbb{R}$. For equations

$$\partial_t u - \partial_x^5 u + c_1 u^2 \partial_x u + c_2 \partial_x u \partial_x^2 u + c_3 u \partial_x^3 u = 0, \qquad x, t \in \mathbb{R}, \tag{1.37}$$

Kwon [27] demonstrated that the solution map is not even uniformly continuous by using the arguments of [23] and [24]. All of these facts result from uncontrollable interactions when both high and low frequencies are present in the initial data. Thus, in contrast to the k-gKdV, equations of the form (1.37) cannot be solved using the contraction principle in $H^s(\mathbb{R})$ alone.

Differences between (1.6) and (1.37) also arise when applying the energy method.

Note that after integrating by parts, smooth solutions u to (1.37) satisfy

$$\begin{aligned}\frac{d}{dt}\int(\partial_x^l u)^2\psi\,dx + 2\int(\partial_x^{l+2}u)^2\psi'\,dx\\ \lesssim \|\partial_x^3 u\|_\infty \int(\partial_x^l u)^2\psi\,dx + \left|\int \partial_x u(\partial_x^{l+1}u)^2\psi\,dx\right| + \cdots\end{aligned} \tag{1.38}$$

for $l \in \mathbb{Z}^+$. After integrating in time, the right-hand side cannot be estimated in terms of $\|u\|_{L_T^\infty H^l}$. Kwon [27] introduced a corrected energy and refined Strichartz estimate to overcome this *loss of derivatives* and obtained the following result.

Theorem B. (*Kwon [27]*) *Let $s > 5/2$. For any $u_0 \in H^s(\mathbb{R})$ there exists a time $T \gtrsim \|u_0\|_{s,2}^{-10/3}$ and a unique real-valued solution u of the IVP associated to* (1.37) *satisfying*

$$u \in C([0,T];H^s(\mathbb{R})) \qquad \textit{and} \qquad \partial_x^3 u \in L^1([0,T];L^\infty(\mathbb{R})). \tag{1.39}$$

Using an auxillary Bourgain space introduced in [3] and [4], the local well-posedness of the IVP associated to (1.37) in the energy space $H^2(\mathbb{R})$ was established simultaneously by Kenig and Pilod [17] and Guo, Kwak and Kwon [10]. Thus global well-posedness follows in the Hamiltonian case, i.e., when $c_2 = 2c_3$.

Kenig and Pilod [16] adapted Kwon's corrected energy method to all equations in the KdV hierarchy, demonstrating local well-posedness of the j-th equation in (1.2) in $H^{s_j}(\mathbb{R})$ for $s_j > 4j - \frac{9}{2}$.

We now state the propagation and persistence results for solutions to the fifth order equation (1.37), the proofs of which incorporate Kwon's corrected energy and refined Strichartz estimate as in Theorem B.

Theorem 1. *Let $s > 5/2$. Suppose $u_0 \in H^s(\mathbb{R})$ and for some $l \in \mathbb{Z}^+, x_0 \in \mathbb{R}$*

$$\|\partial_x^l u_0\|_{L^2(x_0,\infty)}^2 = \int_{x_0}^{\infty} (\partial_x^l u_0)^2(x)\, dx < \infty. \tag{1.40}$$

Then the solution u of IVP (1.37) provided by Theorem B satisfies

$$\sup_{0\le t\le T} \int_{x_0+\epsilon-\nu t}^{\infty} (\partial_x^m u)^2(x,t)\, dx \le c \tag{1.41}$$

for any $\nu \ge 0, \epsilon > 0$ and each $m = 0, 1, \ldots, l$ with

$$c = c(l;\nu;\epsilon;T;\|u_0\|_{H^s};\|\partial_x^l u_0\|_{L^2(x_0,\infty)}), \tag{1.42}$$

where T is given in Theorem B.

Moreover, for any $\nu \ge 0, \epsilon > 0$ and $R > \epsilon$

$$\int_0^T \int_{x_0+\epsilon-\nu t}^{x_0+R-\nu t} (\partial_x^{l+2} u)^2(x,t)\, dxdt \le \tilde{c} \tag{1.43}$$

with

$$\tilde{c} = \tilde{c}(l;\nu;\epsilon;R;T;\|u_0\|_{H^s};\|\partial_x^l u_0\|_{L^2(x_0,\infty)}). \tag{1.44}$$

Remark 1. Estimate (1.43) is a slight generalization of Kato's local smoothing effect (1.21) in that it provides a local estimate for $\partial_x^{l+2}u$ on compact subsets of $\mathbb{R} \times [0,T]$ which are disjoint from the ray $(-\infty, x_0) \times [0,T]$ without assuming $u_0 \in H^l(\mathbb{R})$.

Remark 2. The constants appearing in Theorem 1 have the form of a polynomial in ν. For $l \ge 6$, the degree of this dependence is $d = 8(l-5)$.

For fixed $l \in \mathbb{Z}^+$, Theorem 1 is the base case for the situation where the derivatives of the initial data possess polynomial decay when restricted to the positive half-line. Our second result states that this decay persists.

Theorem 2. *Let $s > 5/2$ and let $n, l \in \mathbb{Z}^+$. Suppose $u_0 \in H^s(\mathbb{R})$ and for each $m = 0, 1, \ldots, l$*

$$\|x^{n/2}\partial_x^m u_0\|^2_{L^2(0,\infty)} = \int_0^\infty x^n(\partial_x^m u_0)^2(x)\, dx < \infty. \tag{1.45}$$

Then the solution u of IVP (1.37) *provided by Theorem B satisfies*

$$\sup_{0\le t\le T}\int_\epsilon^\infty x^n(\partial_x^m u)^2(x,t)\, dx \le c \tag{1.46}$$

for any $\epsilon > 0$ and each $m = 0, 1, \ldots, l$ with

$$c = c(n; l; \epsilon; T; \|u_0\|_{H^s}; \|x^{n/2}\partial_x^k u_0\|_{L^2(0,\infty)}) \tag{1.47}$$

for $k = 0, 1, \ldots, m$, where T is given in Theorem B. By local well-posedness, we may take $\epsilon = 0$ for $m \le s$.

Moreover, for any $\epsilon > 0$

$$\int_0^T\int_0^\infty x^{n-1}(\partial_x^{l+2}u)^2(x,t)\, dxdt \le \tilde{c} \tag{1.48}$$

with $\tilde{c}$ as in (1.47).

A bootstrapping argument yields regularity of the solution for positive times by imposing decay only on the initial data and not its derivatives.

Theorem 3. *Let $s > 5/2$. Suppose $u_0 \in H^s(\mathbb{R})$ and for some $n \in \mathbb{Z}^+$*

$$\|x^{n/2}u_0\|^2_{L^2(0,\infty)} = \int_0^\infty x^n u_0^2(x)\, dx < \infty. \tag{1.49}$$

Then the solution u of IVP (1.37) *provided by Theorem B satisfies*

$$\sup_{\delta \leq t \leq T} \int_{\epsilon-\nu t}^{\infty} (\partial_x^{2n} u)^2(x,t)\, dx + \int_{\delta}^{T} \int_{\epsilon-\nu t}^{R-\nu t} (\partial_x^{2n+2} u)^2(x,\tau)\, dx d\tau \leq c \tag{1.50}$$

for every $\nu \geq 0$, $\epsilon, \delta > 0$ and $R > \epsilon$, with

$$c = c(n; \delta; \nu; \epsilon; R; T; \|u_0\|_{H^s}; \|x^{n/2} u_0\|_{L^2(0,\infty)}). \tag{1.51}$$

The time reversible nature of equation (1.37) yields a number of consequences, as noted in [12]. Combining with the contrapositive of Theorems 1 and 3, we have the following.

Corollary 1. *Assume that $s > 5/2$. Let $u \in C([-T,T]; H^s(\mathbb{R}))$ be a solution of* (1.37) *provided by Theorem B such that*

$$\partial_x^m u(\cdot, \hat{t}) \notin L^2(a, \infty) \quad \textit{for some } \hat{t} \in [-T,T] \textit{ and } a \in \mathbb{R}. \tag{1.52}$$

Then for any $t \in [-T, \hat{t})$ and any $\beta \in \mathbb{R}$

$$\partial_x^m u(\cdot, t) \notin L^2(\beta, \infty) \quad \textit{and} \quad x^{\lceil m/2 \rceil/2} u(\cdot, t) \notin L^2(0, \infty). \tag{1.53}$$

Additional consequences along these lines may be found in [42].

The propagation and persistence results extend to the broader fifth order class

$$\partial_t u - \partial_x^5 u + Q(u, \partial_x u, \partial_x^2 u, \partial_x^3 u) = 0, \qquad x, t \in \mathbb{R}, \tag{1.54}$$

where $Q : \mathbb{R}^4 \to \mathbb{R}$ is a polynomial having no constant or linear terms. Recall that the IVP associated to a particular equation in this class is locally well-posed in $X_{s,m}$ for s, m large enough nonnegative integers ([21], [22]). However, some equations experience a loss of derivatives in the weighted energy method which cannot be accounted for with Kwon's correction technique. By imposing decay on the solutions we arrive at the following extension of Theorem 1.

Theorem 4. *For each equation in the class* (1.54) *there exists nonnegative integers* s, m *such that if* $u_0 \in X_{s,m}$*, then there exists a time* $T > 0$ *and a unique local solution* $u \in C([0,T]; X_{s,m}) \cap \cdots$ *to the associated IVP. Additionally, if for some* $l \in \mathbb{Z}^+, x_0 \in \mathbb{R}$

$$\|\partial_x^l u_0\|_{L^2(x_0,\infty)}^2 = \int_{x_0}^{\infty} (\partial_x^l u_0)^2(x)\, dx < \infty, \tag{1.55}$$

then the corresponding solution u *satisfies*

$$\sup_{0 \le t \le T} \int_{x_0+\epsilon-\nu t}^{\infty} (\partial_x^l u)^2(x,t)\, dx + \int_0^T \int_{x_0+\epsilon-\nu t}^{x_0+R-\nu t} (\partial_x^{l+2} u)^2(x,\tau)\, dx d\tau < \infty \tag{1.56}$$

for any $\nu \ge 0, \epsilon > 0$ *and* $R > \epsilon$.

Remark 3. It will be clear that the proof of Theorem 4 holds when the polynomial Q contains linear terms, provided the equation is locally well-posed in some (possibly weighted) Sobolev space. In particular, the results apply to equation (1.31).

Remark 4. The technique of Theorem 4 applies to all equations in class (1.2) with $Q : \mathbb{R}^{2j} \to \mathbb{R}$ a polynomial having no constant of linear terms. However, we won't provide details of the necessary cutoff function for higher order equations.

Remark 5. The work [16] suggests that Theorem 4 holds for equations in the KdV hierarchy without imposing weight through repeated application of Kwon's corrected energy.

As seen in Theorem 4, the propagation of regularity holds for many equations which are not completely integrable. The second goal of this work is to demonstrate that the propagation phenomenon holds for dispersive equations for which Kato's local smoothing effect can be obtained by the weighted energy method.

To understand the extent of the phenomenon, first consider the quasilinear equation

$$\partial_t u + \partial_x^3(u^2) + \partial_x(u^2) = 0, \quad x, t \in \mathbb{R}, \tag{1.57}$$

which may be written

$$\partial_t u + 2u\partial_x^3 u + 6\partial_x u \partial_x^2 u + \partial_x(u^2) = 0. \tag{1.58}$$

Rosenau and Hyman [40] discovered the compactly supported traveling wave solution, or *compacton*, $u(x,t) = \phi(x - ct)$ with $c > 0$ and

$$\phi(y) = \begin{cases} \frac{4c}{3}\cos^2(y/4), & |y| \le 2\pi \\ 0, & |y| > 2\pi. \end{cases} \tag{1.59}$$

Notice that $\phi \ge 0$, $\phi \in C^1(\mathbb{R})\backslash C^2(\mathbb{R})$ and $\phi \equiv 0$ for $|y| > 2\pi$. In contrast, a solution to an equation in the class (1.2) corresponding to compactly supported initial data becomes spatially smooth for positive times according to either the propagation of regularity phenomenon or by the quasiparabolic smoothing effect.

Motivated by [7] and [8], we treat the quasilinear IVP

$$\begin{cases} \partial_t u + a(u, \partial_x u, \partial_x^2 u)\partial_x^3 u + b(u, \partial_x u, \partial_x^2 u) = 0, \quad x \in \mathbb{R}, t \ge 0, \\ u(x,0) = u_0(x) \end{cases} \tag{1.60}$$

where the functions $a, b : \mathbb{R}^3 \to \mathbb{R}$ satisfy:

(H1) for any compact subset of $\mathbb{R}^3$ there exists $\kappa > 1$ so that $1/\kappa \le a(\cdot) \le \kappa$;

(H2) a, b are C^∞ and thus all derivatives are bounded on compact subsets of $\mathbb{R}^3$;

(H3) $\partial_z b \leq 0$, where $b = b(x, y, z)$.

The compacton solution to (1.57) suggests that the dispersive assumption (H1) is, in some sense, necessary to obtain results along the lines of Theorem 1 and Theorem 3. Assumptions (H1)-(H3) guarantee local well-posedness of (1.60) in the Sobolev scale and, as we shall see, the propagation of regularity within such solutions.

Theorem C. (*Craig, Kappeler, Strauss [8]*) *Let $m \in \mathbb{Z}^+$, $m \geq 7$. For any $u_0 \in H^m(\mathbb{R})$, there exists a time $T = T(\|u_0\|_{7,2}) > 0$ and a unique solution $u = u(x,t)$ of the IVP* (1.60) *satisfying*

$$u \in L^\infty([0,T]; H^m(\mathbb{R})), \tag{1.61}$$

with

$$\partial_x^{m+1} u \in L^2([0,T] \times [-R,R]), \text{ for all } R > 0. \tag{1.62}$$

This theorem does not address continuous dependence of the solution on the initial data, which is necessary to perform a limiting argument. Instead, our result is based on the following local well-posedness theorem.

Theorem D. (*Linares, Ponce, Smith [30]*) *Let $m \in \mathbb{Z}^+$, $m \geq 7$. For any $u_0 \in H^m(\mathbb{R})$ there exists a time $T = T(\|u_0\|_{7,2}) > 0$ and a unique solution $u = u(x,t)$ of the IVP* (1.60) *satisfying*

$$u \in C([0,T]; H^{m-\delta}(\mathbb{R})) \cap L^\infty([0,T]; H^m(\mathbb{R})), \text{ for all } \delta > 0, \tag{1.63}$$

with

$$\partial_x^{m+1} u \in L^2([0,T] \times [-R,R]), \text{ for all } R > 0. \tag{1.64}$$

Moreover, the map data solution $u_0 \mapsto u(\cdot,t)$ is locally continuous from $H^m(\mathbb{R})$ into $C([0,T];H^{m-\delta}(\mathbb{R}))$ for any $\delta > 0$.

We now state that solutions to the quasilinear problem (1.60) propagate regularity. The proof uses the weighted energy method, however, the cutoff function depends on the solution through the coefficient $a(u, \partial_x u, \partial_x^2 u)$. This method was used in [8] to obtain a relationship between one-sided polynomial decay of the initial data and regularity of the corresponding solution to (1.60) at positive times.

Theorem 5. (*Linares, Ponce, Smith [30]*) *Let $s \geq 7, s \in \mathbb{Z}^+$. Suppose $u_0 \in H^s(\mathbb{R})$ and for some $l \in \mathbb{Z}^+, x_0 \in \mathbb{R}$*

$$\|\partial_x^l u_0\|^2_{L^2(x_0,\infty)} = \int_{x_0}^{\infty} (\partial_x^l u_0)^2(x)\, dx < \infty. \tag{1.65}$$

Then the solution u of IVP (1.60) provided by Theorem D satisfies

$$\sup_{0\leq t\leq T} \int_{x_0+\epsilon-\nu t}^{\infty} (\partial_x^l u)^2(x,t)\, dx + \int_0^T \int_{x_0+\epsilon-\nu t}^{x_0+R-\nu t} (\partial_x^{l+1} u)^2(x,\tau)\, dx d\tau < \infty \tag{1.66}$$

for any $\nu \geq 0, \epsilon > 0$ and $R > \epsilon$.

Remark 6. Assumption (H2) is not strictly necessary, it merely sidesteps having to count the number of times the equation is differentiated in the energy method.

Remark 7. Cai [5] weakened the parabolic hypothesis (H3) in extending the work [8], though this is not pursued in the context of propagation of regularity.

The results of Isaza, Linares and Ponce ([12], [14], [13]), along with Theorems 1-5 presented here, demonstrate that the unidirectional propagation of Sobolev-scale regularity is a phenomenon common to many dispersive systems. Finally, we point the reader to solutions to the k-gKdV equation for which regularity in the C^m and $W^{m,p}, p > 2$, sense does not propagate.

Consider data $\phi(x) = e^{-2|x|} \in H^1(\mathbb{R})\backslash C^1(\mathbb{R})$. Due to its symmetric decay, the corresponding solution to the Airy equation (1.7) is of class $C^\infty(\{x,t : x \in \mathbb{R}, t \neq 0\})$. Thus choosing $v_0 = V(-1)\phi \in H^1(\mathbb{R}) \cap C^\infty(\mathbb{R})$ produces a solution $v = v(x,t)$ to the same linear problem which looses C^m-regularity in finite time. Somewhat surprisingly, this linear dispersive blowup occurs in the nonlinear setting.

Theorem E. (*Bona and Saut [2]*) *Let $k \in \mathbb{Z}^+$. There exists*

$$u_0 \in H^1(\mathbb{R}) \cap C^\infty(\mathbb{R})$$

with $\|u_0\|_{1,2} \ll 1$ so that the corresponding solution $u \in C(\mathbb{R}; H^1(\mathbb{R})) \cap \dots$ of the IVP (1.6) *is global in time if $k \geq 4$ and such that u satisfies*

$$\begin{cases} u(\cdot,t) \in C^1(\mathbb{R}), \quad t > 0, \quad t \notin \mathbb{Z}^+, \\ u(\cdot,t) \in C^1(\mathbb{R}\backslash\{0\})\backslash C^1(\mathbb{R}), \quad t \in \mathbb{Z}^+. \end{cases} \tag{1.67}$$

Bona and Saut [2] proved this result by analyzing the IVP for the k-gKdV in one-sided polynomial weighted spaces similar to those found in [26], [7] and [8]. Using the integral formulation

$$u(t) = V(t)u_0 - \int_0^t V(t-t')(u^k \partial_x u)(t')\, dt' \qquad (k \in \mathbb{Z}^+) \tag{1.68}$$

of IVP (1.6), they first demonstrated that for data in $H^m(\langle x_+ \rangle^{2\sigma}\, dx)$ the nonlinear term is of class $C^m(\mathbb{R})$ in the x-variable. Second, they used the fundamental solution (1.11) to construct explicit data lying in these spaces for which the linear part of (1.68) lies in $C^m(\mathbb{R})$ for $0 \leq t < 1$ and is not in $C^m(\mathbb{R})$ for $t = 1$. Linares and Scialom [31] subsequently simplified the argument, eliminating the need for weighted spaces.

The authors in [30] offer an alternative proof of Theorem E, a modification of which yields the following.

Theorem F. (*Linares, Ponce, Smith [30]*)

(a) Fix $k = 2, 3, \dots,$ *let* $p \in (2, \infty)$ *and* $j \geq 1,\ j \in \mathbb{Z}^+$. *There exists*

$$u_0 \in H^{3/4}(\mathbb{R}) \cap W^{j,p}(\mathbb{R}) \tag{1.69}$$

such that the corresponding solution

$$u \in C([-T, T]; H^{3/4}(\mathbb{R})) \cap \dots$$

of (1.6) *satisfies*

$$u(\cdot, \pm t) \notin W^{j,p}(\mathbb{R}^+) \qquad \textit{for some } t \in [0, T]. \tag{1.70}$$

(b) For $k = 1$, *the same result holds for* $j \geq 2,\ j \in \mathbb{Z}^+$.

The remainder of this work is organized as follows: Chapter 2 contains proofs of Theorems 1-4, as well as a short discussion of propagation of regularity within solutions to equations in the KdV hierarchy; Chapter 3 contains the proof of Theorem 5. The material in Chapter 2 as well as portions of this introduction were previously published as [42] and appear with permission of the publisher.

Chapter 2

Propagation and Persistence for Fifth Order Models

2.1 Construction of Cutoff Function

In this section we construct cutoff functions for use with the weighted energy method. Define the polynomial

$$\rho(x) = 2772 \int_0^x y^5(1-y)^5 \, dy$$

which satisfies

$$\rho(0) = 0, \qquad \rho(1) = 1,$$
$$\rho'(0) = \rho''(0) = \cdots = \rho^{(5)}(0) = 0,$$
$$\rho'(1) = \rho''(1) = \cdots = \rho^{(5)}(1) = 0$$

with $0 < \rho, \rho'$ for $0 < x < 1$. Much of the complexity of our construction airses when handling the ratio which appears in (2.13), see Section 2.2 below. Thus we note that the expression

$$\frac{(\rho'''(x))^2}{\rho'(x)} = -277200x(x-1)\left(2-9x+9x^2\right)^2 \tag{2.1}$$

is continuous for $x \in [0,1]$ and vanishes at the endpoints. For $\epsilon, b > 0$, define $\chi \in C^5(\mathbb{R})$ by

$$\chi(x;\epsilon,b) = \begin{cases} 0 & x \leq \epsilon, \\ \rho((x-\epsilon)/b) & \epsilon < x < b+\epsilon, \\ 1 & b+\epsilon \leq x. \end{cases}$$

By construction χ is positive for $x \in (\epsilon, \infty)$ and all derivatives are supported in $[\epsilon, b+\epsilon]$. A scaling argument and (2.1) provides

$$\sup_{x\in[\epsilon,b+\epsilon]} \left| \frac{(\chi'''(x;\epsilon,b))^2}{\chi'(x;\epsilon,b)} \right| \leq c(b) \tag{2.2}$$

and for $j = 1,2,3,4,5$

$$|\chi^{(j)}(x;\epsilon,b)| \leq c(j;b). \tag{2.3}$$

A computation produces

$$\frac{(\chi'''(x;\epsilon,b))^2}{\chi'(x;\epsilon,b)} \cdot \frac{1}{\chi'(x;\epsilon/3,b+\epsilon)} = q_0(x)\frac{(x-\epsilon)(b+\epsilon-x)}{(3x-\epsilon)^5(3b-3x+4\epsilon)^5}$$

and for $j = 1,2,3,4,5$

$$\frac{\chi^{(j)}(x;\epsilon,b)}{\chi'(x;\epsilon/3,b+\epsilon)} = q_j(x)\frac{(x-\epsilon)(b+\epsilon-x)}{(3x-\epsilon)^5(3b-3x+4\epsilon)^5}$$

where $q_0,\ldots,q_5$ are polynomials. In each of the previous two cases, the right-hand side is continuous on the interval $x \in [\epsilon, b+\epsilon]$, hence bounded. These computations lead to the following estimates, which will be used in a later inductive argument:

$$\sup_{x\in[\epsilon,b+\epsilon]}\left|\frac{(\chi'''(x;\epsilon,b))^2}{\chi'(x;\epsilon,b)}\right| \leq c(\epsilon;b)\chi'(x;\epsilon/3,b+\epsilon) \tag{2.4}$$

and for $j = 1,2,3,4,5$

$$\sup_{x\in[\epsilon,b+\epsilon]}\left|\chi^{(j)}(x;\epsilon,b)\right| \leq c(j;\epsilon;b)\chi'(x;\epsilon/3,b+\epsilon). \tag{2.5}$$

Additionally, we define $\chi_n \in C^5(\mathbb{R})$ via the formula

$$\chi_n(x;\epsilon,b) = x^n\chi(x;\epsilon,b).$$

It is helpful to make the auxillary definition

$$p(y) = 462 - 1980y + 3465y^2 - 3080y^3 + 1386y^4 - 252y^5,$$

whose only real root occurs at $y \approx 1.29727$. Note that for $n \in \mathbb{Z}^+$

$$\chi_n'(x;\epsilon,b) = nx^{n-1}\chi(x;\epsilon,b) + x^n\chi'(x;\epsilon,b) \tag{2.6}$$

which is positive for $\epsilon < x \leq b+\epsilon$. Hence the expression

$$\frac{(\chi_n'''(x;\epsilon,b))^2}{\chi_n'(x;\epsilon,b)}$$

is continuous in this interval. To prove that it is bounded in $[\epsilon, b+\epsilon]$, we must only analyze the limit $x \to \epsilon^+$. First observe

$$\chi_n'(x;\epsilon,b) = \left(\frac{x-\epsilon}{b}\right)^5 \left(\frac{n}{b}x^{n-1}(x-\epsilon)p\left(\frac{x-\epsilon}{b}\right) + \frac{2772}{b}x^n\left(1-\frac{x-\epsilon}{b}\right)^5\right)$$

so that

$$\lim_{x\to\epsilon^+}\frac{(\chi_n'''(x;\epsilon,b))^2}{\chi_n'(x;\epsilon,b)} = \left(\frac{b^6}{2772\epsilon^n}\right)\lim_{x\to\epsilon^+}\frac{(\chi_n'''(x;\epsilon,b))^2}{(x-\epsilon)^5}.$$

Each term of χ_n''' has a factor of $(x-\epsilon)^3$ implying the above limit vanishes. Hence

$$\sup_{x\in[\epsilon,b+\epsilon]}\left|\frac{(\chi_n'''(x;\epsilon,b))^2}{\chi_n'(x;\epsilon,b)}\right| \le c(n;b) \tag{2.7}$$

and so

$$\left|\frac{(\chi_n'''(x;\epsilon,b))^2}{\chi_n'(x;\epsilon,b)}\right| \le c(n;b)(1+\chi_n(x;\epsilon,b)). \tag{2.8}$$

Each term of (2.6) is nonnegative and χ' is supported in $[\epsilon, b+\epsilon]$, hence

$$\chi_n'(x;\epsilon,b) \le c(n;b)(1+\chi_n(x;\epsilon,b)).$$

Using the Leibniz rule, it similarly follows for $j = 1,2,3,4,5$ that

$$|\chi_n^{(j)}(x;\epsilon,b)| \le c(n;j;b)(1+\chi_n(x;\epsilon,b)). \tag{2.9}$$

Assuming $n \ge 3$, notice that (2.7) and

$$\frac{(\chi_n'''(x;\epsilon,b))^2}{\chi_n'(x;\epsilon,b)} = (n-1)(n-2)x^{n-5} \qquad (b+\epsilon \le x)$$

imply

$$\left|\frac{(\chi_n'''(x;\epsilon,b))^2}{\chi_n'(x;\epsilon,b)}\right| \le c(n;\epsilon;b)\chi_{n-1}(x;\epsilon/3,b+\epsilon). \tag{2.10}$$

A similar argument holds for $n = 1, 2$. Next we prove for $j = 1, 2, 3, 4, 5$

$$|\chi_n^{(j)}(x; \epsilon, b)| \leq c(n; j; \epsilon; b)\chi_{n-1}(x; \epsilon/3, b + \epsilon). \tag{2.11}$$

This follows by definition when $b + \epsilon \leq x$; thus it suffices to prove

$$\sup_{x\in[\epsilon,b+\epsilon]} \left| \frac{\chi_n^{(j)}(x; \epsilon, b)}{\chi_{n-1}(x; \epsilon/3, b + \epsilon)} \right| \leq c(n, j, \epsilon, b).$$

We demonstrate the details for $j = 1$, the remaining cases being similar. In this case

$$\frac{\chi_n^{(j)}(x; \epsilon, b)}{\chi_{n-1}(x; \epsilon/3, b + \epsilon)} = \frac{n\chi(x; \epsilon, b)}{\chi(x; \epsilon/3, b + \epsilon)} + \frac{x\chi'(x; \epsilon, b)}{\chi(x; \epsilon/3, b + \epsilon)}.$$

Assuming $\epsilon \leq x \leq b + \epsilon$,

$$\frac{n\chi(x; \epsilon, b)}{\chi(x; \epsilon/3, b + \epsilon)} = n \left(\frac{b + \epsilon}{b}\right)^6 \frac{(x - \epsilon)^6 p\left(\frac{x-\epsilon}{b}\right)}{(x - \frac{\epsilon}{3})^6 p\left(\frac{x-\frac{\epsilon}{3}}{b+\epsilon}\right)}.$$

Note that $\frac{x-\frac{\epsilon}{3}}{b+\epsilon} < 1$ so that p does not vanish in $[\epsilon, b + \epsilon]$. Hence this above expression is continuous and bounded on this interval. Similarly for the second term

$$\frac{x\chi'(x; \epsilon, b)}{\chi(x; \epsilon/3, b + \epsilon)} = \frac{2772(b + \epsilon)^6(x - \epsilon)^5(b - x + \epsilon)^5 x}{b^{11}(x - \frac{\epsilon}{3})p\left(\frac{x-\frac{\epsilon}{3}}{b+\epsilon}\right)}.$$

This proves (2.11) in the case $j = 1$.

2.2 Proof of Theorem 1

The proofs of Theorems 1, 2 and 3 require several lemmas. The first is an energy inequality found in [11].

Lemma 1. *Let $u \in C^\infty([0,T] : H^\infty(\mathbb{R}))$ be a solution to the IVP*

$$\begin{cases} \partial_t u - \partial_x^5 u = F & x, t \in \mathbb{R} \\ u(x,0) = u_0(x) \end{cases} \tag{2.12}$$

and let $\psi \in C^5(\mathbb{R}^2)$ satisfy $\partial_x \psi \geq 0$. Then we have

$$\frac{d}{dt}\int u^2 \psi \, dx + \int (\partial_x^2 u)^2 \partial_x \psi \, dx \\ \leq \int u^2 \left\{ \partial_t \psi + \frac{3}{2}\partial_x^5 \psi + \frac{25}{16}\frac{(\partial_x^3 \psi)^2}{\partial_x \psi} \right\} dx + 2\int uF\psi \, dx. \tag{2.13}$$

By interpolation we have the following lemma, which is required to apply the inductive hypothesis.

Lemma 2. *Suppose $u_0 \in L^2(\mathbb{R})$ and for some $l \in \mathbb{Z}^+$, $l \geq 2$, $x_0 \in \mathbb{R}$*

$$\|\partial_x^l u_0\|^2_{L^2(x_0,\infty)} = \int_{x_0}^\infty |\partial_x^l u_0|^2 \, dx < \infty. \tag{2.14}$$

For any $k = 1, 2, \ldots, l-1$ and $\delta > 0$

$$\|\partial_x^k u_0\|^2_{L^2(x_0+\delta,\infty)} = \int_{x_0+\delta}^\infty |\partial_x^k u_0|^2 \, dx < \infty. \tag{2.15}$$

We reproduce for convenience a lemma from [12].

Lemma 3. *Let $j_1, j_2, j_3 \in \mathbb{Z}^+$ and $\epsilon, b > 0$. Suppose $\psi(x;\epsilon,b)$ has support in $[\epsilon,\infty)$,*

$\psi \geq 0$ and $\psi(x;\epsilon,b) \geq 1$ whenever $x \geq b+\epsilon$. Then

$$\begin{aligned}\int |\partial_x^{j_1} u \partial_x^{j_2} u \partial_x^{j_3} u| \psi(x)\, dx \\ \lesssim \left\{ \int (\partial_x^{1+j_1} u)^2 \psi(x)\, dx + \int (\partial_x^{j_1} u)^2 \psi(x)\, dx + \int (\partial_x^{j_1} u)^2 |\psi'(x)|\, dx \right\} \\ \times \int (\partial_x^{j_2} u)^2 \psi(x;\epsilon/5, 4\epsilon/5)\, dx + \int (\partial_x^{j_3} u)^2 \psi(x)\, dx. \qquad (2.16)\end{aligned}$$

In particular, we may choose $\psi = \chi, \chi', \chi_n$ or χ_n'.

Proof. Using Cauchy-Schwarz and Young's inequality, followed by the Sobolev embedding, we have

$$\begin{aligned}&\int |\partial_x^{j_1} u \partial_x^{j_2} u \partial_x^{j_3} u| \psi\, dx \\ &\leq \frac{1}{2}\int (\partial_x^{j_1} u)^2 (\partial_x^{j_2} u)^2 \psi\, dx + \frac{1}{2}\int (\partial_x^{j_3} u)^2 \psi\, dx \\ &\leq \frac{1}{2}\|(\partial_x^{j_1} u)^2 \psi\|_{L_x^\infty} \int_\epsilon^\infty (\partial_x^{j_2} u)^2\, dx + \frac{1}{2}\int (\partial_x^{j_3} u)^2 \psi\, dx \\ &\leq \frac{1}{2}\|\partial_x((\partial_x^{j_1} u)^2 \psi)\|_{L_x^1} \int (\partial_x^{j_2} u)^2 \psi(x;\epsilon/5, 4\epsilon/5)\, dx + \frac{1}{2}\int (\partial_x^{j_3} u)^2 \psi\, dx\end{aligned}$$

since $\psi(x;\epsilon,b)$ is nonnegative, supported on $[\epsilon,\infty)$ and $\psi(x;\epsilon,b) \geq 1$ when $x \geq b+\epsilon$. Furthermore, Young's inequality yields

$$\begin{aligned}\|\partial_x((\partial_x^{j_1} u)^2 \psi)\|_{L_x^1} &\leq 2\int |\partial_x^{j_1} u \partial_x^{1+j_1} u| \psi\, dx + \int (\partial_x^{j_1} u)^2 |\psi'|\, dx \\ &\leq \int (\partial_x^{1+j_1} u)^2 \psi\, dx + \int (\partial_x^{j_1} u)^2 \psi\, dx + \int (\partial_x^{j_1} u)^2 |\psi'|\, dx.\end{aligned}$$

This completes the proof of Lemma 3. □

We now turn to the proof of Theorem 1. As the argument is translation invariant, we consider only $x_0 = 0$. Additionally, the estimates are performed for nonlinearity $u\partial_x^3 u$;

a later remark explains how to control other terms. We invoke constants $c_0, c_1, c_2, \ldots$, depending only on the parameters

$$c_k = c_k(l, T, \epsilon, b, \|u_0\|_{H^s}; \|\partial_x^l u_0\|_{L^2(x_0,\infty)}; \|\partial_x^3 u\|_{L^1_T L^\infty_x}) \tag{2.17}$$

whose value may change from line to line. We explicitly record dependence on the parameter ν using the notation $c(\nu; d)$, which indicates a constant taking the form of a degree-d polynomial in ν:

$$c(\nu; d) = c_d \nu^d + \cdots + c_1 \nu + c_0.$$

We first describe the formal calculations and later provide justification using a limiting argument. Let u be a smooth solution of IVP (1.37), differentiate the equation l-times and apply (2.13) with $\phi(x,t) = \chi(x + \nu t; \epsilon, b)$. Using properties (2.4) and (2.5) to expand the region of integration in the first term, we arrive at

$$\begin{aligned}
\frac{d}{dt}\int (\partial_x^l u)^2 \chi(x+\nu t)\, dx &+ \int (\partial_x^{l+2} u)^2 \chi'(x+\nu t)\, dx \\
&\le \int (\partial_x^l u)^2 \left\{ \nu \chi'(x+\nu t) + \frac{3}{2}\chi^{(5)}(x+\nu t) + \frac{25}{16}\frac{(\chi'''(x+\nu t))^2}{\chi'(x+\nu t)} \right\} dx \\
&\qquad + 2\int \partial_x^l u \partial_x^l (u \partial_x^3 u) \chi(x+\nu t)\, dx \\
&\le A + B,
\end{aligned} \tag{2.18}$$

where

$$\begin{aligned}
A &= \nu \int (\partial_x^l u)^2 \chi'(x+\nu t)\, dx + c(\epsilon; b) \int (\partial_x^l u)^2 \chi'(x+\nu t; \epsilon/3, b+\epsilon)\, dx, \\
B &= 2\int \partial_x^l u \partial_x^l (u \partial_x^3 u) \chi(x+\nu t)\, dx.
\end{aligned}$$

We have used the convention that when ϵ and b are suppressed, $\chi(x) = \chi(x;\epsilon,b)$. The argument proceeds via induction on l where, for fixed l, we integrate (2.18) in time, integrate B by parts and apply a correction to account for the loss of derivatives.

Case $l = 1$ Integrating in the time interval $[0,t]$ and applying (2.3), we obtain

$$\left|\int_0^t A\ d\tau\right| \le c_0(1+\nu)\int_0^t\int(\partial_x u)^2\ dxd\tau \le c_0(1+\nu)T\|u\|^2_{L^\infty_T H^1_x} \tag{2.19}$$

where $0 \le t \le T$. After integrating by parts, we find

$$\begin{aligned} B = &\int \partial_x u(\partial_x^2 u)^2\chi(x+\nu t)\ dx + 3\int u(\partial_x^2 u)^2\chi'(x+\nu t)\ dx \\ &+ \frac{4}{3}\int(\partial_x u)^3\chi''(x+\nu t)\ dx - \int u(\partial_x u)^2\chi'''(x+\nu t)\ dx. \end{aligned} \tag{2.20}$$

The inequality (2.3) and the Sobolev embedding imply

$$\begin{aligned} \left|\int_0^t B\ d\tau\right| &\le c_1(\|\partial_x u\|_{L^\infty_T L^\infty_x} + \|u\|_{L^\infty_T L^\infty_x})\int_0^t\int(\partial_x u)^2 + (\partial_x^2 u)^2\ dxd\tau \\ &\le c_1 T\|u\|^3_{L^\infty_T H^2_x}. \end{aligned} \tag{2.21}$$

Integrating the inequality (2.18) and combining (2.19) and (2.21), we obtain

$$\begin{aligned} &\int(\partial_x u)^2\chi(x+\nu t)\ dx + \int_0^t\int(\partial_x^3 u)^2\chi'(x+\nu\tau)\ dxd\tau \\ &\quad\le \int(\partial_x u_0)^2\chi(x)\ dx + \left|\int_0^t A + B\ d\tau\right| \\ &\quad\le c_0\nu + c_1. \end{aligned}$$

As the right-hand side is independent of t, the result follows.

Case $l=2$ Similar to the previous case, integrating in the time interval $[0,t]$, we find

$$\left|\int_0^t A\,d\tau\right| \le c_0(1+\nu)\int_0^t \int (\partial_x^2 u)^2\,dx d\tau \le c_0(1+\nu)T\|u\|^2_{L_T^\infty H_x^2} \tag{2.22}$$

where $0\le t\le T$. After integrating by parts, we see

$$\begin{aligned} B = & -\int \partial_x u(\partial_x^3 u)^2\chi(x+\nu t)\,dx + 3\int u(\partial_x^3 u)^2\chi'(x+\nu t)\,dx \\ & - \int \partial_x u(\partial_x^2 u)^2\chi''(x+\nu t)\,dx - \int u(\partial_x^2 u)^2\chi'''(x+\nu t)\,dx. \end{aligned} \tag{2.23}$$

This expression exhibits a loss of derivatives in that the term

$$\int \partial_x u(\partial_x^3 u)^2\chi(x+\nu t)\,dx \tag{2.24}$$

can be controlled neither by the well-posedness theory nor by the $l=1$ case (without the technique introduced in Section 7). In [27], Kwon introduced a modified energy to overcome a similar issue. In particular, a smooth solution u to the IVP (1.37) satisfies the following identity:

$$\begin{aligned} & \frac{d}{dt}\int u(\partial_x u)^2\chi\,dx \\ = \;& -5\int \partial_x u(\partial_x^3 u)^2\chi\,dx - 5\int u(\partial_x^3 u)^2\chi'\,dx + \frac{28}{3}\int (\partial_x^2 u)^3\chi'\,dx \\ & +21\int \partial_x u(\partial_x^2 u)^2\chi''\,dx + 5\int u(\partial_x^2 u)^2\chi'''\,dx - \frac{10}{3}\int (\partial_x u)^3\chi^{(4)}\,dx \\ & -\int u(\partial_x u)^2\chi^{(5)}\,dx + 4\int u\partial_x u(\partial_x^2 u)^2\chi\,dx + 3\int u^2(\partial_x^2 u)^2\chi'\,dx \\ & -\frac{9}{4}\int (\partial_x u)^4\chi'\,dx - \int u\partial_x^2 u(\partial_x u)^2\chi'\,dx - 4\int u(\partial_x u)^3\chi''\,dx \\ & -\int u^2(\partial_x u)^2\chi'''\,dx + \nu\int u(\partial_x u)^2\chi'\,dx \end{aligned} \tag{2.25}$$

where $\chi^{(j)}$ denotes $\chi^{(j)}(x+\nu t)$. We use this identity to eliminate (2.24) from (2.23), yielding

$$B = \frac{1}{5}\frac{d}{dt}\int u(\partial_x u)^2\chi(x+\nu t)\, dx + 4\int u(\partial_x^3 u)^2\chi'(x+\nu t)\, dx$$
$$- \frac{4}{5}\int u\partial_x u(\partial_x^2 u)^2\chi(x+\nu t)\, dx - \frac{\nu}{5}\int u(\partial_x u)^2\chi'(x+\nu t)\, dx$$
$$+ \sum_{\substack{0\le j_1,j_2,j_3\le 2\\ 1\le j_4\le 5}} c_{j_1,j_2,j_3,j_4}\int \widetilde{\partial_x^{j_1} u}\partial_x^{j_2} u(\partial_x^{j_3} u)^2\chi^{(j_4)}(x+\nu t)\, dx \tag{2.26}$$

where the notation $\widetilde{\partial_x^{j_1} u}$ indicates this factor may be omitted. That is, since $0 \le j_1, j_2 \le 2$,

$$\|\widetilde{\partial_x^{j_1} u}\partial_x^{j_2} u\|_{L^\infty_T L^\infty_x} \le \|u\|_{L^\infty_T H^s_x} + \|u\|^2_{L^\infty_T H^s_x}.$$

Integrating in the time interval $[0,t]$, applying (2.3) and the Sobolev embedding, we obtain

$$\left|\int_0^t \int \widetilde{\partial_x^{j_1} u}\partial_x^{j_2} u(\partial_x^{j_3} u)^2\chi^{(j_4)}(x+\nu\tau)\, dxd\tau\right|$$
$$\le c_1\|\widetilde{\partial_x^{j_1} u}\partial_x^{j_2} u\|_{L^\infty_T L^\infty_x}\int_0^T\int(\partial_x^{j_3} u)^2\, dxd\tau$$
$$\le c_1 T\|u\|^3_{L^\infty_T H^s_x}(1+\|u\|_{L^\infty_T H^s_x}) \tag{2.27}$$

since $\max\{j_1, j_2, j_3\} \le 2$. The fundamental theorem of calculus and Sobolev embedding yield

$$\begin{aligned}\left|\int_0^t B\,d\tau\right| \leq \left|\int u_0(\partial_x u_0)^2\chi(x)\,dx\right| + \left|\int u(\partial_x u)^2\chi(x+\nu t)\,dx\right| \\ + 4\|u\|_{L^\infty_T H^1_x}\int_0^T\int(\partial_x^3 u)^2\chi'(x+\nu\tau)\,dxd\tau \\ + \frac{4}{5}\|u\|^2_{L^\infty_T H^2_x}\int_0^T\int(\partial_x^2 u)^2\chi(x+\nu\tau)\,dxd\tau \\ + \frac{\nu}{5}\|u\|_{L^\infty_T H^1_x}\int_0^T\int(\partial_x u)^2\chi'(x+\nu\tau)\,dxd\tau \\ + c_1 T\|u\|^3_{L^\infty_T H^s_x}(1+\|u\|_{L^\infty_T H^s_x}).\end{aligned} \tag{2.28}$$

The first term on the right-hand side is controlled by the Sobolev embedding, the hypothesis on the initial data and Lemma 2. The second and third term illustrate the iterative nature of the argument, as they can be bounded by the $l = 1$ result. The two remaining integrals are finite by property (2.3). Therefore

$$\left|\int_0^t B\,d\tau\right| \leq c_0\nu + c_1. \tag{2.29}$$

Integrating inequality (2.18), using (2.22), (2.29) and the hypothesis on the initial data, we have

$$\begin{aligned}\int(\partial_x^2 u)^2\chi(x+\nu t)\,dx + \int_0^t\int(\partial_x^4 u)^2\chi'(x+\nu\tau)\,dxd\tau \\ \leq \int(\partial_x^2 u_0)^2\chi(x)\,dx + \left|\int_0^t A + B\,d\tau\right| \\ \leq c_0\nu + c_1.\end{aligned}$$

As the right-hand side is independent of t, the result follows.

Case $l = 3$ Integrating in the time interval $[0, t]$ and applying the $l = 1$ result, we obtain

$$\begin{aligned}\left|\int_0^t A\, d\tau\right| &\leq \nu \int_0^T \int (\partial_x^3 u)^2 \chi'(x+\nu\tau)\, dxd\tau \\ &\quad + c_0 \int_0^T \int (\partial_x^3 u)^2 \chi'(x+\nu\tau; \epsilon/3, b+\epsilon)\, dxd\tau \\ &\leq c_2\nu^2 + c_1\nu + c_0 \end{aligned} \tag{2.30}$$

where $0 \leq t \leq T$. After integrating by parts, we find

$$\begin{aligned} B = &-3\int \partial_x u(\partial_x^4 u)^2 \chi(x+\nu t)\, dx + 3\int u(\partial_x^4 u)^2 \chi'(x+\nu t)\, dx \\ &+ \int (\partial_x^3 u)^3 \chi(x+\nu t)\, dx - \int u(\partial_x^3 u)^2 \chi'''(x+\nu t)\, dx. \end{aligned} \tag{2.31}$$

This expression exhibits a loss of derivatives in the term

$$\int \partial_x u(\partial_x^4 u)^2 \chi(x+\nu t)\, dx. \tag{2.32}$$

A smooth solution u to the IVP (1.37) satisfies the following identity:

$$\begin{aligned}
&\frac{d}{dt}\int u(\partial_x^2 u)^2\chi\,dx \\
&= -5\int \partial_x u(\partial_x^4 u)^2\chi\,dx - 5\int u(\partial_x^4 u)^2\chi'\,dx \\
&\quad + 5\int (\partial_x^3 u)^3\chi\,dx + 25\int \partial_x^2 u(\partial_x^3 u)^2\chi'\,dx + 15\int \partial_x u(\partial_x^3 u)^2\chi''\,dx \\
&\quad + 5\int u(\partial_x^3 u)^2\chi'''\,dx + 2\int u\partial_x u(\partial_x^3 u)^2\chi\,dx + 3\int u^2(\partial_x^3 u)^2\chi'\,dx \\
&\quad - \frac{25}{3}\int (\partial_x^2 u)^3\chi'''\,dx - 5\int \partial_x u(\partial_x^2 u)^2\chi^{(4)}\,dx - \int u(\partial_x^2 u)^2\chi^{(5)}\,dx \\
&\quad - \int \partial_x u(\partial_x^2 u)^3\chi\,dx - 3\int u(\partial_x^2 u)^2\chi'\,dx - 2\int (\partial_x u)^2(\partial_x^2 u)^2\chi'\,dx \\
&\quad - 4\int u\partial_x u(\partial_x^2 u0^2\chi''\,dx - \int u^2(\partial_x^2 u)^2\chi'''\,dx + \nu\int u(\partial_x^2 u)^2\chi'\,dx
\end{aligned} \tag{2.33}$$

where $\chi^{(j)}$ denotes $\chi^{(j)}(x+\nu t)$, which we use to eliminate (2.32) from (2.31). Thus, ignoring coefficients, we may write

$$\begin{aligned}
B &= \frac{d}{dt}\int u(\partial_x^2 u)^2\chi(x+\nu t)\,dx + \int u(\partial_x^4 u)^2\chi'(x+\nu t)\,dx \\
&\quad + \int (1 + u\partial_x u + \partial_x^3 u)(\partial_x^3 u)^2\chi(x+\nu t)\,dx + \nu\int u(\partial_x^2 u)^2\chi'\,dx \\
&\quad + \sum_{\substack{0\le j_1,j_2\le 2\\ 1\le j_3\le 3}} c_{j_1,j_2,j_3}\int \widetilde{\partial_x^{j_1} u}\,\partial_x^{j_2} u(\partial_x^3 u)^2\chi^{(j_3)}(x+\nu t)\,dx \\
&\quad + \sum_{\substack{0\le j_1,j_2\le 2\\ 1\le j_3\le 5}} c_{j_1,j_2,j_3}\int \widetilde{\partial_x^{j_1} u}\,\partial_x^{j_2} u(\partial_x^2 u)^2\chi^{(j_3)}(x+\nu t)\,dx
\end{aligned} \tag{2.34}$$

where the notation $\widetilde{\partial_x^{j_1} u}$ indicates this factor may be omitted. Integrating in the time interval $[0,t]$, applying (2.5), the Sobolev embedding and the $l=1$ result yields

$$\begin{aligned}
&\left|\int_0^t \int \widetilde{\partial_x^{j_1} u \partial_x^{j_2} u}(\partial_x^3 u)^2 \chi^{(j_3)}(x+\nu\tau)\, dx d\tau\right| \\
&\qquad \le c_1 \|\widetilde{\partial_x^{j_1} u \partial_x^{j_2} u}\|_{L_T^\infty L_x^\infty} \int_0^T \int (\partial_x^3 u)^2 \chi'(x+\nu\tau;\epsilon/3, b+\epsilon)\, dx d\tau \\
&\qquad \le (\|u\|_{L_T^\infty H_x^s} + \|u\|^2_{L_T^\infty H_x^s})(c_0\nu + c_1). \qquad (2.35)
\end{aligned}$$

Similarly, integrating in the time interval $[0,t]$, applying (2.3) and the Sobolev embedding, we find

$$\begin{aligned}
&\left|\int_0^t \int \widetilde{\partial_x^{j_1} u \partial_x^{j_2} u}(\partial_x^2 u)^2 \chi^{(j_3)}(x+\nu\tau)\, dx d\tau\right| \\
&\qquad \le c_1 \|\widetilde{\partial_x^{j_1} u \partial_x^{j_2} u}\|_{L_T^\infty L_x^\infty} \int_0^T \int (\partial_x^2 u)^2 \, dx d\tau \\
&\qquad \le c_1 T \|u\|^3_{L_T^\infty H_x^s}(1 + \|u\|_{L_T^\infty H_x^s}). \qquad (2.36)
\end{aligned}$$

Hence the fundamental theorem of calculus and Sobolev embedding yield

$$\begin{aligned}
\left|\int_0^t B\, d\tau\right| \le & \left|\int u_0(\partial_x^2 u_0)^2 \chi(x)\, dx\right| + \left|\int u(\partial_x^2 u)^2 \chi(x+\nu t)\, dx\right| \\
&+ \|u\|_{L_T^\infty H_x^1} \int_0^T \int (\partial_x^4 u)^2 \chi'(x+\nu\tau)\, dx d\tau \\
&+ \int_0^t (1 + \|u\|^2_{L_T^\infty H_x^2} + \|\partial_x^3 u(\tau)\|_{L_x^\infty}) \int (\partial_x^3 u)^2 \chi(x+\nu\tau)\, dx d\tau \\
&+ (\|u\|_{L_T^\infty H_x^s} + \|u\|^2_{L_T^\infty H_x^s})(c_0\nu + c_1) \\
&+ c_1 T \|u\|^3_{L_T^\infty H_x^s}(1 + \|u\|_{L_T^\infty H_x^s}). \qquad (2.37)
\end{aligned}$$

Similar to the $l=2$ case, the first term on the right-hand side is controlled by the hypothesis on the initial data. The second and third terms are finite by the $l=2$ case.

Therefore

$$\left|\int_0^t B\ d\tau\right| \leq c(\nu;1) + \int_0^t (c_0 + c_1\|\partial_x^3 u(\tau)\|_{L_x^\infty}) \int (\partial_x^3 u)^3 \chi(x+\nu\tau)\ dxd\tau. \tag{2.38}$$

Integrating inequality (2.18), using (2.30), (2.38) and the hypothesis on the initial data, we have

$$\begin{aligned} y(t) &:= \int (\partial_x^3 u)^2 \chi(x+\nu t)\ dx + \int_0^t \int (\partial_x^5 u)^2 \chi'(x+\nu\tau)\ dxd\tau \\ &\leq \int (\partial_x^3 u_0)^2 \chi(x)\ dx + \left|\int_0^t A + B\ d\tau\right| \\ &\leq c(\nu;2) + \int_0^t (c_0 + c_1\|\partial_x^3 u(\tau)\|_{L_x^\infty}) \int (\partial_x^3 u)^2 \chi(x+\nu\tau)\ dxd\tau \\ &\leq c(\nu;2) + \int_0^t (c_0 + c_1\|\partial_x^3 u(\tau)\|_{L_x^\infty}) y(\tau)\ dxd\tau. \end{aligned}$$

Applying Gronwall's inequality produces

$$\begin{aligned} &\sup_{0\leq t\leq T} \int (\partial_x^4 u)^2 \chi(x+\nu t)\ dx + \int_0^T \int (\partial_x^5 u)^2 \chi'(x+\nu\tau)\ dxd\tau \\ &\qquad \leq c(\nu;2) \exp\left(c_0 T + c_1 \|\partial_x^3 u\|_{L_T^1 L_x^\infty}\right). \end{aligned}$$

This proves the desired result with $l = 3$.

<u>Cases $l = 4, 5, 6$</u> Due to the structure of the IVP, the cases $l = 4, 5, 6$ must be handled individually. The analysis is omitted as it is similar to the cases $l = 3$ and $l \geq 7$. It can be proved that

$$\sup_{0\leq t\leq T} \int (\partial_x^l u)^2 \chi(x+\nu t)\ dx + \int_0^T \int (\partial_x^{l+2} u)^2 \chi'(x+\nu\tau)\ dxd\tau \leq c(\nu;d)$$

where the values of d are summarized in the following table.

l	1	2	3	4	5	6
d	1	1	2	2	4	8

Case $l \geq 7$ In the course of this case, we will prove that for $l \geq 7$, the final constant obtained after integrating both sides of (2.18) takes the form of a polynomial in ν with degree $8(l-5)$.

Integrating in the time interval $[0,t]$ and applying the $l-2$ result (assuming $l > 7$) we have

$$\begin{aligned} \left|\int_0^t A \, d\tau\right| &\leq \nu \int_0^T \int (\partial_x^l u)^2 \chi'(x+\nu\tau) \, dx d\tau \\ &\quad + c_0 \int_0^T \int (\partial_x^l u)^2 \chi'(x+\nu\tau; \epsilon/3, b+\epsilon) \, dx d\tau \\ &\leq c(\nu; 1+8(l-7)) \end{aligned} \tag{2.39}$$

where $0 \leq t \leq T$. For $l = 7$, this expression has degree 5 in ν. We write

$$B = B_1 + B_2 \tag{2.40}$$

where

$$\begin{aligned} B_1 = 2\int \partial_x^l u \Big\{ u\partial_x^{l+3}u + \binom{l}{1}\partial_x u \partial_x^{l+2} u + \binom{l}{2}\partial_x^2 u \partial_x^{l+1} u \\ +(1+\binom{l}{3})\partial_x^3 u \partial_x^l u \Big\} \chi(x+\nu t) \, dx \\ B_2 = \sum_{k=1}^{\lceil l/2 \rceil -2} c_{l,k} \int \partial_x^{3+k} u \partial_x^{l-k} u \partial_x^l u \chi(x+\nu t) \, dx \end{aligned}$$

and $3+k \leq l-k < l$ for $1 \leq k \leq \lceil l/2 \rceil - 2$. Integrating by parts, we have

$$B_1 = B_{11} + B_{12}, \tag{2.41}$$

where

$$\begin{aligned}
B_{11} &= (3-2l)\int \partial_x u(\partial_x^{l+1}u)^2\chi(x+\nu t)\ dx,\\
B_{12} &= \int u(\partial_x^{l+1}u)^2\chi'(x+\nu t)\ dx + \int \partial_x^3 u(\partial_x^l u)^2\chi(x+\nu t)\ dx\\
&\quad + \int \partial_x^2 u(\partial_x^l u)^2\chi'(x+\nu t)\ dx + \int \partial_x u(\partial_x^l u)^2\chi''(x+\nu t)\ dx\\
&\quad + \int u(\partial_x^l u)^2\chi'''(x+\nu t)\ dx
\end{aligned}$$

and, in B_{12}, we have omitted coefficients depending only on l using the expression (2.41). Then integrating in the time interval $[0,t]$, where $0 \le t \le T$, we obtain

$$\begin{aligned}
\left|\int_0^t B_{12}\ d\tau\right| \le \|u\|_{L_T^\infty H_x^1}\int_0^T &\int (\partial_x^{l+1}u)^2\chi'(x+\nu\tau)\ dxd\tau\\
&+ \int_0^t \|\partial_x^3 u(\tau)\|_{L_x^\infty}\int(\partial_x^l u)^2\chi(x+\nu\tau)\ dxd\tau\\
&+ c_0\|u\|_{L_T^\infty H_x^s}\int_0^t\int(\partial_x^l u)^2\chi'(x+\nu\tau)\ dxd\tau
\end{aligned}$$

by the Sobolev embedding and (2.5). Applying the result for cases $l-1$ and $l-2$, we have

$$\left|\int_0^t B_{12}\ d\tau\right| \le c(\nu;8(l-6)) + \int_0^t \|\partial_x^3 u(\tau)\|_{L_x^\infty}\int(\partial_x^l u)^2\chi(x+\nu\tau)\ dxd\tau. \tag{2.42}$$

Observe that term B_2 only occurs when $l \ge 5$. For $l > 5$, note that $4+k < l$. The inequality (2.16) produces

$$\begin{aligned}
|B_2| &\leq \sum_{k=1}^{\lceil l/2\rceil-2} c_{l,k}\int |\partial_x^{3+k}u\partial_x^{l-k}u\partial_x^l u|\chi(x+\nu t)\,dx\\
&\leq \int (\partial_x^l u)^2\chi(x+\nu t)\,dx\\
&\quad+\sum_{k=1}^{\lceil l/2\rceil-2}\left\{\int(\partial_x^{4+k}u)^2\chi(x+\nu t)\,dx+\int(\partial_x^{3+k}u)^2\chi(x+\nu t)\,dx\right.\\
&\qquad\left.+\int(\partial_x^{3+k}u)^2\chi'(x+\nu t)\,dx\right\}\int(\partial_x^{l-k}u)^2\chi(x+\nu t;\epsilon/5,4\epsilon/5)\,dx,
\end{aligned} \tag{2.43}$$

after suppressing constants depending on l. Integrating in the time interval $[0,t]$, we have

$$\begin{aligned}
&\left|\int_0^t B_2\,d\tau\right|\\
&\quad\leq \int_0^t\int(\partial_x^l u)^2\chi(x+\nu\tau)\,dx\\
&\qquad+T\sum_{k=1}^{\lceil l/2\rceil-2}\left(\sup_{0\leq t\leq T}\int(\partial_x^{l-k}u)^2\chi(x+\nu t;\epsilon/5,4\epsilon/5)\,dx\right)\\
&\qquad\qquad\times\left(\sup_{0\leq t\leq T}\int(\partial_x^{4+k}u)^2\chi(x+\nu t)\,dx\right)\\
&\qquad+T\sum_{k=1}^{\lceil l/2\rceil-2}\left(\sup_{0\leq t\leq T}\int(\partial_x^{l-k}u)^2\chi(x+\nu t;\epsilon/5,4\epsilon/5)\,dx\right)\\
&\qquad\qquad\times\left(\sup_{0\leq t\leq T}\int(\partial_x^{3+k}u)^2\chi(x+\nu t)\,dx\right)\\
&\qquad+T\sum_{k=1}^{\lceil l/2\rceil-2}\left(\sup_{0\leq t\leq T}\int(\partial_x^{l-k}u)^2\chi(x+\nu t;\epsilon/5,4\epsilon/5)\,dx\right)\\
&\qquad\qquad\times\left(\sup_{0\leq t\leq T}\int(\partial_x^{3+k}u)^2\chi'(x+\nu t)\,dx\right).
\end{aligned}$$

The strongest ν-dependence for B_2 arises from analyzing terms of the form:

$$\left(\sup_{0\le t\le T}\int(\partial_x^{l-k}u)^2\chi(x+\nu t;\epsilon/5,4\epsilon/5)\,dx\right)\left(\sup_{0\le t\le T}\int(\partial_x^{4+k}u)^2\chi(x+\nu t)\,dx\right). \quad (2.44)$$

Each factor in (2.44) is finite by the result for cases $l-k$ and $4+k$. The inductive hypothesis further implies that the ν-dependence has the form of a polynomial in ν having degree

$$\nu^{8(l-k-5)}\cdot\nu^{8(4+k-5)}=\nu^{8(l-6)}.$$

Hence

$$\left|\int_0^t B_2\,d\tau\right|\le c(\nu;8(l-6))+c_0\int_0^t\int(\partial_x^l u)^2\chi(x+\nu\tau)\,dxd\tau. \quad (2.45)$$

Integrating the inequality (2.18) in the time interval $[0,t]$, where $0\le t\le T$, we have

$$\begin{aligned}\int(\partial_x^l u)^2\chi(x+\nu t)\,dx&+\int_0^t\int(\partial_x^{l+2}u)^2\chi'(x+\nu\tau)\,dxd\tau\\ &\le\int(\partial_x^l u_0)^2\chi(x)\,dx+\left|\int_0^t A+B_{11}+B_{12}+B_2\,d\tau\right|\\ &\le c(\nu;8(l-6))\\ &\quad+\left|\int_0^t B_{11}\,d\tau\right|+\int_0^t(c_0+c_1\|\partial_x^3u(\tau)\|_{L_x^\infty})\int(\partial_x^l u)^2\chi(x+\nu\tau)\,dxd\tau \quad (2.46)\end{aligned}$$

using the hypothesis on the initial data, (2.39), (2.42) and (2.45). Thus it only remains to estimate the integral involving

$$B_{11}=(3-2l)\int\partial_x u(\partial_x^{l+1}u)^2\chi(x+\nu t)\,dx,$$

which exhibits a loss of derivatives. Assuming that u satisfies the IVP (1.37), we rewrite this term by considering the correction factor

$$\begin{aligned}
&\frac{d}{dt}\int u(\partial_x^{l-1}u)^2\chi(x+\nu t)\ dx \\
&= \int \partial_x^5 u(\partial_x^{l-1}u)^2\chi(x+\nu t)\ dx + \int u\partial_x^3 u(\partial_x^{l-1}u)^2\chi(x+\nu t)\ dx \\
&\quad +2\int u\partial_x^{l-1}u\partial_x^{l+4}u\chi(x+\nu t)\ dx + 2\int u\partial_x^{l-1}u\partial_x^{l-1}(u\partial_x^3 u)\chi(x+\nu t)\ dx \\
&\quad +\nu\int u(\partial_x^{l-1}u)^2\chi'(x+\nu t)\ dx \\
&=: C_1 + C_2 + \widetilde{C_3} + C_4 + C_5.
\end{aligned} \tag{2.47}$$

Observe that integrating $\widetilde{C_3}$ by parts reveals

$$\widetilde{C_3} = \left(\frac{5}{2l-3}\right) B_{11} + C_3, \tag{2.48}$$

where

$$\begin{aligned}
C_3 = &-5\int u(\partial_x^{l+1}u)^2\chi'\ dx + 5\int \partial_x^3 u(\partial_x^l u)^2\chi\ dx \\
&+ 9\int \partial_x^2 u(\partial_x^l u)^2\chi'\ dx + 15\int \partial_x u(\partial_x^l u)^2\chi''\ dx + \int u(\partial_x^l u)^2\chi'''\ dx \\
&- 5\int \partial_x^5 u(\partial_x^{l-1}u)^2\chi\ dx - 5\int \partial_x^4 u(\partial_x^{l-1}u)^2\chi'\ dx - 9\int \partial_x^3 u(\partial_x^{l-1}u)^2\chi''\ dx \\
&- 10\int \partial_x^2 u(\partial_x^{l-1}u)^2\chi'''\ dx - 5\int \partial_x u(\partial_x^{l-1}u)^2\chi^{(4)}\ dx - \int u(\partial_x^{l-1}u)^2\chi^{(5)}\ dx.
\end{aligned} \tag{2.49}$$

Here $\chi^{(j)}$ denotes $\chi^{(j)}(x+\nu t;\epsilon,b)$. The fundamental theorem of calculus leads to

$$\begin{aligned}
\left(\frac{5}{2l-3}\right)\left|\int_0^t B_{11}\ d\tau\right| \le \left|\int u_0(\partial_x^{l-1}u_0)^2\chi(x)\ dx\right| + \left|\int u(\partial_x^{l-1}u)^2\chi(x+\nu t)\ dx\right| \\
+ \left|\int_0^t C_1 + C_2 + C_3 + C_4 + C_5\ d\tau\right|.
\end{aligned} \tag{2.50}$$

We now concern ourselves with estimating the right-hand side of this expression. By the Sobolev embedding, hypothesis on the initial data, Lemma 2 and the result for case $l-1$, we have

$$\begin{aligned}&\left|\int u_0(\partial_x^{l-1}u_0)^2\chi(x)\,dx\right| + \left|\int u(\partial_x^{l-1}u)^2\chi(x+\nu t)\,dx\right|\\&\qquad\leq \|u_0\|_{H^s}\|\partial_x^{l-1}u_0\|^2_{L^2_x((0,\infty))} + \|u\|_{L^\infty_T H^s_x}\int(\partial_x^{l-1}u)^2\chi(x+\nu t)\,dx, \qquad (2.51)\end{aligned}$$

which is uniformly bounded by the inductive hypothesis. Applying (2.16), we obtain

$$\begin{aligned}|C_1| &\leq \int \partial_x^5 u(\partial_x^{l-1}u)^2\chi(x+\nu t)\,dx\\&\leq \int(\partial_x^{l-1}u)^2\chi(x+\nu t)\,dx\\&\quad+\left\{\int(\partial_x^6u)^2\chi(x+\nu t)\,dx + \int(\partial_x^5u)^2\chi(x+\nu t)\,dx + \int(\partial_x^5u)^2\chi'(x+\nu t)\,dx\right\}\\&\qquad\times\int(\partial_x^{l-1}u)^2\chi(x+\nu t;\epsilon/5,4\epsilon/5)\,dx.\end{aligned}$$

Integrating in the time interval $[0,t]$ and following the argument applied to term B_2, we see that the highest degree ν-dependence for C_1 arises from analyzing the term

$$\left(\sup_{0\leq t\leq T}\int(\partial_x^{l-1}u)^2\chi(x+\nu t;\epsilon/5,4\epsilon/5)\,dx\right)\left(\sup_{0\leq t\leq T}\int(\partial_x^6u)^2\chi(x+\nu t)\,dx\right). \qquad (2.52)$$

Each factor in (2.52) is finite by the result for cases 6 and $l-1$. Hence for the base case $l=7$, the right-hand side is bounded by $c(\nu;16)$. For $l>7$, the inductive hypothesis further yields that the ν-dependence has the form of a polynomial in ν with degree determined by

$$\nu^{8(l-6)}\cdot\nu^8 = \nu^{8(l-5)}.$$

Thus

$$\left|\int_0^t C_1\, d\tau\right| \le c(\nu; 8(l-5)). \tag{2.53}$$

It will be clear from the remainder of the argument that (2.52) produces the *overall* highest degree ν-dependence, hence justifying this inductive calculation.

Integrating in time, using the Sobolev embedding and inductive hypothesis, we find

$$\begin{aligned}\left|\int_0^t C_2\, d\tau\right| &\le \|u\|_{L^\infty_T H^s_x}\int_0^T\int |\partial_x^3 u|(\partial_x^{l-1}u)^2\chi(x+\nu\tau)\, dxd\tau\\ &\le \|u\|_{L^\infty_T H^s_x}\int_0^T \|\partial_x^3 u(\tau)\|_{L^\infty_x}\left(\sup_{0\le t\le T}\int(\partial_x^{l-1}u)^2\chi(x+\nu t)\, dx\right) d\tau\\ &\le c(\nu; 8(l-6))\|u\|_{L^\infty_T H^s_x}\|\partial_x^3 u\|_{L^1_T L^\infty_x}. \end{aligned}\tag{2.54}$$

Integrating in time and using (2.5), (2.16), the Sobolev embedding and the inductive hypothesis, we have

$$\left|\int_0^t C_3\, d\tau\right| \le c(\nu, 8(l-6)) + \int_0^t (c_0 + c_1\|\partial_x^3 u(\tau)\|_{L^\infty_x})\int(\partial_x^l u)^2\chi(x+\nu\tau)\, dxd\tau. \tag{2.55}$$

Expanding but ignoring binomial coeffiecients, we write $C_4 = C_{41} + C_{42}$ with

$$\begin{aligned}C_{41} = &\int u\partial_x u(\partial_x^l u)^2\chi(x+\nu t)\, dx + \int u^2(\partial_x^l u)^2\chi(x+\nu t)\, dx\\ &+\int u\partial_x^3 u(\partial_x^{l-1}u)^2\chi(x+\nu t)\, dx + \int \partial_x u\partial_x^2 u(\partial_x^{l-1}u)^2\chi(x+\nu t)\, dx\\ &+\int u\partial_x^2 u(\partial_x^{l-1}u)^2\chi'(x+\nu t)\, dx + \int \partial_x u\partial_x u(\partial_x^{l-1}u)^2\chi'(x+\nu t)\, dx\\ &+\int u\partial_x u(\partial_x^{l-1}u)^2\chi''(x+\nu t)\, dx - \int u^2(\partial_x^{l-1}u)^2\chi'''(x+\nu t)\, dx \end{aligned}\tag{2.56}$$

and

$$C_{42} = \sum_{k=1}^{\lfloor (l-1)/2\rfloor-2} c_{l,k}\int u\partial_x^{(l-1)-k}u\partial_x^{3+k}u\partial_x^{l-1}u\chi(x+\nu t)\, dx. \tag{2.57}$$

Similar to C_2 and C_3,

$$\left|\int_0^t C_{41}\, d\tau\right| \le c(\nu; 8(l-6)) + c_0 \int_0^t \int (\partial_x^l u)^2 \chi(x+\nu\tau)\, dxd\tau. \tag{2.58}$$

Similar to B_2, ignoring constants we have

$$\left|\int_0^t C_{42}\, d\tau\right| \le \sum_{k=1}^{\lfloor (l-1)/2\rfloor - 2} \int_0^T \int |u\partial_x^{(l-1)-k} u \partial_x^{3+k} u \partial_x^{l-1} u|\chi\, dxd\tau \le c(\nu; 8(l-6)) \tag{2.59}$$

after applying (2.16). Finally, assuming $l > 7$, we obtain

$$\left|\int_0^t C_5\, d\tau\right| \le \nu \|u\|_{L_T^\infty H_x^{5/2+}} \int_0^T \int (\partial_x^{l-1} u)^2 \chi'(x+\nu\tau)\, dxd\tau \le c(\nu; 1+8(l-7))$$

(or $c(\nu;3)$ when $l=7$) using the Sobolev embedding and inductive case $l-3$.

Inserting the above into (2.50) and (2.46), then using nonnegativity of χ, χ', we find

$$\begin{aligned} y(t) &:= \int (\partial_x^l u)^2 \chi(x+\nu t)\, dx + \int_0^t \int (\partial_x^{l+2} u)^2 \chi'(x+\nu\tau)\, dxd\tau \\ &\le c(\nu; 8(l-5)) + \int_0^t (c_0 + c_1 \|\partial_x^3 u(\tau)\|_{L_x^\infty}) \int (\partial_x^l u)^2 \chi(x+\nu\tau)\, dxd\tau \\ &\le c(\nu; 8(l-5)) + \int_0^t (c_0 + c_1 \|\partial_x^3 u(\tau)\|_{L_x^\infty}) y(\tau)\, d\tau. \end{aligned} \tag{2.60}$$

Hence Gronwall's inequality yields

$$\begin{aligned} \sup_{0\le t\le T} \int (\partial_x^l u)^2 \chi(x+\nu t)\, dx &+ \int_0^T \int (\partial_x^{l+2} u)^2 \chi'(x+\nu\tau)\, dxd\tau \\ &\le c(\nu; 8(l-5)) \exp\left(c_0 T + c_1 \|\partial_x^3 u\|_{L_T^1 L_x^\infty}\right). \end{aligned}$$

This concludes the proof for the case of smooth data.

Now we use a limiting argument to justify the previous computations for arbitrary

$u_0 \in H^s(\mathbb{R})$ with $s > 5/2$. Fix $\rho \in C_0^\infty(\mathbb{R})$ with supp $\rho \subseteq (-1,1)$, $\rho \geq 0$, $\int \rho(x)\, dx = 1$ and

$$\rho_\mu(x) = \frac{1}{\mu}\rho\left(\frac{x}{\mu}\right), \quad \mu > 0.$$

The the solution u^μ of IVP (1.37) corresponding to smoothed data $u_0^\mu = \rho_\mu * u_0$, $\mu \geq 0$, satisfies

$$u^\mu \in C^\infty([0,T] : H^\infty(\mathbb{R})).$$

Hence we may conclude

$$\sup_{0\leq t\leq T} \int (\partial_x^l u^\mu)^2 \chi(x+\nu t)\, dx + \int_0^T \int (\partial_x^{l+2} u^\mu)^2 \chi'(x+\nu\tau)\, dx d\tau \leq c.$$

where

$$c = c(l, \nu, \epsilon, R, T; \|u_0^\mu\|_{H^s}; \|\partial_x^l u_0^\mu\|_{L^2(0,\infty)}; \|u^\mu\|_{L_T^\infty H_x^s}; \|\partial_x^3 u^\mu\|_{L_T^1 L_x^\infty}).$$

To see that this bound is independent of $\mu > 0$, first note

$$\|u_0^\mu\|_{H^s} \leq \|\widehat{\rho_\mu}\|_\infty \|u_0\|_{H^s} \leq \|u_0\|_{H^s}.$$

As $\chi \equiv 0$ for $x < \epsilon$, restricting $0 < \mu < \epsilon$ it follows

$$(\partial_x^l u_0^\mu)^2 \chi(x;\epsilon,b) = (\rho_\mu * \partial_x^l u_0 1_{[0,\infty)})^2 \chi(x;\epsilon,b).$$

Thus by Young's inequality

$$\begin{aligned}\int_\epsilon^\infty (\partial_x^l u_0^\mu)^2(x)\, dx &= \int_\epsilon^\infty (\rho_\mu * \partial_x^l u_0 1_{[0,\infty)})^2(x)\, dx \\ &\leq \|\rho_\mu\|_1^2 \int_\epsilon^\infty (\partial_x^l u_0)^2(x)\, dx \\ &\leq \|\partial_x^l u_0\|_{L^2((0,\infty))}^2.\end{aligned}$$

From Kwon's local well-posedness result [27] we have

$$\|u^\mu\|_{L^\infty_T H^s_x} + \|\partial_x^3 u^\mu\|_{L^1_T L^\infty_x} \leq c(\|u_0^\mu\|_{H^s}) \leq c(\|u_0\|_{H^s})$$

and so we may replace the bound $c = c(\mu)$ with $\tilde{c}$ as in (1.42).

As the solution depends continuously on the initial data,

$$\sup_{0\leq t\leq T} \|u^\mu(t) - u(t)\|_{H^{5/2+}} \downarrow 0 \quad \text{as} \quad \mu \downarrow 0.$$

Combining this fact with the μ-uniform bound $\tilde{c}$, weak compactness and Fatou's lemma, the theorem holds for all $u_0 \in H^s(\mathbb{R})$ with $s > 5/2$. This completes the proof of Theorem 1 for nonlinearity $u\partial_x^3 u$.

Including nonlinearity $\partial_x u \partial_x^2 u$, term B in (2.18) will contain a term

$$2\int \partial_x^l u \partial_x^l (\partial_x u \partial_x^2 u)\chi(x+\nu t)\, dx.$$

As this nonlinearity has a total of three derivatives, integrating by parts produces a form very similar to (2.40). The nonlinearity $u^2\partial_x u$, containing only a single derivative, shows no loss of derivatives (see Section 2.5 for a more thorough treatment). This completes the proof of Theorem 1.

2.3 Proof of Theorem 2

Let u be a smooth solution of IVP (1.37), differentiate the equation l-times and apply (2.13) with $\phi(x,t) = \chi_n(x+\nu t;\epsilon,b)$ to arrive at

$$\frac{d}{dt}\int (\partial_x^l u)^2 \chi_n(x+\nu t)\,dx + \int (\partial_x^{l+2} u)^2 \chi_n'(x+\nu t)\,dx$$
$$\leq A + B, \tag{2.61}$$

where

$$\begin{aligned} A &= \int (\partial_x^l u)^2 \left\{ \nu\chi_n'(x+\nu t) + \frac{3}{2}\chi_n^{(5)}(x+\nu t) + \frac{25}{16}\frac{(\chi_n'''(x+\nu t))^2}{\chi_n'(x+\nu t)} \right\} dx, \\ B &= 2\int \partial_x^l u \partial_x^l (u\partial_x^3 u)\chi_n(x+\nu t)\,dx. \end{aligned}$$

The proof proceeds by induction on l, however, for fixed l we induct on n. The base case $n = 0$ coincides with the propagation of regularity result. We invoke constants $c_0, c_1, c_2, \dots,$ depending only on the parameters

$$c_k = c_k(n, l; \|u_0\|_{H^s}; \|\partial_x^3 u\|_{L_T^1 L_x^\infty}; \nu; \epsilon; b; T) \tag{2.62}$$

as well as the decay assumptions on the initial data (1.45).

Case $l = 0$ Using properties (2.8) and (2.9), we see

$$|A| \leq c_0 \int u^2 (1 + \chi_n(x+\nu t))\,dx.$$

and so integrating in the time interval $[0,t]$, we have

$$\left| \int_0^t A\,d\tau \right| \leq c_0 \left\{ T\|u\|_{L_T^\infty L_x^2}^2 + \int_0^t \int u^2 \chi_n(x+\nu\tau)\,dxd\tau \right\} \tag{2.63}$$

where $0 \leq t \leq T$. Additionally,

$$\left| \int_0^t B \, d\tau \right| \leq 2 \int_0^t \|\partial_x^3 u(\tau)\|_{L_x^\infty} \int u^2 \chi_n(x + \nu\tau) \, dx d\tau. \tag{2.64}$$

Integrating (2.61) in the time interval $[0, t]$, combining (2.63) and (2.64), we have

$$\begin{aligned} y(t) &:= \int u^2 \chi_n(x + \nu t) \, dx + \int_0^t \int (\partial_x^2 u)^2 \chi_n(x + \nu\tau) \, dx d\tau \\ &\leq \int u_0^2(x) \chi_n(x) \, dx + \left| \int_0^t A + B \, d\tau \right| \\ &\leq c_0 + \int_0^t (c_1 + c_2 \|\partial_x^3 u(\tau)\|_{L_x^\infty}) \int u^2 \chi_n(x + \nu\tau) \, dx d\tau \\ &\leq c_0 + \int_0^t (c_1 + c_2 \|\partial_x^3 u(\tau)\|_{L_x^\infty}) y(\tau) \, dx d\tau. \end{aligned}$$

using the hypothesis on the initial data. Gronwall's inequality yields

$$\sup_{0 \leq t \leq T} \int u^2 \chi_n(x + \nu t) \, dx + \int_0^T \int (\partial_x^2 u)^2 \chi_n(x + \nu\tau) \, dx d\tau \leq c_0 \exp\left(c_1 T + c_2 \|\partial_x^3 u\|_{L_T^1 L_x^\infty} \right).$$

Note that induction in n was not required in this case.

Case $l = 1$ Using properties (2.8) and (2.9), we have

$$|A| \leq c_0 \int (\partial_x u)^2 (1 + \chi_n(x + \nu t)) \, dx.$$

and so integrating in the time interval $[0, t]$, we find

$$\left| \int_0^t A \, d\tau \right| \leq c_0 \left\{ T \|u\|_{L_T^\infty H_x^1}^2 + \int_0^t \int (\partial_x u)^2 \chi_n(x + \nu\tau) \, dx d\tau \right\} \tag{2.65}$$

where $0 \leq t \leq T$. After integrating by parts, we find

$$\begin{aligned} B = &\int \partial_x u(\partial_x^2 u)^2 \chi_n(x+\nu t)\, dx + 3\int u(\partial_x^2 u)^2 \chi_n'(x+\nu t)\, dx \\ &+ \frac{4}{3}\int (\partial_x u)^3 \chi_n''(x+\nu t)\, dx - \int u(\partial_x u)^2 \chi_n'''(x+\nu t)\, dx. \end{aligned} \tag{2.66}$$

This expression exhibits a loss of derivatives requiring a correction. A smooth solution u to the IVP (1.37) satisfies the following identity

$$\begin{aligned} &\frac{d}{dt}\int u^3 \chi_n\, dx \\ &= -15\int \partial_x u(\partial_x^2 u)^2 \chi_n\, dx - 9\int u(\partial_x^2 u)^2 \chi_n'\, dx \\ &\quad + 10\int (\partial_x u)^3 \chi_n''\, dx + 12\int u(\partial_x u)^2 \chi_n'''\, dx - \int u^3 \chi_n^{(5)}\, dx \\ &\quad + 9\int u(\partial_x u)^3 \chi_n\, dx + \frac{27}{2}\int u^2(\partial_x u)^2 \chi_n'\, dx - \frac{3}{4}\int u^4 \chi_n'''\, dx \\ &\quad + \nu\int u^3 \chi_n'\, dx \end{aligned} \tag{2.67}$$

after integrating by parts, where $\chi_n^{(j)}$ denotes $\chi_n^{(j)}(x+\nu t)$. Substituting (2.67), we can write (2.66) as a linear combination of the following terms

$$\begin{aligned} B = &\frac{d}{dt}\int u^3 \chi_n\, dx + \int u(\partial_x^2 u)^2 \chi_n'\, dx \\ &+ \int (\partial_x u)^3 \chi_n''\, dx + \int u(\partial_x u)^2 \chi_n'''\, dx + \int u^3 \chi_n^{(5)}\, dx \\ &+ \int u(\partial_x u)^3 \chi_n\, dx + \int u^2(\partial_x u)^2 \chi_n'\, dx + \int u^4 \chi_n'''\, dx \\ &+ \nu\int u^3 \chi_n'\, dx \\ =: &\, B_1 + \cdots + B_9. \end{aligned} \tag{2.68}$$

The fundamental theorem of calculus and the Sobolev embedding yield

$$\left|\int_0^t B_1\, d\tau\right| \leq \|u_0\|_{H^1} \int u_0^2(x)\chi_n(x)\, dx + \|u\|_{L_T^\infty H_x^1} \int u^2 \chi_n(x+\nu t)\, dx \tag{2.69}$$

where $0 \leq t \leq T$. This term is finite by hypothesis (1.45) and the case $l = 0$. Next,

$$\left|\int_0^t B_2\, d\tau\right| \leq \|u\|_{L_T^\infty H_x^1} \int_0^T \int (\partial_x^2 u)^2 \chi_n'(x+\nu\tau)\, dx d\tau, \tag{2.70}$$

which is finite by case $l = 0$. Using (2.11) and the Sobolev embedding, we obtain

$$\begin{aligned}
&\left|\int_0^t B_3 + B_4 + B_5\, d\tau\right| \\
&\qquad \leq \|u\|_{L_T^\infty H_x^2} \int_0^T \int (\partial_x u)^2 |\chi_n''(x+\nu\tau)| + (\partial_x u)^2 |\chi_n'''(x+\nu\tau)|\, dx d\tau \\
&\qquad\quad + \|u\|_{L_T^\infty H_x^1} \int_0^T \int u^2 |\chi_n^{(5)}(x+\nu\tau)|\, dx d\tau \\
&\qquad \leq c_0 \|u\|_{L_T^\infty H_x^2} \int_0^T \int (\partial_x u)^2 \chi_{n-1}(x+\nu\tau; \epsilon/3, b+\epsilon)\, dx d\tau \\
&\qquad\quad + c_1 \|u\|_{L_T^\infty H_x^1} \int_0^T \int u^2 \chi_{n-1}(x+\nu\tau; \epsilon/3, b+\epsilon)\, dx d\tau.
\end{aligned} \tag{2.71}$$

The first term is finite by induction on n in the current case $l = 1$, whereas the second term is finite by the case $l = 0$. The Sobolev embedding implies

$$\left|\int_0^t B_6\, d\tau\right| \leq \|u\|_{L_t^\infty H_x^2}^2 \int_0^t \int (\partial_x u)^2 \chi_n(x+\nu\tau)\, dx d\tau. \tag{2.72}$$

Finally the inequality (2.11) and the Sobolev embedding yield

$$\left|\int_0^t B_7 + B_8 + B_9\, d\tau\right| \leq c_2 \|u\|_{L_T^\infty H_x^2}^2 \int_0^T \int u^2 \chi_{n-1}(x+\nu\tau; \epsilon/3, b+\epsilon)\, dx d\tau, \tag{2.73}$$

which is finite by case $l = 0$. Integrating (2.61) in the time interval $[0, t]$ and combining the above, we have

$$\begin{aligned} y(t) &:= \int (\partial_x u)^2 \chi_n(x+\nu t)\,dx + \int_0^t \int (\partial_x^3 u)^2 \chi_n'(x+\nu\tau)\,dxd\tau \\ &\leq \int (\partial_x u_0)^2(x)\chi_n(x)\,dx + \left| \int_0^t A + B\,d\tau \right| \\ &\leq c_0 + c_1 \int_0^t \int (\partial_x u)^2 \chi_n(x+\nu\tau)\,dxd\tau \\ &\leq c_0 + c_1 \int_0^t y(\tau)\,d\tau. \end{aligned}$$

The result follows by Gronwall's inequality.

<u>Cases $l = 2, 3, 4, 5$</u> Due to the structure of the IVP, the cases $l = 2, 3, 4, 5$ must be handled individually. The analysis is omitted, however, as it is similar to the cases presented.

<u>Case $l \geq 6$</u> Integrating in the time interval $[0, t]$ and using properties (2.10) and (2.11), we have

$$\left| \int_0^t A\,d\tau \right| \leq c_0 \int_0^t \int (\partial_x^l u)^2 \chi_{n-1}(x+\nu\tau;\epsilon/3, b+\epsilon)\,dxd\tau, \tag{2.74}$$

which is finite by induction on n. Recall (2.40) and (2.41), wherein we wrote

$$B = B_{11} + B_{12} + B_2,$$

with the term B_{11} exhibiting a loss of derivatives. Integrating in the time interval $[0, t]$, we see

$$\left|\int_0^t B_{12}\, d\tau\right| \leq \|u\|_{L_T^\infty H_x^1}\int_0^T\int(\partial_x^{l+1}u)^2\chi_n'(x+\nu\tau)\, dxd\tau$$
$$+\int_0^t\|\partial_x^3 u(\tau)\|_{L_x^\infty}\int(\partial_x^l u)^2\chi_n(x+\nu\tau)\, dxd\tau$$
$$+ c_0\|u\|_{L_T^\infty H_x^s}\int_0^T\int(\partial_x^l u)^2\chi_{n-1}(x+\nu\tau)\, dxd\tau \tag{2.75}$$

where we have used (2.11). The first term is finite by the case $l-1$ and the third is finite by induction on n, hence

$$\left|\int_0^t B_{12}\, d\tau\right| \leq c_0 + c_1\int_0^t\|\partial_x^3 u(\tau)\|_{L_x^\infty}\int(\partial_x^l u)^2\chi_n(x+\nu\tau)\, dxd\tau$$

Observe that term B_2 only occurs when $l \geq 5$. For $l > 5$, note that $4+k < l$. The inequality (2.16) yields

$$|B_2| \leq \sum_{k=1}^{\lceil l/2\rceil-2} c_{l,k}\int|\partial_x^{3+k}u\partial_x^{l-k}u\partial_x^l u|\chi_n(x+\nu t)\, dx$$
$$\leq \int(\partial_x^l u)^2\chi_n(x+\nu t)\, dx$$
$$+\sum_{k=1}^{\lceil l/2\rceil-2}\left\{\int(\partial_x^{4+k}u)^2\chi_n(x+\nu t)\, dx + \int(\partial_x^{3+k}u)^2\chi_n(x+\nu t)\, dx\right.$$
$$\left. + \int(\partial_x^{3+k}u)^2\chi_n'(x+\nu t)\, dx\right\}\int(\partial_x^{l-k}u)^2\chi_n(x+\nu t;\epsilon/5, 4\epsilon/5)\, dx, \tag{2.76}$$

where we have suppressed constants depending on l. Integrating in the time interval $[0,t]$, we see

$$\left|\int_0^t B_2\, d\tau\right| \leq c_0 + c_1\int_0^t\int(\partial_x^l u)^2\chi(x+\nu\tau)\, dxd\tau, \tag{2.77}$$

as factors in the summation are estimated via (2.11) and the inductive hypothesis.

Assuming that u satisfies the IVP (1.37), we rewrite this term by considering the correction factor

$$\frac{d}{dt}\int u(\partial_x^{l-1}u)^2\chi_n(x+\nu t)\,dx = \widetilde{C_1} + C_2 + C_3 + C_4,$$

where

$$\begin{aligned}
\widetilde{C_1} &= \int \partial_x^5 u(\partial_x^{l-1}u)^2\chi_n(x+\nu t)\,dx + 2\int u\partial_x^{l-1}u\partial_x^{l+4}u\chi_n(x+\nu t)\,dx,\\
C_2 &= \int u\partial_x^3 u(\partial_x^{l-1}u)^2\chi_n(x+\nu t)\,dx,\\
C_3 &= 2\int u\partial_x^{l-1}u\partial_x^{l-1}(u\partial_x^3 u)\chi_n(x+\nu t)\,dx,\\
C_4 &= \nu\int u(\partial_x^{l-1}u)^2\chi_n'(x+\nu t)\,dx.
\end{aligned}$$

Integrating $\widetilde{C_1}$ by parts, we have

$$\widetilde{C_1} = \left(\frac{5}{2l-3}\right)B_{11} + C_1, \tag{2.78}$$

where

$$\begin{aligned}
C_1 = &-5\int u(\partial_x^{l+1}u)^2\chi_n'\,dx + 5\int \partial_x^3 u(\partial_x^l u)^2\chi_n\,dx\\
&+ 15\int \partial_x^2 u(\partial_x^l u)^2\chi_n'\,dx + 15\int \partial_x u(\partial_x^l u)^2\chi_n''\,dx\\
&+ 5\int u(\partial_x^l u)^2\chi_n'''\,dx - 5\int \partial_x^4 u(\partial_x^{l-1}u)^2\chi_n'\,dx\\
&- 10\int \partial_x^3 u(\partial_x^{l-1}u)^2\chi_n''\,dx - 10\int \partial_x^2 u(\partial_x^{l-1}u)^2\chi_n'''\,dx\\
&- 5\int \partial_x u(\partial_x^{l-1}u)^2\chi_n^{(4)}\,dx - \int u(\partial_x^{l-1}u)^2\chi_n^{(5)}\,dx.
\end{aligned} \tag{2.79}$$

Here $\chi_n^{(j)}$ denotes $\chi_n^{(j)}(x+\nu t;\epsilon,b)$. The fundamental theorem of calculus yields

$$\begin{aligned}\left(\frac{5}{2l-3}\right)\left|\int_0^t B_{11}\,d\tau\right| \\ \leq \quad & \left|\int u_0(\partial_x^{l-1}u_0)^2\chi_n(x)\,dx\right| + \left|\int u(\partial_x^{l-1}u)^2\chi_n(x+\nu t)\,dx\right| \\ & + \left|\int_0^t C_1+C_2+C_3+C_4\,d\tau\right|.\end{aligned}$$

We now concern ourselves with estimating the right-hand side of this expression. First note

$$\begin{aligned}&\left|\int u_0(\partial_x^{l-1}u_0)^2\chi_n(x)\,dx\right| + \left|\int u(\partial_x^{l-1}u)^2\chi_n(x+\nu t)\,dx\right| \\ &\qquad \leq \|u_0\|_{H^1}\|x^{n/2}\partial_x^{l-1}u_0\|^2_{L^2_x(\epsilon,\infty)} + \|u\|_{L^\infty_T H^1_x}\int(\partial_x^{l-1}u)^2\chi_n(x+\nu t)\,dx,\end{aligned} \tag{2.80}$$

is bounded by the hypothesis (1.45) and the case $l-1$. Similarly to B_2 and B_{12}, integrating in the time interval $[0,t]$, using (2.16) and property (2.11), we obtain

$$\left|\int_0^t C_1\,d\tau\right| \leq c_0 + \int_0^t (c_1 + c_2\|\partial_x^3 u(\tau)\|_{L^\infty_x})\int(\partial_x^l u)^2\chi_n(x+\nu\tau)\,dxd\tau \tag{2.81}$$

where the term containing $(\partial_x^{l+1}u)^2\chi_n'$ is controlled using the induction case $l-1$, as in (2.75).

Using (2.16) and the inductive hypothesis, we see

$$\left|\int_0^t C_2\,d\tau\right| \leq c_0, \tag{2.82}$$

similar to B_2. The same technique applies to C_3 and C_4.

Integrating (2.61) in the time interval $[0,t]$ and combining the above, we find that there exists constants as in (2.62) such that

$$\begin{aligned} y(t) &:= \int (\partial_x^l u)^2 \chi_n(x+\nu t)\, dx + \int_0^t \int (\partial_x^{l+2} u)^2 \chi_n'(x+\nu\tau)\, dx d\tau \\ &\le \int (\partial_x^l u_0)^2(x) \chi_n(x)\, dx + \left| \int_0^t A + B\, d\tau \right| \\ &\le c_0 + \int_0^t (c_1 + c_2 \|\partial_x^3 u(\tau)\|_{L_x^\infty}) \int (\partial_x^l u)^2 \chi_n(x+\nu\tau)\, dx d\tau \\ &\le c_0 + \int_0^t (c_1 + c_2 \|\partial_x^3 u(\tau)\|_{L_x^\infty}) y(\tau)\, d\tau. \end{aligned}$$

The result follows by Gronwall's inequality. To handle the case of arbitrary data $u_0 \in H^s(\mathbb{R})$ with $s > 5/2$, a limiting argument similar to the proof of Theorem 1 is used. This completes the proof of Theorem 2.

2.4 Proof of Theorem 3

Integration by parts yields the next lemma.

Lemma 4. *Suppose for some* $l \in \mathbb{Z}^+$

$$\sup_{0\le t\le T} \int (\partial_x^l u)^2 \chi_n(x+\nu t)\, dx + \int_0^T \int (\partial_x^{l+2} u)^2 \chi_n'(x+\nu\tau)\, dx d\tau < \infty. \tag{2.83}$$

Then for every $0 < \delta < T$*, there exists* $\hat{t} \in (0,\delta)$ *such that*

$$\int (\partial_x^{l+j} u)^2 \chi_{n-1}(x+\nu\hat{t}; \epsilon^+, b)\, dx < \infty \qquad (j = 0,1,2). \tag{2.84}$$

To prove Theorem3, it suffices to consider an example; fix $n = 9$ in the hypothesis of the theorem. Then we may apply Theorem 2 with $(l,n) = (0,9)$. Thus, after applying

Lemma 4, there exists $t_0 \in (0, \delta/2)$ such that

$$\int (u^2 + (\partial_x u)^2 + (\partial_x^2 u)^2)\chi_8(x + \nu t_0; \epsilon^+, b)\, dx < \infty.$$

Hence we may apply Theorem 2 with $(l, n) = (2, 8)$ and find $t_1 \in (t_0, \delta/2)$ such that

$$\int (u^2 + \cdots + (\partial_x^4 u)^2)\chi_7(x + \nu t_1; \epsilon^+, b)\, dx < \infty.$$

Continuing in this manner, applying Theorem 2 with $(l, n) = (4, 7), (6, 6), \ldots, (18, 0)$ provides the existence of $\hat{t} \in (\delta/2, \delta)$ such that

$$\int (u^2 + \cdots + (\partial_x^{19} u)^2)\chi(x + \nu\hat{t}; \epsilon^+, b)\, dx < \infty.$$

Finally, we can apply Theorem 1 with $l = 19$, completing the proof.

2.5 Proof of Theorem 4

Proof. Though not strictly necessary, we break the proof into cases based on the form of the nonlinearity $Q(u)$. We treat the case $x_0 = 0$ as the argument is translation invariant. Following the proof of Theorem 1, let u be a smooth solution of the IVP (1.54). Differentiating the equation l-times, applying (2.13) and using properties of χ, we arrive at

$$\begin{aligned} \frac{d}{dt}\int (\partial_x^l u)^2\chi(x + \nu t)\, dx &+ \int (\partial_x^{l+2} u)^2\chi'(x + \nu t)\, dx \\ &\lesssim \int (\partial_x^l u)^2\chi'(x + \nu t; \epsilon/3, b + \epsilon)\, dx + \int \partial_x^l u \partial_x^l Q(u)\chi(x + \nu t)\, dx \\ &=: A + B \end{aligned} \tag{2.85}$$

The proof proceeds by induction on $l \in \mathbb{Z}^+$. For a given nonlinearity $Q(u)$, there exists $l_0 \in \mathbb{Z}^+$ such that the cases $l = 0, 1, \ldots, l_0$ can be proved by choosing s large enough. Thus it suffices to prove only the inductive step. We describe the formal calculations, omitting the limiting argument.

Integrating in the time interval $[0, t]$ and applying the $l - 2$ result we have

$$\left| \int_0^t A \, d\tau \right| \leq c(\nu; \epsilon; b) \int_0^T \int (\partial_x^l u)^2 \chi'(x + \nu\tau) \, dx d\tau \leq c_0 \tag{2.86}$$

where $0 \leq t \leq T$ and

$$c_0 = c_0(l; \nu; \epsilon; T; \|u_0\|_{X_{s,m}}; \|\partial_x^l u_0\|_{L^2(x_0,\infty)}). \tag{2.87}$$

We now turn to term B.

Case 1 Suppose Q is independent of both $\partial_x^2 u$ and $\partial_x^3 u$. Then there exists $N \in \mathbb{Z}^+$ such that, after integrating by parts, B is a linear combination of terms of the form

$$\int u^{j_0} (\partial_x u)^{j_1} (\partial_x^2 u)^{j_2} (\partial_x^l u)^2 \chi(x + \nu t) \, dx, \qquad j_0, j_1, j_2 \leq N,$$

and

$$\int u^{j_0} (\partial_x u)^{j_1} (\partial_x^2 u)^{j_2} (\partial_x^k u)^2 \chi^{(j_3)}(x + \nu t) \, dx, \qquad j_0, j_1, j_2 \leq N$$

where $1 \leq j_3 \leq 5$ and $3 \leq k \leq l + 1$. Hence no loss of derivatives occurs. Integrating in the time interval $[0, t]$, applying the induction hypothesis and the Sobolev embedding

$$\left| \int_0^t B \, d\tau \right| \leq c_0 + c_1 \int_0^t \int (\partial_x^l u)^2 \chi(x + \nu\tau) \, dx d\tau$$

provided $s_0 > 7/2$, with c_0 and c_1 as in (2.87). Combining with (2.86), after integrating

(2.85) in time and using the hypothesis on the initial data we have

$$\begin{aligned} y(t) &:= \int (\partial_x^l u)^2 \chi(x+\nu t)\, dx + \int_0^t \int (\partial_x^{l+2} u)^2 \chi'(x+\nu\tau)\, dx d\tau \\ &\leq c_0 + c_1 \int_0^t \int (\partial_x^l u)^2 \chi(x+\nu\tau)\, dx d\tau \\ &\leq c_0 + c_1 \int_0^t y(\tau)\, d\tau. \end{aligned} \tag{2.88}$$

The result follows by an application of Gronwall's inequality. The value of m is determined by the LWP theory.

Case 2 Suppose Q is a linear combination of quadratic terms (with the exception of $u\partial_x^2 u$). After integrating by parts B is a linear combination of terms of the form

$$\int \partial_x^j u (\partial_x^{l+1} u)^2 \chi(x+\nu t)\, dx, \qquad 1 \leq j \leq 4$$

as well as lower order terms. The correction technique of Theorem 1 can be modified to account for this loss of derivatives. For example, if $Q(u) = \partial_x^2 u \partial_x^3 u$, then integrating by parts and supressing coefficients

$$B = \int \partial_x^2 u (\partial_x^{l+1} u)^2 \chi(x+\nu t)\, dx + \int \partial_x^4 u (\partial_x^l u)^2 \chi(x+\nu t)\, dx + \tilde{B}$$

where $\tilde{B}$ is controlled by induction. For the second term, we impose $s_0 > 9/2$ to control $\|\partial_x^4 u\|_{L_x^\infty}$. For the first term, consider the correction

$$\frac{d}{dt} \int \partial_x u (\partial_x^{l-1} u)^2 \chi(x+\nu t)\, dx.$$

In general, more than one correction may be necessary. The remainder of the proof is similar to Theorem 1, thus the value of m is determined by the LWP theory. Note that if Q additionally contained higher degree terms independent of $\partial_x^2 u$ and $\partial_x^3 u$, the above argument applies. Equations in the class (1.37) are of this form.

Case 3 The remaining nonlinearities in the class (1.54) exhibit a loss of derivatives which, in general, cannot be controlled by the correction technique. We illustrate the argument in this case by focusing on the example equation

$$\partial_t u - \partial_x^5 u = u\partial_x^2 u. \tag{2.89}$$

The IVP associated to this equation is locally well-posed in $H^s(\mathbb{R}), s \geq 2$, using the contraction mapping principle. However, our modification to the proof of Theorem 1 will require the use of weighted Sobolev spaces.

After integrating by parts and supressing coefficients

$$B = \int u(\partial_x^{l+1} u)^2 \chi(x + \nu t)\, dx + \int \partial_x^2 u(\partial_x^l u)^2 \chi(x + \nu t)\, dx + \tilde{B} \tag{2.90}$$

where $\tilde{B}$ is controlled by induction. Combining with (2.86), after integrating (2.85) in time and using the hypothesis on the initial data we have

$$\begin{aligned} y(t) &:= \int (\partial_x^l u)^2 \chi(x + \nu t)\, dx + \int_0^t \int (\partial_x^{l+2} u)^2 \chi'(x + \nu\tau)\, dx d\tau \\ &\leq c_0 + \int_0^t \int \partial_x^2 u(\partial_x^l u)^2 \chi(x + \nu\tau)\, dx d\tau + \left| \int_0^t \int u(\partial_x^{l+1} u)^2 \chi(x + \nu\tau)\, dx d\tau \right| \\ &\leq c_0 + c_1 \int_0^t y(\tau)\, d\tau + \left| \int_0^t \int u(\partial_x^{l+1} u)^2 \chi(x + \nu\tau)\, dx d\tau \right|. \end{aligned} \tag{2.91}$$

Focusing on the last term in the above line,

$$\left|\int_0^t \int u(\partial_x^{l+1}u)^2\chi(x+\nu\tau)\,dxd\tau\right|$$
$$\leq \left(\sum_{j\in\mathbb{Z}} \sup_{\substack{0\leq t\leq T\\ j\leq x\leq j+1}} |u(x,t)|\right)\left(\sup_{j\in\mathbb{Z}}\int_0^T\int_j^{j+1}(\partial_x^{l+1}u)^2\chi(x+\nu\tau)\,dxd\tau\right). \quad (2.92)$$

We check three cases to show the inductive case $l-1$ bounds the second factor. First, the integral vanishes for $j+1<\epsilon-\nu T$. For $\epsilon<j$ we apply the inductive hypothesis with $\nu=0$. Otherwise we utilize a pointwise bound on χ

$$\int_0^T\int_j^{j+1}(\partial_x^{l+1}u)^2\chi(x+\nu\tau)\,dxd\tau \lesssim \int_0^T\int(\partial_x^{l+1}u)^2\chi'(x+\nu\tau;\epsilon/5,\nu T+\epsilon)\,dxd\tau.$$

The technique for bounding the first factor is described in the next theorem. In general, there exists a nonnegative integer n depending on the form of the polynomial Q such that the following quantities must be estimated:

$$\sum_{j\in\mathbb{Z}} \sup_{\substack{0\leq t\leq T\\ j\leq x\leq j+1}} |\partial_x^k u(x,t)|, \quad k=0,1,\ldots,n,$$

assuming u is a Schwarz solution of IVP (1.54). With such an estimate in hand, the result follows by an application of Gronwall's inequality. □

Theorem 6. *Let $k\in\mathbb{Z}^+\cup\{0\}$ and u be a Schwartz solution of the IVP* (1.54) *corresponding to initial data $u_0\in\mathscr{S}(\mathbb{R})$. Then there exists a nonnegative integer m_0 (depending on Q and k) and positive real number $s_0\geq 2m_0$ such that*

$$\sum_{j\in\mathbb{Z}} \sup_{\substack{0\leq t\leq T\\ j\leq x\leq j+1}} |\partial_x^k u(x,t)| \leq c(T;\|u_0\|_{X_{s_0,m_0}}).$$

The idea is to apply a Sobolev type inequality in the t-variable and show that the resulting summation converges by imposing enough spatial decay on the solution. Acheiving this goal requires the following lemma.

Lemma 5. *If $f \in C^2(\mathbb{R}^2)$, then*

$$\sup_{\substack{0\le t\le T\\ 0\le x\le L}} |f(x,t)| \le \int_0^T \int_0^L |\partial_{xt} f(y,s)|\, dyds + \frac{1}{TL}\int_0^T \int_0^L |f(y,s)|\, dyds$$
$$\frac{1}{L}\int_0^T \int_0^L |\partial_t f(y,s)|\, dyds + \frac{1}{T}\int_0^T \int_0^L |\partial_x f(y,s)|\, dyds$$

for any $L, T > 0$.

We now turn to the proof of Theorem 6.

Proof. For concreteness, we show details for $k = 0$. Applying Lemma 5,

$$\sum_{j\in\mathbb{Z}} \sup_{\substack{0\le t\le T\\ j\le x\le j+1}} |u(x,t)| \lesssim_T \|\partial_{xt} u\|_{L^1_T L^1_x} + \|\partial_x u\|_{L^1_T L^1_x} + \|\partial_t u\|_{L^1_T L^1_x} + \|u\|_{L^1_T L^1_x}.$$

Focusing on the worst term $\|\partial_{xt} u\|_{L^1_T L^1_x}$ and applying

$$\|f\|_1 \le \|f\|_2 + \|xf\|_2$$

we arrive at

$$\|\partial_{xt} u\|_{L^1_T L^1_x} \lesssim_T \|\partial_{xt} u\|_{L^\infty_T L^2_x} + \|x\partial_{xt} u\|_{L^\infty_T L^2_x}.$$

Looking at the second term and using the differential equation we have

$$\|x\partial_{xt} u\|_2 \le \|x\partial_x^6 u(t)\|_2 + \|x\partial_x(u\partial_x^2 u)\|_2 =: A + B.$$

Then

$$\begin{aligned}
A^2 &= \int x^2(\partial_x^6 u)^2 dx \\
&= \int u\partial_x^6(x^2\partial_x^6 u)dx \\
&= \int x^2 u\partial_x^{12} u dx + 12\int xu\partial_x^{11} u dx + 30\int u\partial_x^{10} u dx \\
&\lesssim \|x^2 u\|_2\|\partial_x^{12} u\|_2 + \|xu\|_2\|\partial_x^{11} u\|_2 + \|u\|_2\|\partial_x^{10} u\|_2.
\end{aligned}$$

and so we impose $s_0 \geq 12, m_0 \geq 4$ (compared to the $H^2(\mathbb{R})$ local well-posedness). The estimates for the remaining terms are similar, completing the case $k = 0$. □

Chapter 3

Propagation for Quasilinear Equations

In this chapter we treat the quasilinear IVP

$$\begin{cases} \partial_t u + a(u, \partial_x u, \partial_x^2 u)\partial_x^3 u + b(u, \partial_x u, \partial_x^2 u) = 0, , \quad x \in \mathbb{R}, t \geq 0, \\ u(x, 0) = u_0(x) \end{cases} \tag{3.1}$$

where the functions $a, b : \mathbb{R}^3 \to \mathbb{R}$ satisfy:

(H1) for any compact subset of $\mathbb{R}^3$ there exists $\kappa > 1$ so that $1/\kappa \leq a(\cdot) \leq \kappa$;

(H2) a, b are C^∞ and thus all derivatives are bounded on compact subsets of $\mathbb{R}^3$;

(H3) $\partial_z b \leq 0$, where $b = b(x, y, z)$.

For $a = a(y_0, y_1, y_2)$ we write $\partial_j a := \partial_{y_j} a$ for $j = 0, 1, 2$, and similarly for $b = b(y_0, y_1, y_2)$.

3.1 Energy Identity and Cutoff Function

We now state the energy identity for equation (3.1), which is similar to that in [8].

Lemma 6. *Let $u \in C([0,T] : H^{\infty}(\mathbb{R}))$ be a solution to IVP (3.1) and let $\psi \in C^3(\mathbb{R}^2)$. Then for each $l \in \mathbb{Z}^+, l \geq 8$,*

$$\frac{d}{dt}\int (\partial_x^l u)^2 \psi \, dx + \int (\partial_x^{l+1} u)^2 \{3\partial_x(a\,\psi) - 2(l+1)\partial_x a\,\psi - \partial_2 b\,\psi\} \, dx$$
$$= \mathscr{O}(u, \partial_x u, \dots, \partial_x^l u). \tag{3.2}$$

Details of this identity will be provided in the next section. First, we construct a cutoff function which satisfies

$$3\partial_x(a\ \psi) - 2(l+1)\partial_x a\ \psi \geq 0. \tag{3.3}$$

Along with (H3), this will facilitate an inductive proof of Theorem 5.

Begin by defining the polynomial

$$\rho(x) = 140 \int_0^x y^3(1-y)^3 \, dy \tag{3.4}$$

which satisfies

$$\rho(0) = 0, \qquad \rho(1) = 1,$$
$$\rho'(0) = \rho''(0) = \rho'''(0) = 0,$$
$$\rho'(1) = \rho''(1) = \rho'''(1) = 0,$$

with $0 < \rho, \rho'$ for $0 < x < 1$. For $\epsilon, b > 0$ define $\chi \in C^3(\mathbb{R})$ via

$$\chi(x;\epsilon,b) = \begin{cases} 0 & x \le \epsilon, \\ \rho((x-\epsilon)/b) & \epsilon < x < b+\epsilon, \\ 1 & b+\epsilon \le x. \end{cases}$$

All derivatives of χ are supported in $[\epsilon, b+\epsilon]$, thus continuity yields

$$\sup_{x\in[\epsilon,b+\epsilon]} |\chi^{(j)}(x;\epsilon,b)| \le c(j;b) \tag{3.5}$$

for $j = 1, 2, 3$. A direct computation provides

$$\sup_{x\in[\epsilon,b+\epsilon]} |\chi^{(j)}(x;\epsilon,b)| \le c(j;\epsilon;b)\chi'(x+\nu t;\epsilon/3, b+\epsilon) \tag{3.6}$$

for $j = 1, 2, 3$. When the parameters ϵ and b are suppressed, $\chi(x)$ denotes $\chi(x;\epsilon,b)$.

Motivated by [8], we construct a "rough" version of χ by defining for $\nu \ge 0$

$$\psi(x,t) = \frac{1}{3}[a(u,\partial_x u,\partial_x^2 u)]^{-1+\frac{2}{3}(l+1)} \int_{-\infty}^{x} [a(u,\partial_x u,\partial_x^2 u)]^{-\frac{2}{3}(l+1)}\chi'(y+\nu t)\, dy. \tag{3.7}$$

As with χ, we generally suppress the dependence of ψ on l, ϵ, b and ν. Note that ψ depends on x, t through the solution u to IVP (1.60); we will often abbreviate $a = a(x,t)$.

We next list some properties of ψ. Assumption (H1) implies

$$c \cdot \chi(x+\nu t) \le \psi(x,t) \le \tilde{c} \cdot \chi(x+\nu t) \tag{3.8}$$

for constants $0 < c, \tilde{c}$ depending only on κ and l. As χ' is supported in $[\epsilon, b+\epsilon]$, we see that property (3.8) leads to

$$\mathcal{S} := \text{supp}\ \psi(x,t) = \{(x,t) : x + \nu t \leq \epsilon\}. \tag{3.9}$$

Denote by $\mathbb{1}_{\mathcal{S}}$ the characteristic function of this set. Computation shows

$$\partial_x \psi(x,t) = \left(\frac{2(l+1)-3}{3}\right) \partial_x a(x,t)[a(x,t)]^{-1}\psi(x,t) + \frac{1}{3}[a(x,t)]^{-1}\chi'(x+\nu t) \tag{3.10}$$

and, crucially,

$$3\partial_x(a\psi) - 2(l+1)\partial_x a\psi = \chi'(x+\nu t). \tag{3.11}$$

Using (H1), (3.8), (3.10) and the Sobolev embedding

$$|\partial_x \psi(x,t)| \leq c(l;\kappa;\|u\|_{L^\infty_T H^4_x}) \left\{\chi(x+\nu t) + \chi'(x+\nu t)\right\}. \tag{3.12}$$

Similarly,

$$|\partial_x^2 \psi(x,t)| \leq c(l;\kappa;\|u\|_{L^\infty_T H^5_x}) \left\{\chi + \chi' + \chi''\right\}\big|_{x+\nu t} \tag{3.13}$$

and

$$|\partial_x^3 \psi(x,t)| \leq c(l;\kappa;\|u\|_{L^\infty_T H^6_x}) \left\{\chi + \chi' + \chi'' + \chi'''\right\}\big|_{x+\nu t}. \tag{3.14}$$

Combining these inequalities with (3.6) shows for $j = 1, 2, 3$,

$$|\partial_x^{(j)} \psi(x,t)| \leq c(l;\kappa;\epsilon;b;\|u\|_{L^\infty_T H^6_x}) \left\{\chi(x+\nu t) + \chi'(x+\nu t;\epsilon/3, b+\epsilon)\right\} \tag{3.15}$$

which is the core inequality of the inductive technique.

By using the equation (3.1), (H1) and (H2),

$$\begin{aligned}\partial_t a(u,\partial_x u,\partial_x^2 u) &= \partial_0 a\,\partial_t u + \partial_1 a\,\partial_t\partial_x u + \partial_2 a\,\partial_t\partial_x^2 u\\ &= -\partial_0 a\,[a(u,\partial_x u,\partial_x^2 u)\partial_x^3 u + b(u,\partial_x u,\partial_x^2 u)]\\ &\quad - \partial_1 a\,\partial_x[a(u,\partial_x u,\partial_x^2 u)\partial_x^3 u + b(u,\partial_x u,\partial_x^2 u)]\\ &\quad - \partial_2 a\,\partial_x^2[a(u,\partial_x u,\partial_x^2 u)\partial_x^3 u + b(u,\partial_x u,\partial_x^2 u)] \end{aligned} \tag{3.16}$$

and so applying the Sobolev embedding produces

$$|\partial_t a(u)| \le c(\|u\|_{L_T^\infty H_x^6}). \tag{3.17}$$

Analogous to (3.15),

$$|\partial_t \psi(x,t)| \le c(l;\kappa;\nu;\epsilon;b;\|u\|_{L_T^\infty H_x^6})\left\{\chi(x+\nu t) + \chi'(x+\nu t;\epsilon/3,b+\epsilon)\right\}. \tag{3.18}$$

3.2 Proof of Theorem 5

The proof will be carried out for the IVP

$$\begin{cases}\partial_t u + \big(1 + (\partial_x^2 u)^2\big)\partial_x^3 u = 0, \;\; x\in\mathbb{R},\; t>0,\\ u(x,0) = u_0(x) \in H^m(\mathbb{R}), \quad m \ge 7.\end{cases} \tag{3.19}$$

Proof. Without loss of generality we shall assume that $m = 7$ and that $x_0 = 0$. Let $u(\cdot)$ be a smooth solution of the IVP (3.19) provided by Theorem D corresponding to smooth data u_0 satisfying for some $l \in \mathbb{Z}^+$, $l \ge 8$,

$$\|\partial_x^l u_0\|_{L^2(0,\infty)}^2 = \int_0^\infty (\partial_x^l u_0)^2(x)\,dx < \infty. \tag{3.20}$$

We have that the solution $u(\cdot)$ satisfies

$$u \in C([0,T] : H^{7-\delta}(\mathbb{R})) \cap L^{\infty}([0,T] : H^{7}(\mathbb{R})), \quad \text{for all} \;\; \delta > 0 \tag{3.21}$$

and

$$\int_{0}^{T}\int_{-R}^{R} (\partial_x^8 u)^2(x,t)\; dxdt \le c(T, R, \|u_0\|_{7,2}). \tag{3.22}$$

We carry out the details for $a(u, \partial_x u, \partial_x^2 u) = 1 + (\partial_x^2 u)^2$ and $b(u, \partial_x u, \partial_x^2 u) \equiv 0$. Applying $2\psi\partial_x^l u \partial_x^l$ to the equation and integrating in the x-variable yields

$$\begin{aligned} &\frac{d}{dt}\int (\partial_x^l u)^2 \psi \; dx - \int (\partial_x^l u)^2 \partial_t \psi \; dx \\ &\qquad + \sum_{k=0}^{l} 2\binom{l}{k} \int \partial_x^l u \partial_x^k a \partial_x^{l-k+3} u\psi \; dx = 0. \end{aligned} \tag{3.23}$$

Denote by A_k each term in the summation above. After integrating by parts,

$$\begin{aligned} A_1 &= 3\int (\partial_x^{l+1} u)^2 \partial_x (a\,\psi)\; dx - \int (\partial_x^l u)^2 \partial_x^3 (a\,\psi)\; dx \\ A_2 &= -2l \int (\partial_x^{l+1} u)^2 \partial_x a\,\psi \; dx + l\int (\partial_x^l u)^2 \partial_x^2 (\partial_x a\,\psi)\; dx \\ A_3 &= -\binom{l}{2}\int (\partial_x^l u)^2 \partial_x (\partial_x^2 a\,\psi)\; dx \\ A_4 &= 2\binom{l}{3}\int (\partial_x^l u)^2 \partial_x^3 a\,\psi\; dx \end{aligned} \tag{3.24}$$

and

$$\begin{aligned}A_l &= 2\int \partial_x^3 u \partial_x^l u \partial_x^l (1+(\partial_x^2 u)^2)\psi \, dx \\ &= -2\int (\partial_x^{l+1}u)^2 \partial_x a \, \psi \, dx + 2\int (\partial_x^l u)^2 \partial_x^2(\partial_x^2 u \partial_x^3 u \, \psi)\, dx \\ &\quad + 4\sum_{k=1}^{l-1}\binom{l-1}{k}\int \partial_x^3 u \partial_x^{2+k} u \partial_x^{l+2-k} u \partial_x^l u \, \psi \, dx. \end{aligned} \tag{3.25}$$

Therefore, ignoring combinatorial coefficients and taking into consideration the form of $a(u, \partial_x u, \partial_x^2 u)$, the energy identity (3.2) becomes

$$\begin{aligned}&\frac{d}{dt}\int (\partial_x^l u)^2 \, \psi \, dx + \int (\partial_x^{l+1} u)^2 \, \chi'(x+\nu t)\, dx \\ &\quad = \int (\partial_x^l u)^2 \{\partial_t \psi + P((\partial_x^j u)_{j=2,3,4,5}; (\partial_x^j \psi)_{j=0,1,2,3})\}\, dx \\ &\qquad + \sum_{k=1}^{\lfloor l/2 \rfloor} \int \partial_x^3 u \partial_x^{2+k} u \partial_x^{l+2-k} u \partial_x^l u \, \psi \, dx \\ &\quad =: E_1 + E_2. \end{aligned} \tag{3.26}$$

with $l \in \mathbb{Z}^+$, $l \geq 8$ and ψ chosen as in (3.7). Here $P(\cdot)$ is a polynomial in its variables, linear in the components $\psi, \partial_x \psi, \partial_x^2 \psi, \partial_x^3 \psi$ and quadratic in $\partial_x^2 u, \ldots, \partial_x^5 u$.

Case $l = 8$

Using property (3.18), followed by (3.8),

$$\begin{aligned}\left|\int (\partial_x^8 u)^2 \, \partial_t \psi \, dx\right| &\leq \int (\partial_x^8 u)^2 \, |\partial_t \psi| \, dx \\ &\lesssim \int (\partial_x^8 u)^2 \{\chi(x+\nu t) + \chi'(x+\nu t; \epsilon/3, b+\epsilon)\}\, dx \\ &\lesssim \int (\partial_x^8 u)^2 \, \psi \, dx + \int (\partial_x^8 u)^2 \, \chi'(x+\nu t; \epsilon/3, b+\epsilon)\}\, dx. \end{aligned} \tag{3.27}$$

Using the Sobolev embedding, (3.15) and (3.8),

$$\left|\int (\partial_x^8 u)^2 P((\partial_x^j u)_{j=2,3,4,5}; (\partial_x^j \psi)_{j=0,1,2,3})\} \, dx\right| \tag{3.28}$$

$$\leq c_0 \|u\|^2_{L_T^\infty H_x^7} \int (\partial_x^8 u)^2 \{\chi(x+\nu t) + \chi'(x+\nu t; \epsilon/3, b+\epsilon)\} \, dx$$

$$\lesssim \int (\partial_x^8 u)^2 \, \psi \, dx + \int (\partial_x^8 u)^2 \, \chi'(x+\nu t; \epsilon/3, b+\epsilon)\} \, dx. \tag{3.29}$$

Combining these two estimates yields

$$|E_1| \lesssim \int (\partial_x^8 u)^2 \, \psi \, dx + \int (\partial_x^8 u)^2 \, \chi'(x+\nu t; \epsilon/3, b+\epsilon)\} \, dx. \tag{3.30}$$

For $k = 1, 2$, the terms of E_2 may be estimated with similar arguments after integrating by parts. For $k = 3, 4$, using the Sobolev embedding and Cauchy-Schwarz inequality

$$\left|\int \partial_x^3 u \partial_x^{2+k} u \partial_x^{10-k} u \partial_x^8 u \, \psi \, dx\right| \tag{3.31}$$

$$\leq \|u\|^2_{L_T^\infty H_x^7} \left\{\int (\partial_x^{10-k} u)^2 \, \psi \, dx + \int (\partial_x^8 u)^2 \, \psi \, dx\right\}$$

$$\lesssim 1 + \int (\partial_x^8 u)^2 \, \psi \, dx. \tag{3.32}$$

Therefore

$$|E_2| \lesssim 1 + \int (\partial_x^8 u)^2 \, \psi \, dx + \int (\partial_x^8 u)^2 \, \chi'(x+\nu t; \epsilon/3, b+\epsilon)\} \, dx. \tag{3.33}$$

Integrating the identity (3.26) in the time interval $[0, t]$, for $0 \le t \le T$ produces

$$\begin{aligned}
\int (\partial_x^8 u)^2 \, \psi \, dx + \int_0^t \int (\partial_x^9 u)^2 \, \chi'(x+\nu\tau) \, dx d\tau \\
&\le \int_0^t |E_1| + |E_2| \, d\tau \\
&\le \int (\partial_x^8 u_0)^2 \, \psi(x,0) \, dx \\
&\quad + cT + c \int_0^T \int (\partial_x^8 u)^2 \, \chi'(x+\nu\tau; \epsilon/3, b+\epsilon)\} \, dx d\tau \\
&\quad + \int_0^t \int (\partial_x^8 u)^2 \, \psi(x,\tau) \, dx d\tau. \qquad (3.34)
\end{aligned}$$

The integral involving the data, u_0, is finite by hypothesis; the integral containing χ' is finite by the local smoothing property (3.22). Thus

$$\begin{aligned}
y(t) := \int (\partial_x^8 u)^2 \, \psi \, dx + \int_0^t \int (\partial_x^9 u)^2 \, \chi'(x+\nu\tau) \, dx d\tau \\
&\le c_0 + c_1 \int_0^t \int (\partial_x^8 u)^2 \, \psi(x,\tau) \, dx d\tau \\
&\le c_0 + C_1 \int_0^t y(\tau) \, d\tau. \qquad (3.35)
\end{aligned}$$

Applying Gronwall's inequality

$$\sup_{0 \le t \le T} \int (\partial_x^8 u)^2 \, \psi(x,t) \, dx + \int_0^T \int (\partial_x^9 u)^2 \, \chi'(x+\nu\tau) \, dx d\tau < \infty. \qquad (3.36)$$

Case $l \ge 9$

The estimates for E_1 are identical to the $l = 8$ case and yield

$$|E_1| \lesssim \int (\partial_x^l u)^2 \, \psi \, dx + \int (\partial_x^l u)^2 \, \chi'(x+\nu t; \epsilon/3, b+\epsilon)\} \, dx. \qquad (3.37)$$

After integrating by parts, the $k = 1, 2$ terms of E_2 may be estimated as before,

$$\left|\int \partial_x^3 u \partial_x^{2+k} u \partial_x^{l+2-k} u \partial_x^l u \, \psi \; dx\right| \lesssim 1 + \int (\partial_x^l u)^2 \, \psi \; dx. \tag{3.38}$$

However, an inductive argument is used to control lower order terms present in E_2. Using the Sobolev embedding and (2.16), we find

$$\begin{aligned}
&\left|\int \partial_x^3 u \partial_x^{2+k} u \partial_x^{l+2-k} u \partial_x^l u \, \psi \; dx\right| \\
&\qquad \le \|\partial_x^3 u\|_{L_x^\infty} \int |\partial_x^{2+k} u \partial_x^{l+2-k} u \partial_x^l u| \, \psi \; dx \\
&\qquad \lesssim \left\{ \int (\partial_x^{3+k} u)^2 \psi \; dx + \int (\partial_x^{2+k} u)^2 \psi \; dx + \int (\partial_x^{2+k} u)^2 |\partial_x \psi| \; dx \right\} \\
&\qquad\qquad \times \int (\partial_x^{l+2-k} u)^2 \psi(x + \nu t; \epsilon/5, 4\epsilon/5) \; dx + \int (\partial_x^l u)^2 \psi(x + \nu t) \; dx.
\end{aligned} \tag{3.39}$$

Since $k = 3, 4, \ldots, \lfloor l/2 \rfloor$, notice that $3 + k < l$ and $l + 2 - k < l$. Therefore, the induction hypothesis shows that all terms (except for the last) in the above inequality are finite. Collecting these estimates,

$$|E_2| \le \sum_{k=1}^{\lfloor l/2 \rfloor} \left|\int \partial_x^3 u \partial_x^{2+k} u \partial_x^{l+2-k} u \partial_x^l u \, \psi \; dx\right| \lesssim 1 + \int (\partial_x^l u)^2 \, \psi \; dx. \tag{3.40}$$

Integrating the identity (3.26) in the time interval $[0,t]$, for $0 \le t \le T$ produces

$$\begin{aligned}
\int (\partial_x^l u)^2\, \psi\; dx &+ \int_0^t \int (\partial_x^{l1} u)^2\, \chi'(x+\nu\tau)\; dx d\tau \\
&\le \int_0^t |E_1| + |E_2|\; d\tau \\
&\le \int (\partial_x^l u_0)^2\, \psi(x,0)\; dx \\
&\quad + cT + c\int_0^T \int (\partial_x^l u)^2\, \chi'(x+\nu\tau; \epsilon/3, b+\epsilon)\}\; dx d\tau \\
&\quad + \int_0^t \int (\partial_x^l u)^2\, \psi(x,\tau)\; dx d\tau. \qquad (3.41)
\end{aligned}$$

The integral involving the data, u_0, is finite by hypothesis; the integral containing χ' is finite by the inductive hypothesis. Thus

$$\begin{aligned}
y(t) := \int (\partial_x^l u)^2\, \psi\; dx &+ \int_0^t \int (\partial_x^{l+1} u)^2\, \chi'(x+\nu\tau)\; dx d\tau \\
&\le c_0 + c_1 \int_0^t \int (\partial_x^l u)^2\, \psi(x,\tau)\; dx d\tau \\
&\le c_0 + C_1 \int_0^t y(\tau)\; d\tau. \qquad (3.42)
\end{aligned}$$

Applying Gronwall's inequality

$$\sup_{0\le t\le T} \int (\partial_x^l u)^2\, \psi(x,t)\; dx + \int_0^T \int (\partial_x^{l+1} u)^2\, \chi'(x+\nu\tau)\; dx d\tau < \infty. \qquad (3.43)$$

Finally, to justify the previous formal computation we recall that the argument in the proof of Theorem D shows that u in (3.21)-(3.22) is the limit in the $C([0,T] : H^{7-\delta}(\mathbb{R}))$-norm (for any $\delta > 0$) of smooth solutions (weak form of the continuous dependence upon the data). In particular, we have that u is the uniform limit of smooth solutions in $\mathbb{R} \times [0,T]$. Hence, by performing the above (formal) argument in the smooth solutions one obtains a uniform bounded sequence in the norms described in (3.21) and (3.22).

Hence, considering the uniform boundedness, the weak convergence and passing to the limit we obtain the desired result. □

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