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The Structure of Fundamental Groups of Smooth Metric Measure Spaces

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Abstract

The Structure of Fundamental Groups of Smooth Metric Measure Spaces

Maree Trisha Afaga Jaramillo

In this dissertation, we investigate the structure of fundamental groups of smooth metric measure spaces with Bakry-Émery Ricci tensor bounded from below. In particular, we generalize a result of Gabjin Yun to show that if a smooth metric measure space has almost nonnegative Bakry-Émery Ricci tensor and a lower bound on volume, then its fundamental group is almost abelian. We also generalize a result of Vitali Kapovitch and Burkhard Wilking to show that there is a uniform bound on the number of generators of the fundamental groups of smooth metric measure spaces with Bakry-Émery Ricci tensor bounded from below. In order to utilize the proof techniques of Yun and Kapovitch-Wilking, we extend many valuable tools for studying Riemannian manifolds with Ricci curvature bounded from below to the smooth metric measure space setting. In particular, we extend Jeff Cheeger and Tobias Colding's Splitting Theorem, which plays a key role in the proofs of our results on fundamental groups.

Contents

Chapter 1

Introduction

Geometry and topology are two areas in mathematics that are intricately linked. It is then unsurprising that there exist many theorems which relate the geometry of a manifold to its topology. For example, Myers' Theorem (see Theorem 2.1) states that if the Ricci curvature of a complete Riemannian manifold is bounded below by a positive constant, then M is compact. As another example, John Milnor proved that any finitely generated subgroup of the fundamental group of a complete n-dimensional Riemannian manifold with nonnegative Ricci curvature has polynomial growth of degree $\leq n$ (see Theorem 2.5). These classic theorems, especially that of Milnor, suggest the broad theme which we wish to explore in this dissertation. We wish to investigate the topology, in particular, the fundamental group, of a space whose curvature is bounded from below.

The classic theorems mentioned above deal with curvature on a Riemannian manifold. A Riemannian manifold M is one which is endowed with a Riemannian metric g. This metric is used to derive and study different types of curvature on M, such as Ricci curvature. This metric also induces a natural volume form $dvol_g$ which allows us to compute volumes of subsets of M. The Bishop-Gromov volume comparison, below, is a classic theorem which relates Ricci curvature and volume. It gives control over the volume of a ball in a Riemannian manifold in terms of the volume of a ball in the model space M^n_H, that is, the simply connected Riemannian manifold of dimension n with constant sectional curvature H.

Theorem 1.1 (Bishop-Gromov Volume Comparison)**.** *If M^n has $Ric(M) \geq (n-1)H$, then*

$$\frac{Vol(B(p,R))}{Vol^n_H(B(R))}$$

is nonincreasing in R.

Here, $B(R)$ is a ball of radius R in M^n_H and $\mathrm{V}ol^n_H(B(R))$ denotes the volume of that ball. The proof of the Bishop-Gromov volume comparison is contained in many texts, see for example [Pet98, Lemma 9.1.6]. This volume comparison proves to be an indispensable tool in much of the work that motivates this dissertation. In fact, Milnor utilized this tool in proving his theorem regarding the growth of finitely generated subgroups of the fundamental group.

In this dissertation we explore the fundamental groups of Riemannian manifolds with a weighted volume element, which are called smooth metric measure spaces. In particular, we have the following definition.

Definition 1.2. *A smooth metric measure space is a triple $(M^n, g, e^{-f}\,dvol_g)$, where M^n is a complete n-dimensional Riemannian manifold equipped with metric g, volume density $dvol_g$, and $f : M^n \to \mathbb{R}$ is smooth.*

Smooth metric measure spaces occur naturally as collapsed measured Gromov-Hausdorff limits of sequences of warped products. Specifically, $(M^n \times F^m, g_\epsilon, \widetilde{\mathrm{dvol}_{g_\epsilon}})$ $\to (M^n, g, e^{-f}\mathrm{dvol}_{g_M})$ as $\epsilon \to 0$, where $\widetilde{\mathrm{dvol}_{g_\epsilon}}$ is the renormalized Riemannian measure and $g_\epsilon = g_M + (\epsilon e^{\frac{-f}{m}})^2 g_F$ is the warped product metric with g_M and g_F the metrics on M and F, respectively. The Ricci curvature of the warped product metric g_ϵ in the M direction is given by $\mathrm{Ric} + \mathrm{Hess}\, f - \frac{1}{m} df \otimes df$. This leads to the definition of the m-Bakry-Émery Ricci tensor on the smooth metric measure space $(M^n, g, e^{-f}\mathrm{dvol}_g)$.

Definition 1.3. *The m-Bakry-Émery Ricci tensor is given by*

$$Ric_f^m = Ric + Hess\, f - \frac{1}{m} df \otimes df, \qquad 0 < m \leq \infty.$$

When $m = \infty$, we have the Bakry-Émery Ricci tensor, given by

$$Ric_f = Ric + Hess\, f.$$

In this sense, the Bakry-Émery Ricci curvature is a natural analogue to Ricci curvature on $(M^n, g, e^{-f}\mathrm{dvol}_g)$. This tensor has appeared in the work on diffusion processes by Dominique Bakry and Michel Émery [BE85]. It also appears in the study of Ricci flow. When $\mathrm{Ric}_f = \lambda g$ for some $\lambda \in \mathbb{R}$, one recognizes this equation as the gradient Ricci soliton equation. Because of this, the function f is sometimes referred to as the potential function. Note that when f is constant, the usual Ricci curvature tensor is recovered. As we have seen, much can be said about the topology of a Riemannian manifold when it has Ricci curvature bounded from below. Thus it is natural to ask if the same information holds true for smooth metric measure spaces with Bakry-Émery Ricci curvature bounded from below. This has been a field of active study. See, for example, [Lic70], [Qia97], [WW09], [FLZ09].

In this dissertation we will show that we can extend theorems regarding fundamental groups of Riemannian manifolds with Ricci curvature bounded from below to smooth metric measure spaces with Bakry-Émery Ricci curvature bounded from below. In particular, we show that the fundamental group of manifolds in the class of compact smooth metric measure spaces which have volume bounded from below, diameter bounded from above, and almost nonnegative Bakry-Émery Ricci curvature contains an abelian subgroup of finite index.

Theorem 1.4. *For any constants $D, k, v > 0$, there exists $\epsilon = \epsilon(D, k, n, v) > 0$ such that if a smooth metric measure space $(M^n, g, e^{-f}dvol_g)$ with $|f| \leq k$ admits a metric under which it satisfies the conditions*

$$Ric_f \geq -\epsilon, \tag{1.1}$$

$$diam(M) \le D, \tag{1.2}$$

$$Vol_f(M) \ge v, \tag{1.3}$$

then $\pi_1(M)$ is almost abelian, i.e. $\pi_1(M)$ contains an abelian subgroup of finite index.

This result may be viewed as an extension of a result of Jeff Cheeger and Detlef Gromoll (see Theorem 2.6) to a class of manifolds in smooth metric measure space setting. We also prove the following regarding generation of the fundamental group of compact smooth metric measure spaces.

Theorem 1.5. *Given n, k, and D there is a constant $C = C(n, k, D)$ such that the following holds. Suppose $(M^n, g, e^{-f} dvol_g)$ is a smooth metric measure space with $|f| \le k$, $diam(M) \le D$, and $Ric_f \ge -(n-1)$. Then $\pi_1(M)$ can be generated by at most C elements.*

This theorem is an extension of Vitali Kapovitch and Burkhard Wilking's result (see Theorem 2.10) to the smooth metric measure space setting. The result of Kapovitch-Wilking stems from the larger question of finding a uniform bound on the index of a nilpotent subgroup contained in the fundamental group of a compact manifold of Ricci curvature bounded from below.

Note that when f is constant, Theorems 1.4 and 1.5 recover the original result in the case that Ricci curvature is bounded from below. In order to obtain these theorems in the smooth metric measure space setting, we use tools developed for smooth metric measure spaces as in [WW09] and follow the proof techniques used in [Yun97] and [KW11]. Both proofs utilize, in part, the Splitting Theorem of Jeff Cheeger and Tobias Colding (see Theorem 2.12). Before proving Theorems 1.4 and 1.5, we show that the following holds for smooth metric measure spaces.

Theorem 1.6. *Let $(M_i^n, g_i, e^{-f_i}dvol_{g_i})$ be a sequence satisfying the following: M_i has $Ric_{f_i}(M_i) \geq -(n-1)\delta_i$, where $\delta_i \to 0$, $|f_i| \leq k$, and $M_i^n \to Y$ in the Gromov-Hausdorff sense. If Y contains a line, then Y splits as an isometric product, $Y = \mathbb{R} \times X$ for some length space X.*

The proof of the Cheeger-Colding Splitting Theorem in the case for Ricci curvature bounded from below is quite involved. As will be discussed in Chapter 3, the Cheeger-Colding Splitting Theorem hinges upon several invaluable, well-established tools for Riemannian manifolds with Ricci curvature bounded from below, among which is the Cheng-Yau gradient estimate. Though we were able to obtain a gradient estimate appropriate for our purposes (see Proposition 3.2), there were difficulties to the full generalization of this estimate to the smooth metric measure space setting.

We note here that Feng Wang and Xiaohua Zhu [WZ13] also have an almost splitting theorem for limits of sequences of spaces with Bakry-Émery Ricci tensor bounded from below. As we see in Theorem 1.6, our splitting theorem requires a uniform bound on $|f_i|$, whereas the hypotheses of Wang-Zhu's splitting theorem require uniform bounds on both $|f_i|$ and $|\nabla f_i|$. The additional hypothesis of Wang and Zhu is due in part to the gradient estimate used.

The structure of the remaining parts of this dissertation are as follows. In Chapter 2, we discuss some of the work which motivates this thesis and provide some background to the main theorems that we are interested in proving. In Chapter 3, we extend the Cheeger-Colding Splitting Theorem to the smooth metric measure space setting. In Chapter 4, we give a new absolute volume comparison (Proposition 4.1) and use this to prove Theorem 1.4. In Chapter 5, we conclude with proof of Theorem 1.5.

Chapter 2

Background

In this chapter, we first review results regarding fundamental groups of manifolds with Ricci curvature bounded from below, including those results which we extend in this dissertation. We also state some of the key definitions and results on smooth metric measure spaces which will be used in later discussions. Finally, we review the notion of Gromov-Hausdorff and equivariant Gromov-Hausdorff convergence of sequences of manifolds.

2.1 Fundamental Groups of Manifolds with Ricci Curvature Bounded from Below

The question of how curvature affects the topology of manifolds has long been studied. A classical result which shows how a lower bound on Ricci curvature affects topology of a manifold is Myers' Theorem:

Theorem 2.1. *[Mye41] Let M^n be a complete n-dimensional Riemannian manifold. Suppose $Ric(M) \geq (n-1)H > 0$. Then $diam(M) \leq \frac{\pi}{\sqrt{H}}$. In particular, M is compact and $\pi_1(M)$ is finite.*

The rigidity result given by Shiu Yuen Cheng [Che75] states if M satisfies the conditions of Myers' Theorem and if, in addition, $\mathrm{diam}(M) = \frac{\pi}{\sqrt{H}}$, then M is isometric to the sphere with constant curvature H and radius $\frac{1}{\sqrt{H}}$. These results demonstrate how a lower Ricci curvature bound can have strong topological implications.

Of these topological implications, the work in this dissertation concerns how curvature affects the fundamental group. A lower Ricci curvature bound has been shown to give control over the fundamental group of a manifold. When the Ricci curvature condition on a manifold M is relaxed to $\mathrm{Ric}(M) \geq 0$, then M may no longer be compact. We note here that it is a conjecture of John Milnor [Mil68] that if $\mathrm{Ric}(M) \geq 0$, then $\pi_1(M)$ is finitely generated. Though this conjecture remains open, the curvature condition has been used to give other characteristics of the fundamental group of M. In particular, we have results regarding the growth of finitely generated subgroups of $\pi_1(M)$. Recall that for a finitely generated group $\Gamma = \langle g_1, \dots, g_k \rangle$, any $g \in \Gamma$ can be written as a word

$$g = \Pi_i g_{k_i}^{n_i},$$

where $k_i \in \{1, \dots, k\}$. Define the length of this word with this representation to be $\sum_i |n_i|$. Let $|g|$ be the minimum of the lengths of all word representations of g using the specified set of generators. Note that $|\cdot|$ depends on the choice of generators.

Definition 2.2. *Fix a set of generators for a finitely generated group Γ. Then the growth function of Γ is given by*

$$\Gamma(s) = \#\{g \in \Gamma : |g| \leq s\}.$$

With this definition for the growth function of the group, we can now state what it means for a group to have polynomial growth.

Definition 2.3. *A finitely generated group Γ is said to have polynomial growth if for each set of generators the growth function $\Gamma(s) \leq as^n$ for some $a > 0$.*

Though the definition of a finitely generated group having polynomial growth requires that the condition on the growth function holds for each set of generators, we have the following lemma.

Lemma 2.4. *If there exists a set of generators for Γ such that $\Gamma(s) \leq as^n$ for some $a > 0$, then Γ has polynomial growth of degree $\leq n$.*

Thus, in order to show that a finitely generated group has polynomial growth, one only needs to show that the condition holds for a set of generators. With these definitions we now recall the following theorem of John Milnor.

Theorem 2.5. *[Mil68] Let M^n be a complete n-dimensional Riemannian manifold. Suppose $Ric(M) \geq 0$. Then any finitely generated subgroup of $\pi_1(M)$ is of polynomial growth of degree $\leq n$.*

Note that Mikhael Gromov [Gro81] proved that a finitely generated group has polynomial growth if and only if it is almost nilpotent, that is, it contains a nilpotent subgroup of finite index. Gromov's result combined with Theorem 2.5 then tells us that any finitely generated subgroup of the fundamental group is almost nilpotent. It is natural then to ask if such groups can always be realized as fundamental groups of manifolds with nonnegative Ricci curvature. In this direction, Guofang Wei [Wei88] showed that every finitely generated torsion-free nilpotent group can be realized as the fundamental group of a complete Riemannian manifold with strictly positive Ricci curvature. Burkhard Wilking [Wil00] then showed that the same holds true for any finitely generated almost nilpotent group.

If M^n is compact, then $\pi_1(M)$ is finitely generated. Thus, if M is compact and has nonnegative Ricci curvature, one can say that $\pi_1(M)$ itself has polynomial growth of index $\leq n$ and is therefore almost nilpotent. This result for compact manifolds can be strengthened further. In particular, we have the following theorem of Jeff Cheeger and Detlef Gromoll.

Theorem 2.6. *[CG71] If M is a complete compact manifold with nonnegative Ricci curvature, then $\pi_1(M)$ is almost abelian, that is, it contains an abelian subgroup of finite index.*

One may then ask what happens if we relax the curvature condition further and allow the lower Ricci curvature bound to be negative. Michael Anderson proved the following finiteness result for a class of n-dimensional compact manifolds with Ricci curvature bounded from below.

Theorem 2.7. *[And90, Theorem 2.2] In the class of manifolds M^n with $Ric(M) \geq (n-1)H$, $Vol_M \geq V$ and $diam(M) \leq D$, there are only finitely many isomorphsim types of $\pi_1(M)$.*

Utilizing Anderson's finiteness result, Wei was able to generalize Theorem 2.5 to a class of compact manifolds with almost nonnegative Ricci curvature in the following way.

Theorem 2.8. *[Wei90] For any constant $v > 0$, there exists $\epsilon = \epsilon(n, v) > 0$ such that if a complete manifold M^n admits a metric satisfying the conditions $Ric(M) \geq -\epsilon$, $diam(M) = 1$, and $Vol(M) \geq v$, then $\pi_1(M)$ is of polynomial growth with degree $\leq n$.*

In light of Theorem 2.6, it is natural to ask if Theorem 2.8 can be strengthened to show that the fundamental group is almost abelian. This question was answered in the positive by Gabjin Yun.

Theorem 2.9. *[Yun97] For any constant $v > 0$, there exists $\epsilon = \epsilon(n, v) > 0$ such that if a complete manifold M^n admits a metric satisfying the conditions $Ric(M) \geq -\epsilon$, $diam(M) = 1$, and $Vol(M) \geq v$, then $\pi_1(M)$ is almost abelian.*

In a slightly different but related direction, one may ask about the index of the nilpotent subgroup contained in the fundamental group of a compact manifold with appropriate bounds on Ricci curvature. Vitali Kapovitch and Burkhard Wilking show that there exist $\epsilon(n) > 0$ and $C(n) > 0$ such that for M in the class of compact n-dimensional manifolds with $\mathrm{Ric}(M) > -(n-1)$ and $\mathrm{diam}\,(M) \leq \epsilon(n)$, $\pi_1(M)$ contains a nilpotent subgroup of index of $\leq C(n)$ [KW11, Corollary 2]. In order to prove this result, Kapovitch and Wilking show the existence of a uniform bound on the number of generators of $\pi_1(M)$.

Theorem 2.10. *[KW11, Theorem 3] Given n and D there exists $C = C(n, D)$ such that for any n-dimensional manifold M with $Ric(M) \geq -(n-1)$ and $diam(M) \leq D$, the fundamental group $\pi_1(M)$ can be generated by at most C elements.*

We note here that a uniform bound had been given previously in the case when the conjugate radius is bounded from below in [Wei97].

In this dissertation we study the fundamental group of smooth metric measure spaces with Bakry-Émery Ricci curvature bounded from below. As we will see in Section 2.3, some of these results regarding fundamental groups have already been extended to this setting. It is our aim to extend more of the results mentioned here to this setting.

2.2 Structure Results: The Splitting Theorems

As mentioned earlier, the Bishop-Gromov volume comparison, Theorem 1.1, is a key tool that is used in proving many of the results on fundamental group. It is utilized, for example, in the proof of Theorems 2.5, 2.7, and 2.8. An important idea in the proof of these theorems is to view elements of the fundamental group as deck transformations of the universal cover and to use estimates on lengths and Theorem 1.1 to obtain appropriate bounds.

In addition to Theorem 1.1, structure results for the manifold itself were used to give information on the fundamental group. Recall that Theorem 2.6 tells us that the fundamental group of a compact manifold with nonnegative Ricci curvature is almost abelian. This theorem for the fundamental group follows from the Cheeger-Gromoll Splitting Theorem:

Theorem 2.11. *[CG71, Theorem 2] Let M be a complete manifold of nonnegative Ricci curvature. If M contains a line, then M splits isometrically as $X \times \mathbb{R}$.*

We note here that versions of the Cheeger-Gromoll Splitting Theorem already exist for smooth metric measure spaces. Andre Lichnerowicz [Lic70] showed that if $\mathrm{Ric}_f \geq 0$ on M for bounded f, and M contains a line, then $M = N^{n-1} \times \mathbb{R}$ and f is constant along the line. See also Wei-Wylie [WW09]. In fact, Fuquan Fang, Xiang-Dong Li, and Zhenlei Zhang [FLZ09] have shown that the splitting theorem holds when f is only bounded from above.

In order to extend Theorem 2.6 to the class of compact manifolds as in Theorem 2.9, Yun [Yun97] utilizes another type of splitting theorem, the Cheeger-Colding Splitting Theorem, which gives a structure result on the Gromov-Hausdorff limit of a sequence of manifolds.

Theorem 2.12. *[CC96, Theorem 6.64] Let $M_i^n \to Y$ in the Gromov-Hausdorff sense. Suppose further that $Ric(M_i^n) \geq -(n-1)\delta_i$, where $\delta_i \to 0$ as $i \to \infty$. If Y contains a line, then Y is isometric to $X \times \mathbb{R}$ for some length space X.*

The notion of Gromov-Hausdorff convergence will be discussed in more detail in last section of this chapter.

Theorem 2.12 settles Conjecture 1.7 of [FY92]. The proof of this theorem is quite complicated. The idea of the proof is to construct a harmonic function, $\bar{b}_+$, which is related to a distance function, b. In particular, Cheeger and Colding show that the function $\bar{b}$ is pointwise close to b. Then, they show that $\nabla\bar{b}$ is close to ∇b in an L^2-sense, and finally that $\mathrm{Hess}\,\bar{b}$ is small in an L^2-sense. It is with the Hessian estimate that Cheeger and Colding develop a Quantitative Pythagorean

Theorem, with which they are able to show that a ball in the manifold is Gromov-Hausdorff close to a ball in the product space $X \times \mathbb{R}$. Their arguments hinge upon the use of some well-established tools, as well as the formulation new tools. The key tools and arguments used to prove the Cheeger-Colding Splitting Theorem will be discussed in more detail in Chapter 3.

2.3 Primer on Smooth Metric Measure Spaces

As mentioned previously, one of the aims of this dissertation is to extend theorems regarding the structure of fundamental groups to smooth metric measure spaces with Bakry-Émery Ricci curvature bounded from below. As part of our motivation, we see that William Wylie [Wyl08, Theorem 1.1] has proven that smooth metric measure spaces with Bakry-Émery Ricci curvature bounded from below by $\lambda > 0$ has finite fundamental group. This extends part of Theorem 2.1 directly to its analogous smooth metric measure spaces setting.

For comparison, let us consider the following example. Let $\mathbb{R}^n$ be endowed with Euclidean metric g_0 and let the potential function $f : \mathbb{R}^n \to \mathbb{R}$ be given by $f(x) = \frac{\lambda}{2}|x|^2$, for some $\lambda > 0$. Then we see $\mathrm{Ric}_f = \lambda g_0$. Since $\mathbb{R}^n$ is not compact, this simple example shows that the compactness conclusion of Theorem 2.1 does not extend directly.

Although results for manifolds with Ricci curvature bounded from below do not necessarily extend directly to smooth metric measure spaces with Bakry-Émery

Ricci curvature bounded from below, we do know that results do extend when additional assumptions are placed on the potential function f. We have seen, for example, that Theorem 2.11 extends when f is bounded from above.

In order to extend theorems regarding fundamental groups to the smooth metric measure space setting, we wish to have analogues of the tools used for studying manifolds with Ricci curvature bounded from below. As we noted earlier, the Bishop-Gromov volume comparison, Theorem 1.1, is a key tool in studying manifolds with Ricci curvature bounded from below. Fortunately, Guofang Wei and William Wylie [WW09] have shown that a Bishop-Gromov type volume comparison holds for smooth metric measure spaces, assuming additional assumptions on f. Before stating their result, we first note that in a smooth metric measure space, the volume of a ball will be computed with respect to the weighted measure on the space, as follows.

Definition 2.13. *The weighted volume (or f-volume) of a ball $B(p,R)$ in a smooth metric measure space $(M^n, g, e^{-f}\, dvol_g)$ is given by*

$$Vol_f(B(p,R)) = \int_{B(p,R)} e^{-f}\, dvol_g.$$

The volume comparison is then given by the following.

Theorem 2.14. *(Volume Comparison)[WW09, Theorem 1.2] Let $(M^n, g, e^{-f}\, dvol_g)$ be a complete smooth metric measure space with $Ric_f \geq (n-1)H$. Fix $p \in M^n$.*

a. *If $\partial_r f \geq -a$ along all minimal geodesic segments from p, then for $R \geq r > 0$ (assume $R \leq \pi/2\sqrt{H}$ if $H > 0$),*

$$\frac{Vol_f(B(p,R)}{Vol_f(B(p,r))} \leq e^{aR}\frac{Vol_H^n(R)}{Vol_H^n(r)}. \tag{2.1}$$

Moreover, equality holds if and only if the radial sectional curvatures are equal to H and $\partial_r f \equiv -a$. In particular if $\partial_r f \geq 0$ and $Ric_f \geq 0$ then M has f-volume growth of degree at most n.

b. *If $|f(x)| \leq k$ then for $R \geq r > 0$ (assume $R \leq \pi/4\sqrt{H}$ if $H > 0$),*

$$\frac{Vol_f(B(p,R))}{Vol_f(B(p,r))} \leq \frac{Vol_H^{n+4k}(R)}{Vol_H^{n+4k}(r)}. \tag{2.2}$$

In particular, if f is bounded and $Ric_f \geq 0$ then M has polynomial f-volume growth.

This volume comparison can be used to extend Milnor's theorem, Theorem 2.5, to the smooth metric measure space setting [WW09, Theorem 4.4]. Notice that when the potential function is bounded, $|f| \leq k$, use of the volume comparison Theorem 2.14(b.) will show that finitely generated subgroups have polynomial growth of degree $\leq n + 4k$. Ning Yang [Yan09] later improved this degree of growth to n. In this dissertation, we will briefly discuss how one may apply Theorem 2.14 to extend Wei's result, Theorem 2.8, to the smooth metric measure space setting to obtain polynomial growth of degree $n+4k$. We will also develop an absolute volume comparison (see Proposition 4.1) to improve this growth degree to n.

As mentioned in the introduction, we wish to extend the Cheeger-Colding Splitting Theorem to smooth metric measure spaces in order to utilize the arguments of Yun and Kapovitch-Wilking. One of the major tools which we will need to show that this result indeed extends to the desired setting is a Laplacian comparison.

Due to the weighted measure in a smooth metric measure space, one replaces the usual Laplace-Beltrami operator on Riemannian manifolds with an analogous version.

Definition 2.15. *Let $(M^n, g, e^{-f}dvol_g)$ be a smooth metric measure space and $u \in C^2(M)$. The* <u>*f-Laplacian*</u> *of u is given by*

$$\Delta_f(u) = \Delta(u) - \langle \nabla u, \nabla f \rangle. \tag{2.3}$$

This operator is a natural analogue to the Laplace-Beltrami operator on Riemannian manifolds in the sense that it is self-adjoint with respect to the weighted measure $e^{-f}\mathrm{dvol}_g$.

From the Mean Curvature Comparison of Wei and Wylie [WW09, Theorem 1.1] and the definition of the f-Laplacian, equation (2.3), one immediately obtains the following f-Laplacian comparison.

Proposition 2.16. **(f-Laplacian Comparison)** *Suppose $Ric_f \geq (n-1)H$ with $|f| \leq k$. Let Δ_H^{n+4k} denote the Laplacian of the simply connected model space of dimension $n+4k$ with constant sectional curvature H. For radial functions u,*

1. $\Delta_f(u) \leq \Delta_H^{n+4k} u$ *if* $u' \geq 0$.

2. $\Delta_f(u) \geq \Delta_H^{n+4k} u$ *if* $u' \leq 0$.

Recall that the classical Bochner formula, another important tool for studying Riemannian manifolds with Ricci curvature bounded from below, is given by

$$\frac{1}{2}\Delta(|\nabla u|^2) = |\mathrm{Hess}\, u|^2 + \langle \nabla u, \nabla \Delta u \rangle + \mathrm{Ric}(\nabla u, \nabla u).$$

Combining this Bochner formula with the definition of the f-Laplacian, equation 2.3, one obtains a smooth metric measure space version of the Bochner formula:

$$\frac{1}{2}\Delta_f(|\nabla u|^2) = |\mathrm{Hess}\, u|^2 + \langle \nabla u, \nabla \Delta_f u \rangle + \mathrm{Ric}_f(\nabla u, \nabla u), \tag{2.4}$$

for any $u \in C^3(M)$.

As we will see, these tools of a volume comparison and Laplacian comparison on smooth metric measure spaces will become invaluable in our work to extend the desired theorems to this setting.

2.4 Gromov-Hausdorff Convergence

In order to prove Theorems 1.4 and 1.5, we closely follow the arguments of Yun [Yun97] and Kapovitch-Wilking [KW11] for the case in which Ricci curvature is bounded from below. Common to both arguments is the utilization of "contradicting sequences," that is, a sequence of manifolds for which the conclusion of the theorem does not hold. Studying sequences of manifolds, in particular those which converge in the Gromov-Hausdorff sense, has indeed become a useful technique in analyzing spaces with Ricci curvature bounded from below. As alluded to previously, Cheeger-Colding's Splitting (Theorem 2.12) is an important tool in analyzing the limit spaces of these sequences. Before we discuss the extension of this splitting theorem, we take a moment to recall some of the basic concepts regarding sequences of manifolds.

To begin discussing convergence of a sequence of manifolds, one must first establish the notion of distance between two arbitrary manifolds. Certainly, one has the notion of a distance between two subsets of the same metric space.

Definition 2.17. *Let (Z, d) be a metric space and $X, Y \subset Z$. Let*

$$B(X, \epsilon) = \{z \in Z : \inf_{x \in X} d(z, x) < \epsilon\}.$$

The Hausdorff distance between X and Y is given by

$$d_H^Z(X, Y) = \inf\{\epsilon > 0 : X \subseteq B(Y, \epsilon), Y \subseteq B(X, \epsilon)\}.$$

In [Gro81, Section 6], Gromov introduces a definition of distance between two arbitrary compact metric spaces. In order to do so, he first defines when a metric on the disjoint union is admissible.

Definition 2.18. *Let* (X, d_X) *and* (Y, d_Y) *be metric spaces. A metric* d *on the disjoint union* $X \sqcup Y$ *is said to be* <u>*admissible*</u> *if* $d|_X = d_X$ *and* $d|_Y = d_Y$.

Using admissible metrics, Gromov then defines the distance between two arbitrary compact metric spaces as follows.

Definition 2.19. *Let* (X, d_X) *and* (Y, d_Y) *be compact metric spaces. The* <u>*Gromov-Hausdorff distance*</u> *between* X *and* Y *is given by*

$$d_{GH}(X, Y) = \inf\{d_H^{X \sqcup Y}(X, Y) : \textit{admissible metrics } d \textit{ on } X \sqcup Y\}.$$

Using this Gromov-Hausdorff distance, one can show that if $d_{GH}(X, Y) = 0$, then X and Y are in fact isometric, see for example [Pet98, Lemma 10.1.3].

If two spaces are close in the Gromov-Hausdorff sense, one may find a map between the two spaces which is almost an isometry and almost surjective.

Definition 2.20. *Let* (X, d_X) *and* (Y, d_Y) *be metric spaces. A map* $f : X \to Y$, *not necessarily continuous, is called an* <u>ϵ*-Gromov-Hausdorff approximation*</u> *if for all* $x_1, x_2 \in X$,

$$|d_X(x_1, x_2) - d_Y(f(x_1), f(x_2))| < \epsilon \quad \textit{and} \quad Y \subseteq B(f(X), \epsilon).$$

The Gromov-Hausdorff distance between (X, d_X) and (Y, d_Y) may alternatively be defined as the infimum among all $\epsilon > 0$ such that there are ϵ-Gromov-Hausdorff approximations from $X \to Y$ and $Y \to X$.

Now, let $(\mathcal{M}, d_{GH})$ denote the collection of isometry classes of all compact metric spaces. This collection forms a separable and complete metric space, see for example [Pet98, Proposition 10.1.7]. A sequence of metric spaces X_i converges in the Gromov-Hausdorff sense to a limit space X if

$$\lim_{i\to\infty} d_{GH}(X_i, X) = 0.$$

If we measure the distance between two noncompact metric spaces in the same manner, the distance will likely be infinite. It is then difficult to speak of Gromov-Hausdorff convergence of sequences of noncompact metric spaces. For noncompact metric spaces, one instead uses the notion of pointed convergence. In order to discuss pointed convergence, we first give the definition for the pointed Gromov-Hausdorff distance. Here, (X, x, d_X) and (Y, y, d_Y) denote metric spaces (X, d_X) with specified base point $x \in X$ and (Y, d_Y) with specified base point $y \in Y$, respectively. When computing the distance between these two pointed metric spaces, one takes into account the distance between these specified points.

Definition 2.21. *Let (X, x, d_X) and (Y, y, d_Y) be pointed metric spaces. Then the pointed Gromov-Hausdorff distance between these two spaces is given by*

$$d_{GH}((X, x, d_X), (Y, y, d_Y)) = \inf\{d_H^{X\sqcup Y}(X, Y) + d(x, y) : \textit{admissible metrics } d \textit{ on } X \sqcup Y\}.$$

Again, the equivalence classes of isometric proper pointed metric spaces form a metric space in its own right. Following the notation in [Pet98], we denote this collection by $\mathcal{M}_*$.

Definition 2.22. *Let* (X_i, x_i, d_i) *be a sequence of pointed metric spaces in* $\mathcal{M}_*$. *We say* (X_i, x_i, d_i) *converges in the pointed Gromov-Hausdorff topology if for all* $R > 0$ *the closed metric balls* $(\overline{B(x_i, R)}, x_i, d_i)$ *converge to* $(\overline{B(x, R)}, x, d)$ *with respect to the pointed Gromov-Hausdorff metric.*

We note that Gromov's Precompactness Theorem implies that the following classes of Riemannian manifolds are precompact under the Gromov-Hausdorff topology and pointed Gromov-Hausdorff topology respectively; for statement see, for example [Pet98, Lemma 10.1.9].

Theorem 2.23 (Gromov (1980))**.** *For any integer* $n \geq 2$, $k \in \mathbb{R}$, *and* $D > 0$, *we have that the following classes are precompact:*

1. *The collection of closed Riemannian* n*-manifolds with* $Ric \geq (n-1)k$ *and* $diam \leq D$.

2. *The collection of pointed complete Riemannian* n*-manifolds with* $Ric \geq (n-1)k$.

Recall that a main goal of this dissertation is to discuss the fundamental group of smooth metric measure spaces with Bakry-Émery Ricci curvature bounded from below. As in [Yun97] and [KW11] we will not only be considering sequences of manifolds, but also group actions on each manifold in the sequence. Specifically, for a sequence of pointed smooth metric measure spaces $(M_i, g_i, e^{-f_i}\mathrm{dvol}_{g_i}, p_i)$ we look at the action of the fundamental group $\pi_1(M_i)$ on the universal cover $\widetilde{M_i}$, and consider the sequence $(\widetilde{M_i}, \pi_1(M_i), \tilde{p}_i)$. We would like then to discuss not only a convergent sequence of manifolds, but also the limit of a group action on a manifold. To this end, we look to the notion of equivariant pointed Gromov-Hausdorff convergence as discussed by Kenji Fukaya and Takao Yamaguchi in their work on Riemannian manifolds with sectional curvature bounded from below [FY92, Section I.3].

Let $\mathcal{M}_{eq}$ denote the set of all triples (X, Γ, x) so that $(X, x) \in \mathcal{M}_*$ and Γ is a closed subgroup of isometries of X. Set

$$\Gamma(D) = \{\gamma \in \Gamma : d(\gamma p, p) < D\}.$$

Definition 2.24. *[FY92, Definition 3.3] Let $(X, \Gamma, x), (Y, \Lambda, y) \in \mathcal{M}_{eq}$. An ϵ-equivariant Hausdorff approximation is a triple of maps $f : B(x, 1/\epsilon) \to Y$, $\phi : \Gamma(1/\epsilon) \to \Lambda(1/\epsilon)$ and $\psi : \Lambda(1/\epsilon) \to \Gamma(1/\epsilon)$ such that*

1. $f(x) = y$;
2. $B(y, 1/\epsilon) \subseteq Y$ *is contained in the ϵ-neighborhood* $B(\epsilon, f(B(x, 1/\epsilon))) \subseteq Y$.
3. $p, q \in B(x, 1/\epsilon) \Rightarrow |d(f(p), f(q)) - d(p, q)| < \epsilon$;
4. $\gamma \in \Gamma(1/\epsilon)$, $p \in B(x, 1/\epsilon)$, $\gamma p \in B(x, 1/\epsilon) \Rightarrow$
$$d(f(\gamma p), (\phi(\gamma))(f(p))) < \epsilon;$$
5. $\mu \in \Lambda(1/\epsilon)$, $p \in B(x, 1/\epsilon)$, $(\psi(\mu))(p) \in B(x, 1/\epsilon) \Rightarrow$

$$d(f((\psi(\mu))(p)), \mu(f(p))) < \epsilon.$$

One says that a sequence (X_i, Γ_i, x_i) in $\mathcal{M}_{eq}$ converges to (X, Γ, x) in the equivariant Gromov-Hausdorff sense if there exists ϵ_i-equivariant Hausdorff approximations between (X_i, Γ_i, x_i) and (X, Γ, x) where $\epsilon_i \to 0$ as $i \to \infty$.

Of particular use will be two results of Fukaya and Fukaya-Yamaguchi which we include below. The first result relates a sequence and its equivariant pointed Gromov-Hausdorff limit to a sequence of the the orbit space.

Theorem 2.25. *[Fuk86, Theorem 2.1] If (X_i, Γ_i, x_i) converges to (Y, Λ, y) in the equivariant pointed Gromov-Hausdorff sense, then $(X_i/\Gamma_i, \bar{x}_i)$ converges to $(Y/\Lambda, \overline{y})$ in the pointed Gromov-Hausdorff sense.*

The next theorem guarantees the existence of a subsequence which converges pointed in the equivariant Gromov-Hausdorff sense.

Theorem 2.26. *[FY92, Proposition 3.6] Let $(X_i, \Gamma_i, x_i) \in \mathcal{M}_{eq}$, and $(Y, y) \in \mathcal{M}_*$. Suppose that (X_i, x_i) converges to (Y, y) in the pointed Gromov-Hausdorff sense. Then there exists a group G and subsequence k_i such that for $(Y, G, y) \in \mathcal{M}_{eq}$ we have that $(X_{k_i}, \Gamma_{k_i}, x_{k_i})$ converges to (Y, G, y) in the equivariant pointed Gromov-Hausdorff sense.*

Chapter 3

The Splitting Theorem

Recall that the Cheeger-Gromoll Splitting Theorem (see Theorem 2.11) implies that a compact Riemannian manifold with nonnegative Ricci curvature has an almost abelian fundamental group. It may not be surprising then that in order to extend this result regarding fundamental groups to spaces in the class of compact Riemannian manifolds with almost nonnegative Ricci curvature, one utilizes the Cheeger-Colding Splitting Theorem (see Theorem 2.12), which may be viewed as a extension of Theorem 2.11 to limit spaces with nonnegative Ricci curvature in a generalized sense.

In the proof of Theorem 2.11, one constructs a function b such that $|\nabla b| = 1$ and $\mathrm{Hess}\, b \equiv 0$ [CG71]. In the proof of Theorem 2.12, one constructs a harmonic function $\bar{b}$ whose Hessian is small in the L^2-sense [CC96, Proposition 6.60]. In order to extend Theorem 2.12 to smooth metric measure spaces, following the proof of Cheeger-Colding, we construct f-harmonic functions $\bar{b}_{\pm}$ and obtain L^2-Hessian estimates for such functions with respect to the conformally changed volume density $e^{-f}\mathrm{dvol}_g$. This desired Hessian estimate is given in Theorem 3.8.

We begin this chapter by discussing and extending some of the key tools used by Cheeger-Colding to develop their Hessian estimate. Once we have these tools, we will prove Theorem 3.8. We then use this Hessian estimate to give a smooth metric measure space version of the Quantitative Pythagorean Theorem (see [Che07, Lemma 9.16] for original statement). We conclude this chapter with the proof of Theorem 1.6.

3.1 Preliminary Estimates

In order to obtain their Hessian estimate and splitting theorem, Cheeger and Colding utilized a number of indispensable tools. For example, the Abresch-Gromoll Inequality [AG90] gives an upper bound on the excess function defined in equation 3.9 and is used to establish the fact that Cheeger-Colding's harmonic function $\bar{b}$ is pointwise-close to a distance-like function. This proof of closeness is the first step toward their Hessian estimate. The Cheng-Yau Gradient Estimate [CY75] is used to show the existence of an appropriate cutoff function which is central in the proof of the Hessian estimate. Cheeger and Colding also developed key tools, such as the Segment Inequality, [CC96, Theorem 2.11] to aid in their proof. We wish to extend such tools to the smooth metric measure space setting.

With these tools we can then generalize the arguments of Cheeger and Colding [CC96], see also [Che07], to obtain Theorems 3.8 and 1.6.

In order to obtain a gradient estimate appropriate for our use, we draw upon the work of Kevin Brighton [Bri11] in which he obtains a gradient estimate for an f-harmonic function, that is, a function u such that $\Delta_f u \equiv 0$. We can generalize Brighton's methods to obtain a gradient estimate for positive functions $u \in C^3(M)$ with $\Delta_f u \equiv c$ where c is a nonnegative constant. The difficulty in extending this proof method to the case for which c is negative is addressed after the proof of Proposition 3.2.

Before giving the statement of the gradient estimate which is utilized in our arguments, we note that Brighton's gradient estimate for f-harmonic functions requires no additional assumption on f. Generalizing his arguments to functions $u \in C^3(M)$ with $\Delta_f u \equiv c$ indeed gives a gradient estimate which also does not require additional assumptions on f. We include this gradient estimate below.

Proposition 3.1. (Gradient Estimate) *Let $(M^n, g, e^{-f} dvol_g)$ be a complete smooth metric measure space with $Ric_f \geq -(n-1)H^2$ where $H \geq 0$. If u is a positive function defined on $\overline{B(q, 2R)}$ with $\Delta_f u = c$, $c \geq 0$, then for any $q_0 \in \overline{B(q, R)}$, we have*

$$|\nabla u| \leq \sqrt{c_1(\alpha, n, H, R) \sup_{p \in B(q,2R)} u(p)^2 + c_2(c, n) \sup_{p \in B(q,2R)} u(p)}$$

where $\alpha = \max_{p \in p: d(p,q) = r_0} \Delta_f r(p)$ for any $r_0 \leq R$ and $r(p) = d(p, q)$.

The constant c_1 then depends on the f-Laplacian of $r(p)$, which will then affect the subsequent estimates which depend on this gradient estimate. Since for our purposes the potential function f is bounded, we will use the following gradient estimate instead.

Proposition 3.2. *Let $(M^n, g, e^{-f}\, dvol_g)$ be a complete smooth metric measure space with $|f|\leq k$ and $Ric_f \geq -(n-1)H^2$ where $H \geq 0$. If u is a positive function defined on $\overline{B(q, 2R)}$ with $\Delta_f u = c$, for a constant $c \geq 0$, then for any $q_0 \in \overline{B(q, R)}$, we have*

$$|\nabla u|\leq \sqrt{c_1(n, k, H, R) \sup_{p\in B(q,2R)} u(p)^2 + c_2(c, n) \sup_{p\in B(q,2R)} u(p)}.$$

Proof. Let $h = u^\epsilon$ where $\epsilon \in (0, 1)$. Applying the Bochner formula for smooth metric measure spaces, equation (2.4), to h gives

$$\frac{1}{2}\Delta_f|\nabla h|^2 = |\mathrm{Hess}\, h|^2 + \langle \nabla h, \nabla(\Delta_f h)\rangle + \mathrm{Ric}_f(\nabla h, \nabla h).$$

Using the Schwartz inequality, we have

$$\begin{aligned} |\mathrm{Hess}\, h|^2 &\geq \frac{|\Delta h|^2}{n} \\ &= \frac{1}{n}(\Delta_f h + \langle \nabla f, \nabla h\rangle)^2 \\ &= \frac{1}{n}\left(\epsilon u^{\epsilon-1}\Delta_f u + \frac{(\epsilon-1)|\nabla h|^2}{\epsilon h} + \langle \nabla f, \nabla h\rangle\right)^2 \\ &= \frac{1}{n}\left(\epsilon u^{\epsilon-1} c + \frac{(\epsilon-1)|\nabla h|^2}{\epsilon h} + \langle \nabla f, \nabla h\rangle\right)^2 \end{aligned}$$

where in the last equality we used the fact that $\Delta_f u = c$. This, together with the lower bound on Bakry-Émery Ricci curvature, gives

$$\begin{aligned}\frac{1}{2}\Delta_f|\nabla h|^2 \geq & \frac{(\epsilon-1)^2}{\epsilon^2 h^2 n}|\nabla h|^4 + \frac{2c(\epsilon-1)}{h^{1/\epsilon}n}|\nabla h|^2 + \frac{2(\epsilon-1)}{\epsilon h n}|\nabla h|^2\langle\nabla f,\nabla h\rangle + \frac{\epsilon^2c^2}{n}(h^{2-2/\epsilon}) \\ & + \frac{2c\epsilon}{n}(h^{1-1/\epsilon})\langle\nabla f,\nabla h\rangle + \frac{1}{n}\langle\nabla f,\nabla h\rangle^2 + \frac{(\epsilon-1)}{\epsilon h}\langle\nabla h,\nabla|\nabla h|^2\rangle \\ & - \frac{(\epsilon-1)}{\epsilon h^2}|\nabla h|^4 + \frac{c(\epsilon-1)}{h^{1/\epsilon}}|\nabla h|^2 - (n-1)H^2|\nabla h|^2\end{aligned} \tag{3.1}$$

As in [Bri11], in order to control the mixed term $2\frac{(\epsilon-1)}{\epsilon h n}|\nabla h|^2\langle\nabla f,\nabla h\rangle$ in (3.1), we consider two cases according to whether $|\nabla h|^2$ dominates over $\langle\nabla h,\nabla f\rangle$, or vice versa. In the first case, suppose that $p\in\overline{B(q,2R)}$ such that $\langle\nabla h,\nabla f\rangle\leq a\frac{|\nabla h|^2}{h}$ for some $a>0$ to be determined. At this point we have

$$\begin{aligned}\frac{1}{2}\Delta_f|\nabla h|^2 a \geq & \left[\frac{(\epsilon-1)^2+2\epsilon(\epsilon-1)a-\epsilon(\epsilon-1)n}{\epsilon^2 n}\right]\frac{|\nabla h|^4}{h^2} + \left[\frac{c(\epsilon-1)(2+n)}{n}\right]\frac{|\nabla h|^2}{h^{1/\epsilon}} \\ & + \frac{1}{n}(\epsilon c h^{1-1/\epsilon} + \langle\nabla f,\nabla h\rangle)^2 + \frac{\epsilon-1}{\epsilon h}\langle\nabla h,\nabla|\nabla h|^2\rangle - (n-1)H^2|\nabla h|^2 \\ \geq & \left[\frac{(\epsilon-1)^2+2\epsilon(\epsilon-1)a-\epsilon(\epsilon-1)n}{\epsilon^2 n}\right]\frac{|\nabla h|^4}{h^2} + \left[\frac{c(\epsilon-1)(2+n)}{n}\right]\frac{|\nabla h|^2}{h^{1/\epsilon}} \\ & + \frac{\epsilon-1}{\epsilon h}\langle\nabla h,\nabla|\nabla h|^2\rangle - (n-1)H^2|\nabla h|^2.\end{aligned} \tag{3.2}$$

In the case that $p \in \overline{B(q,2R)}$ such that $\langle \nabla h, \nabla f \rangle \geq a\frac{|\nabla h|^2}{h}$, we have

$$\begin{aligned}
\frac{1}{2}\Delta_f|\nabla h|^2 \geq &\left[\frac{(\epsilon-1)^2-\epsilon(\epsilon-1)n}{\epsilon^2 n}\right]\frac{|\nabla h|^4}{h^2}+\left[\frac{c(\epsilon-1)(2+n)}{n}\right]\frac{|\nabla h|^2}{h^{1/\epsilon}} \\
&+\left[\frac{2(\epsilon-1)+\epsilon a}{\epsilon n a}\right]\langle \nabla f, \nabla h\rangle^2+\frac{\epsilon^2c^2}{n}(h^{2-2/\epsilon})+\frac{2c\epsilon a}{nh^{1/\epsilon}}|\nabla h|^2 \\
&+\frac{\epsilon-1}{\epsilon h}\langle \nabla h, \nabla|\nabla h|^2\rangle-(n-1)H^2|\nabla h|^2 \\
\geq &\left[\frac{(\epsilon-1)^2-\epsilon(\epsilon-1)n}{\epsilon^2 n}\right]\frac{|\nabla h|^4}{h^2}+\left[\frac{c(\epsilon-1)(2+n)}{n}\right]\frac{|\nabla h|^2}{h^{1/\epsilon}} \\
&+\left[\frac{2(\epsilon-1)+\epsilon a}{\epsilon n a}\right]\langle \nabla f, \nabla h\rangle^2+\frac{\epsilon-1}{\epsilon h}\langle \nabla h, \nabla|\nabla h|^2\rangle-(n-1)H^2|\nabla h|^2
\end{aligned} \tag{3.3}$$

Note that in (3.3) the assumption that $c \geq 0$ is necessary to have $\frac{2c\epsilon a}{nh^{1/\epsilon}}|\nabla h|^2 \geq 0$ which allows us to obtain the second inequality.

As in Brighton's proof [Bri11], we see that choosing $\epsilon = \frac{7}{8}$ and $a = \frac{1}{2}$ will make the coefficient of the $\frac{|\nabla h|^4}{h^2}$ term positive in both cases. This choice also gives a positive coefficient of the $\langle \nabla f, \nabla h\rangle^2$ term in the second case. With this choice of ϵ and a, we see that for every $p \in \overline{B(q,2R)}$, we have

$$\frac{1}{2}\Delta_f|\nabla h|^2 \geq \frac{7n-6}{49n}\frac{|\nabla h|^4}{h^2}-\frac{c(2+n)}{8n}\frac{|\nabla h|^2}{h^{8/7}}-\frac{1}{7h}\langle \nabla h, \nabla|\nabla h|^2\rangle-(n-1)H^2|\nabla h|^2. \tag{3.4}$$

Let $g : [0,2R] \to [0,1]$ have the properties

- $g|_{[0,R]} = 1$
- $\operatorname{supp}(g) \subseteq [0,2R)$

- $\frac{-K}{R}\sqrt{g} \leq g' \leq 0$

- $|g''| \leq \frac{K}{R^2}$

where the last two properties hold for some $K > 0$. Define $\phi : \overline{B(q,2R)} \to [0,1]$ by $\phi(x) = g(d(x,q))$. Set $G = \phi|\nabla h|^2$. Then (3.4) can be written as

$$\begin{aligned}\frac{1}{\phi}\Delta_f G \geq \frac{G}{\phi^2}\Delta_f \phi + 2\langle\frac{\nabla\phi}{\phi}, \frac{\nabla G}{\phi} - \frac{\nabla\phi}{\phi^2}G\rangle + \frac{14n-12}{49nh^2}\frac{G^2}{\phi^2} \\ - \frac{c(2+n)}{4nh^{8/7}}\frac{G}{\phi} - \frac{2}{7h}\langle\nabla h, \frac{\nabla G}{\phi} - \frac{\nabla\phi}{\phi^2}G\rangle - 2(n-1)H^2\frac{G}{\phi}.\end{aligned} \tag{3.5}$$

Next, we consider the point $q_0 \in \overline{B(q,2R)}$ at which G achieves its maximum. At such a point, (3.5) can be rewritten as

$$\frac{14n-12}{49nh^2}G \leq -\Delta_f\phi + 2\langle\frac{\nabla\phi}{\phi}, \nabla\phi\rangle + \frac{c(2+n)}{4nh^{8/7}}\phi - \frac{2}{7h}\langle\nabla h, \nabla\phi\rangle + 2(n-1)H^2\phi. \tag{3.6}$$

If $q_0 \in B(q,R)$, then (3.6) can be rewritten as

$$|\nabla u|^2 \leq \frac{8c(2+n)}{7n-6}u + \frac{64n(n-1)}{7n-6}H^2u^2 \tag{3.7}$$

when evaluated at q_0.

If $q_0 \in \overline{B(q,2R)} \setminus B(q,R)$, one uses Proposition 2.16 to see that

$$\Delta_f r(q_0) \leq (n+4k-1)H\coth(Hr(q_0)) \leq (n+4k-1)H\coth(HR)$$

since $R \leq r(q_0) \leq 2R$. Set $\alpha = (n+4k-1)H\coth(HR)$. This, along with the properties of ϕ, and (3.6) we see

$$|\nabla u|^2 \leq \left[\frac{64[K(R\alpha+1+3K)+2(n-1)R^2H^2]}{(13n-12)R^2}\right]u^2 + \frac{16c(2+n)}{n(13n-12)}u. \tag{3.8}$$

Restricting now to $B(q,R)$ and taking the supremum of u and u^2 over $\overline{B(q,2R)}$ in (3.7) and (3.8) yields the desired form of the gradient estimate. $\square$

Note that Proposition 3.2 only holds for nonnegative c. If we consider the case $c<0$, the term $\frac{2c\epsilon}{n}(h^{1-1/\epsilon})\langle\nabla f,\nabla h\rangle$ in (3.1) becomes problematic. In particular, when $p\in\overline{B(q,2R)}$ is such that $\langle\nabla h,\nabla f\rangle$ dominates over $|\nabla h|^2$, we replace the term $\frac{2(\epsilon-1)}{\epsilon hn}|\nabla h|^2\langle\nabla f,\nabla h\rangle$ by $\frac{2(\epsilon-1)}{\epsilon hna}\langle\nabla f,\nabla h\rangle^2$. In order to control this term, we group it with $\frac{1}{n}\langle\nabla f,\nabla h\rangle^2$. Then we can no longer group the $\langle\nabla f,\nabla h\rangle$ term with other terms to create a perfect square, as in (3.2). Moreover, since its coefficient is negative, we must keep this term in the estimate. Thus, without any additional assumptions, such as a bound on $|\nabla f|$, there is no way to control this term. As noted in the introduction, the assumption of a bound on $|f|$ rather than a bound on $|\nabla f|$ in this gradient estimate is one of the reasons that our Theorem 1.6 only requires a bound on $|f_i|$ whereas the splitting theorem of Wang and Zhu [WZ13, Theorem 3.1] requires bounds on both $|f_i|$ and $|\nabla f_i|$.

In order to convert estimates of integrals of functions over a ball to estimates of integrals of functions along a geodesic segment, we need a Segment Inequality similar to that developed by Cheeger and Colding in [CC96, Theorem 2.11].

Proposition 3.3. (Segment Inequality) *Let $(M^n, g, e^{-f}dvol_g)$ be a complete smooth metric measure space with $Ric_f \geq (n-1)H$ and $|f(x)| \leq k$. Let $A_1, A_2 \in M^n$ be open sets and assume for all $y_1 \in A_1$, $y_2 \in A_2$, there is a minimal geodesic, γ_{y_1,y_2} from y_1 to y_2, such that for some open set, W,*

$$\bigcup_{y_1,y_2} \gamma_{y_1,y_2} \subset W.$$

If v_i is a tangent vector at y_i, $i = 1, 2$, and $|v_i| = 1$, set

$$I(y_i, v_i) = \{t|\gamma(t) \in A_{i+1}, \gamma|[0,t] \text{ is minimal}, \gamma'(0) = v_i\}.$$

Let $|I(y_i, v_i)|$ denote the measure of $I(y_i, v_i)$ and put

$$\mathcal{D}(A_i, A_{i+1}) = \sup_{y_i, v_i} |I(y_i, v_i)|.$$

Here $A_{2+1} := A_1$. Let h be a nonnegative integrable function on M. Let $D = \max d(y_1, y_2)$. Then

$$\int_{A_1 \times A_2} \int_0^{d(y_1,y_2)} h(\gamma_{y_1,y_2}(s))ds(e^{-f}dvol_g)^2 \leq$$
$$c(n+4k, H, D)[\mathcal{D}(A_1, A_2)Vol_f(A_1) + \mathcal{D}(A_2, A_1)Vol_f(A_2)] \times \int_W he^{-f}dvol_g.$$

where $c(n+4k, H, D) = \sup_{0<s/2\leq u\leq s} \mathcal{A}_H^{n+4k}(s)/\mathcal{A}_H^{n+4k}(u)$, where $\mathcal{A}_H^{n+4k}(r)$ denotes the area element on $\partial B(r)$ in the model space with constant curvature H and dimension $n + 4k$.

To obtain this result for smooth metric measure spaces one may follow the arguments of the proof in the original setting as given by Cheeger and Colding in [CC96, Theorem 2.11], where integrals are computed with respect to the conformally changed volume element, $e^{-f}\mathrm{dvol}_g$, and use Wei-Wylie's volume element comparison which follows from [WW09, Theorem 1.1].

Finally, the Abresch-Gromoll Quantitative Maximal Principle was necessary in the proof of the Abresch-Gromoll Inequality in the case when $\mathrm{Ric}(M) \geq (n-1)H$ and also in obtaining an appropriate cutoff function needed to prove the Hessian estimate. Since this proof varies slightly from the exposition contained in Abresch-Gromoll's [AG90] or Cheeger's [Che07] works, we retain the proof here.

Proposition 3.4. (Quantitative Maximal Principle) *If $Ric_f \geq (n-1)H$, $(H \leq 0)$, $|f| \leq k$ and $U : \overline{B(y,R)} \subset M^n \to \mathbb{R}$ is a Lipschitz function with*

1. *$Lip(U) \leq a$, $U(y_0) = 0$ for some $y_0 \in B(y,R)$,*
2. *$\Delta_f U \leq b$ in the barrier sense, $U|_{\partial B(y,R)} \geq 0$.*

Then $U(y) \leq ac + bG_R(c)$ for all $0 < c < R$, where $G_R(r(x))$ is the smallest function on the model space M_H^{n+4k} such that:

1. *$G_R(r) > 0$, $G_R'(r) < 0$ for $0 < r < R$*
2. *$\Delta_H G_R \equiv 1$ and $G_R(R) = 0$.*

Proof. Let $G_R(r)$ be the comparison function in the model space M_H^{n+4k} as given in the statement of the theorem. By the f-Laplacian Comparison, one has

$$\Delta_f G_R \geq \Delta_H^{n+4k} G_R = 1.$$

Consider the function $V = bG_R - U$. Then

$$\Delta_f V = b\Delta_f G_R - \Delta_f U \geq b\Delta_H^{n+4k} G_R - \Delta_f U = b - \Delta_f U \geq 0.$$

Then the maximal principle on $V : \overline{A(y,c,R)} \to \mathbb{R}$ gives

$$V(x) \leq \max\{V|_{\partial B(y,R)}, V|_{\partial B(y,c)}\}$$

for all $x \in \overline{A(y,c,R)}$. By assumption, we have

$$V|_{\partial B(y,R)} = bG_R|_{\partial B(y,R)} - U|_{\partial B(y,R)} \leq 0$$

and

$$V(y_0) = bG_R(y_0) - U(y_0) = bG_R(y_0) > 0.$$

Then there are two cases.

If $y_0 \in A(y,c,R)$, then $\max V|_{\partial B(y,c)} > 0$ so $V(y') > 0$ for some $y' \in \partial B(y,c)$. Since

$$U(y) - U(y') \leq a \cdot d(y,y') = ac$$

and

$$bG_R(c) - U(y') = V(y') > 0,$$

it follows that

$$U(y) \leq ac + U(y') \leq ac + bG_R(c).$$

On the other hand, if $d(y,y_0) \leq c$, we may use the Lipschitz condition directly:

$$U(y) = U(y) - U(y_0) \leq a \cdot d(y,y_0) \leq ac \leq ac + bG_R(c).$$

In either case, we have $U(y) \leq ac + bG_R(c)$ for all $0 < c < R$, as desired. $\square$

For any point $x \in M$, the excess function at x is given by

$$e(x) = d(x, q_+) + d(x, q_-) - d(q_+, q_-), \tag{3.9}$$

where $q_+, q_- \in M$ are fixed. For the excess function, we have the following Abresch-Gromoll Inequality, which gives an upper bound on the excess function in terms of a function

$$\Psi = \Psi(\epsilon_1, \ldots, \epsilon_k | c_1, \ldots, c_N) \tag{3.10}$$

such that $\Psi \geq 0$ and for any fixed $c_1, \ldots, c_N$,

$$\lim_{\epsilon_1, \ldots, \epsilon_k \to 0} \Psi = 0.$$

Such ϵ_i, c_i will be given explicitly below.

Proposition 3.5. (Abresch-Gromoll Inequality) *Let $(M^n, g, e^{-f} dvol_g)$ be a smooth metric measure space and $p, q_+, q_- \in M^n$. Given $R > 0$, $L > 2R + 1$ and $\epsilon > 0$, if*

$$|f| \leq k, \tag{3.11}$$

$$Ric_f \geq -(n-1)H \quad (H \geq 0), \tag{3.12}$$

$$\min\{d(p, q_+), d(p, q_-)\} \geq L, \tag{3.13}$$

$$e(p) = d(p, q_+) + d(p, q_-) - d(q_+, q_-) \leq \epsilon, \tag{3.14}$$

then

$$e(x) \leq \Psi(H, L^{-1}, \epsilon | n, k, R)$$

on $B(p, R)$.

Proof. The proof of the Abresch-Gromoll Inequality for smooth metric measure spaces runs parallel to the proof one finds in Cheeger [Che07, Theorem 9.1], with the modification that one uses the Quantitative Maximal Principal 3.4 together with the f-Laplacian comparison, Proposition 2.16 in place of their Riemannian counterparts. $\square$

We note that an excess estimate for smooth metric measure spaces with $\mathrm{Ric}_f \geq 0$ and $|f| \leq k$ is given by Wei-Wylie [WW09, Theorem 6.1].

3.2 The Hessian Estimate

For fixed $p, q_+, q_- \in M$, define the function $b_\pm : M \to \mathbb{R}$ by

$$b_\pm(x) = d(x, q_\pm) - d(p, q_\pm).$$

Let $\bar{b}_\pm : M \to \mathbb{R}$ be the function such that

$$\Delta_f \bar{b}_\pm = 0 \qquad \text{and} \qquad b_\pm|_{\partial B(p,R)} = \bar{b}_\pm|_{\partial B(p,R)}. \tag{3.15}$$

Lemma 3.6. *Let* $(M^n, g, e^{-f}\, dvol_g)$ *be a smooth metric measure space and* $p, q_+, q_- \in M^n$. *Given* $R > 0$, $L > 2R+1$ *and* $\epsilon > 0$, *if* (3.11) - (3.14) *hold then*

$$|b_\pm - \bar{b}_\pm| \leq \Psi(H, L^{-1}, \epsilon | n, k, R) \tag{3.16}$$

on $B(p, R)$.

Proof. The Abresch-Gromoll Inequality 3.5 along with the f-Laplacian Comparison 2.16 and the Maximal Principle 3.4, allow one to follow the proof of [CC96, Lemma 6.15] to obtain the above. $\square$

For the notation used below, we have $\fint_U he^{-f}\mathrm{dvol}_g = \frac{1}{\mathrm{Vol}_f(U)}\int he^{-f}\mathrm{dvol}_g$.

Lemma 3.7. *Let $(M^n, g, e^{-f}\,dvol_g)$ be a smooth metric measure space and $p, q_+, q_- \in M^n$. Given $R > 0$, $L > 2R + 1$ and $\epsilon > 0$, for any $p, q_+, q_- \in M^n$, if (3.11) - (3.14) hold, then*

$$\fint_{B(p,R)} |\nabla b_\pm - \nabla \bar{b}_\pm|^2 e^{-f}\,dvol \le \Psi(H, L^{-1}, \epsilon|n, k, R). \tag{3.17}$$

Proof. Use the above pointwise estimate on $\bar{b}_\pm$, Lemma 3.6, along with the Gradient Estimate, Theorem 3.2, and integration by parts to obtain (3.17). $\square$

Lemmas 3.6 and 3.7 now allow one to obtain the key estimate for $\mathrm{Hess}\,\bar{b}_\pm$. We will now prove Theorem 3.8.

Theorem 3.8. *Given $R > 0$, $L > 2R + 1$ and $\epsilon > 0$, let $p, q_+, q_- \in M^n$. If $(M^n, g, e^{-f}\,dvol_g)$ satisfies (3.11) - (3.14)*

$$\fint_{B(p,\frac{R}{2})} |Hess\bar{b}_\pm|^2 e^{-f}\,dvol_g \le \Psi(H, L^{-1}, \epsilon|k, n, R). \tag{3.18}$$

Proof. Applying the Bochner formula (2.4) to the f-harmonic function $\bar{b}_\pm$ yields

$$\frac{1}{2}\Delta_f|\nabla\bar{b}_\pm|^2 = |\mathrm{Hess}\,\bar{b}_\pm|^2 + \mathrm{Ric}_f(\nabla\bar{b}_\pm, \nabla\bar{b}_\pm).$$

Multiply by a cutoff function ϕ that has the following properties:

- $\phi|_{B(p,\frac{R}{2})} \equiv 1$,
- $supp(\phi) \subset B(p,R)$,
- $|\nabla\phi| \leq C(n,H,R,k)$,
- $|\Delta_f\phi| \leq C(n,H,R,k)$.

To construct such a cutoff function, one begins with a function $h : A(p,\frac{R}{2},R) \to \mathbb{R}$ such that $\Delta_f h \equiv 1$, $h|_{\partial B(p,\frac{R}{2})} = G_R(R/2)$, $h|_{\partial B(p,R)} = 0$, where G_R is as specified in Proposition 3.4. Then let $\psi : [0, G_R(R/2)] \to [0,1]$ such that ψ is 1 near $G_R(R/2)$ and ψ is 0 near 0. The function $\phi = \psi(h)$ extended to all of M by setting $\phi = 1$ inside $B(p,R/2)$ and $\phi = 0$ outside of $B(p,R)$, is the cutoff function desired. The gradient estimate of Proposition 3.2 guarantees that $|\nabla\phi|$ and $|\Delta_f\phi|$ are bounded away from the boundary of the annulus on which h was originally defined.

Then the above equation may be rewritten as

$$\phi|\mathrm{Hess}\,\bar{b}_\pm|^2 = \frac{1}{2}\phi\Delta_f|\nabla\bar{b}_\pm|^2 - \phi\mathrm{Ric}_f(\nabla\bar{b}_\pm, \nabla\bar{b}_\pm)$$

Integrating both sides of this equality over $B(p,R)$ gives

$$\begin{aligned}
\int_{B(p,R)} \phi|\mathrm{Hess}\,\bar{b}_\pm|^2 e^{-f}dvol_g &= \int_{B(p,R)} \left(\frac{1}{2}\phi\Delta_f|\nabla\bar{b}_\pm|^2 - \phi\mathrm{Ric}_f(\nabla\bar{b}_\pm,\nabla\bar{b}_\pm)\right) e^{-f}dvol_g \\
&\leq \int_{B(p,R)} \left(\frac{1}{2}\phi\Delta_f|\nabla\bar{b}_\pm|^2 + (n-1)H|\nabla\bar{b}_\pm|^2\right) e^{-f}dvol_g \\
&= \frac{1}{2}\int_{B(p,R)} \phi\Delta_f|\nabla\bar{b}_\pm|^2 e^{-f}dvol_g
\end{aligned}$$

$$+\int_{B(p,R)}(n-1)H|\nabla\bar{b}_\pm|^2 e^{-f}dvol_g$$

For the first integrand, we have

$$\begin{aligned}\int_{B(p,R)}\phi\Delta_f|\nabla\bar{b}_\pm|^2 e^{-f}dvol_g &= \int_{B(p,R)}\phi\Delta_f(|\nabla\bar{b}_\pm|^2-1)e^{-f}dvol_g\\ &= \int_{B(p,R)}\Delta_f\phi(|\nabla\bar{b}_\pm|^2-1)e^{-f}dvol_g\end{aligned}$$

Thus

$$\begin{aligned}\int_{B(p,R)}\phi|\mathrm{Hess}\,\bar{b}_\pm|^2 e^{-f}dvol_g &\le \int_{B(p,R)}\left[\frac{1}{2}\Delta_f\phi(|\nabla\bar{b}_\pm|^2-1)+(n-1)H|\nabla\bar{b}_\pm|^2\right]e^{-f}dvol_g\\ &\le \int_{B(p,R)}\left[\frac{1}{2}|\Delta_f\phi||\,|\nabla\bar{b}_\pm|^2-1|+(n-1)H|\nabla\bar{b}_\pm|^2\right]e^{-f}dvol_g\end{aligned}$$

Since $|\nabla b_\pm|=1$,

$$||\nabla\bar{b}_\pm|^2-1|=||\nabla\bar{b}_\pm|-|\nabla b_\pm||(|\nabla\bar{b}_\pm|+1)\le|\nabla\bar{b}_\pm-\nabla b_\pm|(|\nabla\bar{b}_\pm|+1),$$

we have

$$⨍_{B(p,R)}\phi|\mathrm{Hess}\,\bar{b}_\pm|^2 e^{-f}dvol_g\le\Psi.$$ □

3.3 Proof of the Almost Splitting Theorem

The Hessian estimate, Theorem 3.8, is important because it, together with the Segment Inequality, Proposition 3.3, allow us to extend the Quantitative Pythagorean Theorem [Che07, Lemma 9.16] to the smooth metric measure space setting. It is with this Quantitative Pythagorean Theorem that a quantitative version of the splitting theorem follows [CC96, Theorem 6.62]. Once we obtain a quantitative version of the splitting theorem for smooth metric measure spaces, the extension of the Cheeger-Colding Splitting Theorem follows.

Proposition 3.9. (Quantitative Pythagorean Theorem) *Given $R > 0$, $L > 2R+1$ and $\epsilon > 0$, with $p, q_+, q_- \in M^n$, such that (3.11) - (3.14) hold. Let $x, z, w \in B(p, \frac{R}{8})$, with $x \in \overline{b}_+^{-1}(a)$, and z a point on $\overline{b}_+^{-1}(a)$ closest to w. Then $|d(x,z)^2 + d(z,w)^2 - d(x,w)^2| \leq \Psi$.*

Proof. We begin by applying Proposition 3.3 to find points x^*, z^*, w^* such that

$$d(x^*, x) \leq \Psi, \qquad d(z^*, z) \leq \Psi, \qquad d(w^*, w) \leq \Psi.$$

Moreover, the segment inequality gives

$$\int_{B(p,\frac{R}{8})\times B(p,\frac{R}{8})} \inf_{\gamma} \int_0^{\ell(s)} |\mathrm{Hess}\,\overline{b}_+(\gamma(s))| ds \leq \\ c(n+4k, H, R)\frac{R}{2}\left(Vol_f B(p, \frac{R}{8})\right) \int_{B(p,\frac{R}{8})} |\mathrm{Hess}\,\overline{b}_+| e^{-f} \mathrm{dvol}_g$$

where the infimum runs along segments, $\gamma : [0, \ell(s)] \to M^n$ connecting points in $B(p, \frac{R}{8})$ to other points in $B(p, \frac{R}{8})$. Note that by the Hölder inequality, we have

$$\int_{B(p,\frac{R}{8})} |\mathrm{Hess}\,\overline{b}_+| e^{-f} dvol_g \leq \left(\int_{B(p,\frac{R}{8})} |\mathrm{Hess}\,\overline{b}_+|^2 e^{-f} dvol_g\right)^{\frac{1}{2}} \left(Vol_f(B(p, \frac{R}{8}))\right)^{\frac{1}{2}}.$$

This, along with Theorem 3.8, gives

$$\int_{B(p,\frac{R}{8})\times B(p,\frac{R}{8})} \inf_{\gamma} \int_0^{\ell(s)} |\mathrm{Hess}\,\bar{b}_+(\gamma(s))|ds \leq \Psi.$$

Thus along almost every segment $\gamma(s)$ connecting pairs of points in $B(p,\frac{R}{8})$, we may conclude

$$\int_{\gamma(s)} |\mathrm{Hess}\,\bar{b}_+(\gamma(s))|ds \leq \Psi.$$

In particular, consider a minimal geodesic $\sigma : [0,e] \to M^n$ from z^* to w^*. Let $A = \{\sigma(s)|s \in [0,e]\}$. Then again by the segment inequality we may deduce that along almost all geodesics $\tau_s : [0,\ell(s)] \to M$ (except possibly along a set of measure zero) connecting x^* to $\sigma(s) \in A$, we must have

$$\int_0^{\ell(s)} |\mathrm{Hess}\,\bar{b}_+(\tau_s(t))|ds \leq \Psi.$$

Thus there exists a set $U \subset [0,e]$ of full measure such that for all $s \in U$

$$\int_U \int_0^{\ell(s)} |\mathrm{Hess}\,\bar{b}_+(\tau_s(t))|dt\, ds \leq \Psi. \tag{3.19}$$

Similarly, by Lemma 3.7, the segment inequality also gives

$$\int_0^e ||\nabla \bar{b}_+(\sigma(s))| - 1|ds \leq \Psi.$$

Then we have

$$\begin{aligned}
\frac{1}{2}d^2(z,w) &= \int_0^e s ds \pm \psi \\
&= \int_0^e (\bar{b}_+(\sigma(s)) - \bar{b}_+(\sigma(0)))ds \pm \psi \\
&= \int_0^e (\bar{b}_+(\tau_s(\ell(s))) - \bar{b}_+(\tau_s(0)))ds \pm \psi \\
&= \int_U \int_0^{\ell(s)} \langle \nabla \bar{b}_+(\tau_s(t)), \tau_s'(t)\rangle dtds \pm \psi
\end{aligned}$$

Now, for all $t \in [0, \ell(s)]$, we have

$$\begin{aligned}
|\langle \nabla \bar{b}_+(\tau_s(\ell(s))), \tau_s'(\ell(s))\rangle - \langle \nabla \bar{b}_+(\tau_s(t)), \tau_s'(t)\rangle &= \left| \int_t^{\ell(s)} \operatorname{Hess} \bar{b}_+(\tau_s'(u), \tau_s'(u))du \right| \\
&\leq \int_t^{\ell(s)} \left| \operatorname{Hess} \bar{b}_+(\tau_s'(u), \tau_s'(u)) \right| du \\
&\leq \int_0^{\ell(s)} \left| \operatorname{Hess} \bar{b}_+(\tau_s'(u), \tau_s'(u)) \right| du \\
&\leq \psi
\end{aligned}$$

where the last inequality follows from inequality 3.19. This implies

$$\begin{aligned}
\int_U \int_0^{\ell(s)} \langle \nabla \bar{b}_+(\tau_s(t)), \tau_s'(t)\rangle dtds \pm \psi &= \int_U \int_0^{\ell(s)} \langle \nabla \bar{b}_+(\tau_s(\ell(s))), \tau_s'(\ell(s))\rangle dtds \pm \psi \\
&= \int_U \langle \nabla \bar{b}_+(\tau_s(\ell(s))), \tau_s'(\ell(s))\rangle \ell(s) ds \pm \psi \\
&= \int_U \langle \nabla \bar{b}_+(\tau_s(\sigma(s))), \tau_s'(\ell(s))\rangle \ell(s) ds \pm \psi \\
&= \int_U \langle \nabla \sigma'(s), \tau_s'(\ell(s))\rangle \ell(s) ds \pm \psi
\end{aligned}$$

By the first variation formula, we have $\ell'(s) = \langle \sigma'(s), \tau_s'(\ell(s)) \rangle$. Then, since $U \subset [0, e]$ has full measure, we have

$$\begin{aligned}
\int_U \langle \nabla \sigma'(s), \tau_s'(\ell(s)) \rangle \ell(s) ds \pm \psi &= \int_U \ell'(s)\ell(s) ds \pm \psi \\
&= \frac{1}{2} \left[\ell^2(s) \right]_0^e \pm \psi \\
&= \frac{1}{2} d^2(x, z) - \frac{1}{2} d^2(x, w) \pm \psi
\end{aligned}$$

Thus

$$\frac{1}{2} d^2(z, w) = \frac{1}{2} d^2(x, z) - \frac{1}{2} d^2(x, w) \pm \psi$$

as desired. $\square$

From this Quantitative Pythagorean Theorem for smooth metric measure spaces, one may establish the following Almost Splitting Theorem.

Theorem 3.10. (Almost Splitting Theorem) *Let $R > 0$, $L > 2R + 1$ and $\epsilon > 0$ be given. If $(M^n, g, e^{-f} dvol_g)$ and $p, q_+, q_- \in M^n$ satisfy (3.11) - (3.14), then there is a length space X such that for some ball $B((0, x), \frac{R}{4}) \subset \mathbb{R} \times X$ with the product metric, we have*

$$d_{GH}\left(B\left(p, \frac{R}{4}\right), B\left((0, x), \frac{R}{4}\right) \right) \leq \Psi(H, L^{-1}, \epsilon | k, n, R).$$

Proof. Let $q \in B(p, R/4) \setminus \bar{b}_+^{-1}(0)$. Let Ω denote the side of $\bar{b}_+^{-1}(0)$ containing q. For $x \in B(p, R/4)$, $z \in \bar{b}_+^{-1}(0)$, define

$$\hat{d}(x,z) = \begin{cases} d(x,z), & x \in \Omega \\ -d(x,z), & x \in \Omega^C \\ 0, & x \in \bar{b}_+^{-1}(0). \end{cases}$$

Let $\theta : B(p, R/4) \to \mathbb{R} \times \bar{b}_+^{-1}(0)$ be the map taking $x \in B(p, R/4)$ to $(\hat{d}(x,z), z) \in \mathbb{R} \times \bar{b}_+^{-1}(0)$ where z is a point on $\bar{b}_+^{-1}(0)$ closest to x. We will show that θ is a Ψ-Gromov-Hausdorff approximation. Note, if $\sigma : [0, s] \to M^n$ is a geodesic, connecting a point $x \in M$ to a point z on a level set of $\bar{b}_+$ which is closest to x, then

$$|\bar{b}_+(\sigma(s)) - \bar{b}_+(\sigma(0)) - d(x,z)| \leq \psi. \tag{3.20}$$

Let $x_1, x_2 \in B(p, R/4)$ with $\theta(x_1) = (d(x_1, z_1), z_1)$ and $\theta(x_2) = (d(x_2, z_2), z_2)$. If $x_1, x_2 \in \bar{b}_+^{-1}(0)$, then θ preserves distance. Without loss of generality, suppose that $d(x_1, z_1) \leq d(x_2, z_2)$. If $x_1 \in \bar{b}_+^{-1}(0)$, then direct application of Proposition 3.9 shows that $|d(x_1, x_2) - d(\theta(x_1), \theta(x_2))| \leq \Psi$. Suppose then that $x_1 \in \bar{b}_+^{-1}(a)$ and $x_2 \in \bar{b}_+^{-1}(b)$, with $a, b \neq 0$.

Case 1. The points x_1 and x_2 lie on the same side of $\bar{b}_+^{-1}(0)$, and the minimal geodesic connecting x_2 to z_2 intersects $\bar{b}_+^{-1}(a)$.

Without loss of generality, suppose $x_1, x_2 \in \Omega$. Let x_2' be the point at which the geodesic connecting x_2 to z_2 intersects $\bar{b}_+^{-1}(a)$, and $\overline{x_2}$ be the point on $\bar{b}_+^{-1}(a)$ closest to x_2. By equation (3.20), we have

$$d(x_1, z_1) = d(\bar{x}_2, z_2) \pm \Psi. \tag{3.21}$$

Let y_2 be the point on $\bar{b}_+^{-1}(a)$ closest to z_2. Proposition 3.9 and equation (3.20) together give $d(\bar{x_2}, y_2) \leq \Psi$. Applying Proposition 3.9 to the triples of points x_1, z_1, z_2 and x_1, y_2, z_2 gives

$$d(x_1, z_1)^2 + d(z_1, z_2)^2 = d(x_1, z_2)^2 \pm \Psi, \tag{3.22}$$

$$d(z_2, y_2)^2 + d(y_2, x_1)^2 = d(x_1, z_2)^2 \pm \Psi. \tag{3.23}$$

But by equation (3.20), we see $d(x_1, z_1)$ and $d(y_2, z_2)$ are ψ-close. This fact, together with equations (3.22) and (3.23) gives

$$d(z_1, z_2)^2 = d(x_1, x_2)^2 \pm \Psi. \tag{3.24}$$

Now, let x_2' be the point on $\bar{b}_+^{-1}(a)$ closest to x_2. Applying Proposition 3.9 to the triples x_2, $\bar{x}_2$, x_2' and x_2, x_2', x_1 give

$$d(x_2, x_2')^2 + d(x_2', \bar{x}_2)^2 = d(x_2, \bar{x}_2)^2 \pm \Psi, \tag{3.25}$$

$$d(x_2', x_1)^2 + d(x_2, x_2')^2 = d(x_1, x_2)^2 \pm \Psi. \tag{3.26}$$

Let $x_2 \in \bar{b}_+^{-1}(b)$. Then, by equation (3.20) we see

$$d(x_2, x_2') = b - a \pm \Psi. \tag{3.27}$$

Moreover, since $\bar{x}_2$ lies on the minimal geodesic connecting x_2 and z_2, we see $d(x_2, z_2) = d(x_2, \bar{x}_2) + d(\bar{x}_2, z_2)$. By equation (3.20) and we can rewrite this as

$$d(x_2, \bar{x}_2) = d(x_2, z_2) - d(\bar{x}_2, z_2) = b - a \pm \Psi. \tag{3.28}$$

Then, by equations (3.25), (3.27), (3.28), we have

$$d(x_2, x_2') = d(x_2, \bar{x}_2) \pm \Psi, \tag{3.29}$$

$$d(x_2', \bar{x}_2)^2 \leq \Psi. \tag{3.30}$$

By equations (3.29), (3.21), and the fact that $\bar{x}_2$ lies on the minimal geodesic connecting x_2 to z_2, we have

$$d(x_2, x_2') = d(x_2, z_2) - d(x_1, z_1) \pm \Psi. \tag{3.31}$$

Moreover, by equation (3.30) and (3.24) we see

$$d(x_2', x_1) = d(z_1, z_2) \pm \psi. \tag{3.32}$$

Substituting equations (3.30) and (3.32) into (3.26), we see

$$d(x_1, x_2)^2 = (d(x_2, z_2) - d(x_1, z_1))^2 + d(z_1, z_2)^2 \pm \Psi.$$

Case 2. The points x_1 and x_2 on lie on opposite sides of the level set $\bar{b}_+^{-1}(0)$, and the minimal geodesic connecting x_1 to x_2 intersects $\bar{b}_+^{-1}(0)$.

Let $\hat{z}_1$ be the point on $\bar{b}_+^{-1}(b)$ which is closest to x_1 and let σ be the minimal geodesic connecting x_1 to $\hat{z}_1$, with x_1' a point on σ which intersects $\bar{b}_+^{-1}(0)$. Let z_1' be the point on $\bar{b}_+^{-1}(b)$ closest to z_1. Then by equation (3.20), we have

$$d(z_1, z_1') = d(x_1', \hat{z}_1) \pm \psi = d(z_2, x_2) \pm \Psi. \tag{3.33}$$

Moreover, by Proposition 3.9, we have

$$d(z_1, z_2)^2 + d(z_2, x_2)^2 = d(z_1, x_2)^2 \pm \Psi, \tag{3.34}$$

$$d(z_1, z_1')^2 + d(z_1', x_2)^2 = d(z_1, x_2)^2 \pm \Psi. \tag{3.35}$$

Then, by equations (3.33), (3.34), (3.35), we have

$$d(z_1, z_2) = d(x_2, z_1') \pm \Psi. \tag{3.36}$$

Claim. $d(\hat{z}_1, z_1') \leq \Psi$.

Let y' and $\hat{y}$ be the point on $\bar{b}_+^{-1}(0)$ closest to z_1' and $\hat{z}_1$ respectively. Applying Proposition 3.9 and equation (3.20) to the triples z_1, y', z_1', and x_1', $\hat{y}$, $\hat{z}_1$ give $d(z_1, y') \leq \Psi$ and $d(x_1', \hat{y}) \leq \Psi$. Applying Proposition 3.9 to the triples $\hat{y}$, y', $\hat{z}_1$ and $\hat{z}_1$, z_1', y' and combining these results will show that

$$d(\hat{z}_1, z_1') = d(x_1', z_1) \pm \Psi. \tag{3.37}$$

To see that $d(x_1', z_1) \leq \psi$, let z_1'' be the point on $\bar{b}_+^{-1}(a)$ closest to z_1. Applying Proposition 3.9 to the triples x_1, z_1'', z_1, and x_1, z_1, x_1', then show that $d(x_1, z_1'') = d(x_1', z_1) \pm \Psi$. Moreover, by equation (3.20), we see $d(x_1, z_1'') \leq \Psi$. These facts, combined with equation (3.37) give

$$d(\hat{z}_1, z_1') \le \Psi,$$

as desired.

With this claim and equation (3.36) we see that

$$d(z_1, z_2) = d(x_2, \hat{z}_1) \pm \Psi. \tag{3.38}$$

By the proof of the claim, we also see that $d(x_1, x_1') = d(x_1, z_1) \pm \Psi$. This, together with equation (3.33) give

$$d(x_1, \hat{z}_1) = d(x_1, x_1') + d(x_1', \hat{z}_1). \tag{3.39}$$

Proposition 3.9 applied to the triple x_1, $\hat{z}_1$, x_2 combined with equations (3.38) and (3.39) give

$$\begin{aligned} d(x_1, x_2)^2 &= d(x_1, \hat{z}_1)^2 + d(\hat{z}_1, x_2)^2 \pm \Psi \\ &= d(z_1, z_2)^2 + (d(x_1, z_1) + d(x_2, z_2))^2 \pm \Psi. \end{aligned}$$

Thus θ is a Ψ-almost isometry.

To see that θ is Ψ-almost onto, let $(l, y) \in B((0, x), R/4) \subset \mathbb{R} \times \bar{b}_+^{-1}(0)$. Let γ be the geodesic such that $\gamma(0) = y$ and $\dot{\gamma}(0) = \nabla \bar{b}_+(y)$. Set $x = \gamma(l)$. If y is the point on the level set $\bar{b}_+^{-1}(0)$ closest to x, then we are done. So, suppose $z \neq y$ is the point on $\bar{b}_+^{-1}(0)$ closest to x. Then $\theta(x) = (d(z, x), x)$. Applying Proposition 3.9 to z,x, and y, we see that $d(y, z)^2 + d(z, x)^2 = d(y, x)^2 \pm \Psi$. But since $d(z,x) = d(x,y) \pm \Psi = l \pm \Psi$ by equation (3.20), it follows that $d(z,y) \leq \Psi$. Thus

$$
\begin{aligned}
d(\theta(x),(l,y))^2 &= d((d(x,z),z),(l,y))^2 \\
&= |d(x,z) - l|^2 + d(z,y)^2 \\
&\leq \Psi
\end{aligned}
$$

Thus θ is a Gromov-Hausdorff approximation. □

From this Almost Splitting Theorem for smooth metric measure spaces, it follows that the splitting theorem extends to the limit of a sequence of smooth metric measure spaces in the following manner.

Proof of Theorem 1.6. Let $\epsilon > 0$ and $R > 0$ be given. Let $\ell : (-\infty, \infty) \to Y$ denote the line contained in Y. Choose points p, q_-, q_+ which lie on ℓ and satisfy $\min\{d(p,q_-), d(p,q_+)\} \geq L$, and $e(p) = 0$. Since $M_i \to Y$ in the Gromov-Hausdorff sense, we have $d_{GH}(M_i, Y) = \epsilon_i \to 0$ as $i \to \infty$. Thus for each $i \in \mathbb{N}$ there exist ϵ_i-Gromov-Hausdorff approximations from $f_i : M_i \to Y$ and $g_i : Y \to M_i$. In particular, $|d_Y(p, q_\pm) - d_{M_i}(g_i(p), g_i(q_\pm)| \leq \epsilon_i$ and $|d_Y(q_+, q_-) - d_{M_i}(g_i(q_+), g_i(q_-))| \leq \epsilon_i$. Then for each i, we have

$$
d_{M_i}(g_i(p), g_i(q_\pm)) \geq 2R + 1 - \epsilon_i = L \quad \text{and} \quad e(g_i(p)) \leq 3\epsilon_i + \epsilon.
$$

Then for $\bar{R} = (2R - \epsilon_i)/2$, $L \geq 2\bar{R} + 1$, and $3\epsilon_i + \epsilon = \bar{\epsilon}_i$ the Quantitative Splitting Theorem 3.10 gives

$$d_{GH}\left(B\left(g_i(p), \frac{\bar{R}}{4}\right), B\left((0,x), \frac{\bar{R}}{4}\right)\right) \leq \Psi_i(\delta_i, L^{-1}, \bar{\epsilon}_i | k, n, \bar{R}).$$

$B\left(g_i(p), \frac{\bar{R}}{4}\right)$ and $B\left((0,x), \frac{\bar{R}}{4}\right)$. In particular, we have a Ψ-Gromov-Hausdorff approximation $h_i : B\left((0,x), \frac{\bar{R}}{4}\right) \to B\left(g_i(p), \frac{\bar{R}}{4}\right)$. The composition $f_i h_i$ is then a $\Psi_i + \epsilon_i$ approximation between $B((0,x), \bar{R}/4)$ and a subset of Y. Letting $i \to \infty$ and $L^{-1} \to 0$ gives that the Gromov-Hausdorff distance between these two sets tends to 0. Since this holds for all $R > 0$, we have that Y splits isometrically as a product $X \times \mathbb{R}$ for some length space X. $\square$

Again, we note that a splitting theorem for limit spaces of sequences of smooth metric measure spaces has also been proven by Wang-Zhu, see [WZ13, Theorem 3.1]. The gradient estimate, Proposition 3.2, used for the proof of our Theorem 1.6 allows us to relax the conditions on the potential functions in the sequence, requiring only that $|f_i|$ for each i is bounded, rather than both $|f_i|$ and $|\nabla f_i|$ as in Theorem 3.1 of Wang-Zhu.

Chapter 4

Almost Abelian Fundamental Groups

In this chapter, we develop a new absolute volume comparison (Proposition 4.1). With this absolute volume comparison, we extend the Anderson's finiteness theorem to the smooth metric measure space setting (Proposition 4.3). We then show that a manifold in a certain class of smooth metric measure spaces has a fundamental group with polynomial growth (Theorem 4.4). We conclude with the proof of 1.4.

4.1 An Absolute Volume Comparison

In order to show that the fundamental group of a certain class of compact smooth metric measure spaces with almost nonnegative Bakry-Émery Ricci curvature is almost abelian, i.e. that it contains an abelian subgroup of finite index, we first show that under the same conditions, the fundamental group is of polynomial growth. Let us recall that this result is an extension Theorem 2.8 to the smooth metric measure space setting.

As noted previously, the Bishop-Gromov volume comparison, Theorem 1.1 is one of the key tools used in the proof of the above. A key feature of this volume comparison is that it yields an absolute volume comparison. In particular, since

$$\frac{\mathrm{V}ol(B(p,R))}{\mathrm{V}ol_H^n(B(R))} \leq \frac{\mathrm{V}ol(B(p,r))}{\mathrm{V}ol_H^n(B(r))}$$

for $0 < r \leq R$, and the right-hand side of the inequality tends to 1 as r tends to 0, we have $\mathrm{V}ol(B(p,R)) \leq \mathrm{V}ol_H^n(B(R))$ for $R > 0$.

The relative volume comparison on smooth metric measure spaces formulated in Theorem 2.14 only yields a volume growth estimate for $R > 1$ since, as noted by Wei and Wylie [WW09], the right hand side of

$$\frac{\mathrm{V}ol_f(B(p,R))}{\mathrm{V}ol_H^{n+4k}(R)} \leq \frac{\mathrm{V}ol_f(B(p,r))}{\mathrm{V}ol_H^{n+4k}(r)}$$

blows up as $r \to 0$. Using this type of estimate to extend Wei's proof methods to the smooth metric measure space setting is indeed possible. If one assumes that for the potential function f we have $|f|\leq k$ and $|\nabla f|\leq k$, then we can apply Wei's proof techniques to show that the fundamental group of a smooth metric measure space in the class we are considering will have polynomial growth of degree at most

$n+4k$. In order to improve the degree to n with only the additional assumption that $|f|\leq k$, we formulate the following volume estimate.

Proposition 4.1. *Let $(M^n, g, e^{-f}dvol_g)$ be a smooth metric measure space with $Ric_f \geq (n-1)H$, $H<0$, and $|f|\leq k$. Let $p\in M$. Then*

$$Vol_f(B(p,R)) \leq k\int_0^R \mathcal{A}_H(r)e^{2k[\cosh(2\sqrt{-H}r)+1]}dr, \tag{4.1}$$

where $\mathcal{A}_H(r)dr$ denotes the volume element on the model space with constant curvature H.

Proof. Let $sn_H(r)$ be the solution to $sn_H'' + Hsn_H = 0$ such that $sn_H(0)=0$ and $sn_H'(0)=1$. When $H<0$, this solution is given by

$$\frac{1}{\sqrt{-H}}\sinh\sqrt{-H}r. \tag{4.2}$$

From the proof [WW09, Theorem 1.1, inequality (2.17)], we have

$$sn_H^2(r)m_f(r) \leq sn_H^2(r)m_H(r) - f(r)(sn_H^2(r))' + \int_0^r f(t)(sn_H^2)''(t)dt. \tag{4.3}$$

Then integrating both sides of (4.3) from $r=r_1$ to r_2 gives

$$\begin{aligned}\int_{r_1}^{r_2} m_f(r)dr \leq \int_{r_1}^{r_2} m_H(r)dr &- \int_{r_1}^{r_2} f(r)\frac{(sn_H^2(r))'}{sn_H^2(r)}dr \\ &+ \int_{r_1}^{r_2}\frac{1}{sn_H^2(r)}\left\{\int_0^r f(t)(sn_H^2)''(t)dt\right\}dr \\ = \int_{r_1}^{r_2} m_H(r)dr &- 2\sqrt{-H}\int_{r_1}^{r_2} f(r)\coth\sqrt{-H}rdr \\ + 2(-H)\int_{r_1}^{r_2}&\operatorname{csch}^2\sqrt{-H}r\left\{\int_0^r f(t)[\sinh^2\sqrt{-H}t+\cosh^2\sqrt{-H}t]dt\right\}dr\end{aligned}$$

$$
\begin{aligned}
&= \int_{r_1}^{r_2} m_H(r)dr - 2\sqrt{-H}\int_{r_1}^{r_2} f(r)\coth\sqrt{-H}rdr \\
&\qquad + 2(-H)\int_{r_1}^{r_2} \operatorname{csch}^2\sqrt{-H}r\left\{\int_0^r f(t)\cosh 2\sqrt{-H}tdt\right\}dr \\
&= \int_{r_1}^{r_2} m_H(r)dr - 2\sqrt{-H}\int_{r_1}^{r_2} f(r)\coth\sqrt{-H}rdr \\
&\qquad + 2(-H)\left[-\frac{\coth\sqrt{-H}r}{\sqrt{-H}}\int_0^r f(t)\cosh 2\sqrt{-H}tdt\right]_{r_1}^{r_2} \\
&\qquad\quad + 4(-H)\int_{r_1}^{r_2}\frac{\coth\sqrt{-H}r}{\sqrt{-H}}f(r)\sinh^2\sqrt{-H}rdr \\
&\qquad\qquad + 2(-H)\int_{r_1}^{r_2}\frac{\coth\sqrt{-H}r}{\sqrt{-H}}f(r)dr \\
&\leq \int_{r_1}^{r_2} m_H(r)dr + k\coth\sqrt{-H}r_2\sinh 2\sqrt{-H}r_2 \\
&+ k\coth\sqrt{-H}r_1\sinh 2\sqrt{-H}r_1 + 2k[\sinh^2\sqrt{-H}r_2 - \sinh^2\sqrt{-H}r_1] \\
&= \int_{r_1}^{r_2} m_H(r)dr + 2k[\cosh(2\sqrt{-H}r_2) + 1],
\end{aligned}
$$

where the first equality is obtained by substituting (4.2) for sn_H, and the third equality is obtained through integration by parts.

Using exponential polar coordinates around p, we may write the volume element of M as $\mathcal{A}(r,\theta)\wedge d\theta_{n-1}$ where $d\theta_{n-1}$ is the standard volume element of the unit sphere $\mathbb{S}^{n-1}$. Let $\mathcal{A}_f(r,\theta) = e^{-f}\mathcal{A}(r,\theta)$ and $\mathcal{A}_H(r)$ denotes the volume element for the model space with constant curvature H. The mean curvatures on the smooth metric measure space and on the model space can be written, respectively, as

$$
m_f(r) = (\ln(\mathcal{A}_f(r,\theta))' \quad \text{and} \quad m_H(r) = (\ln(\mathcal{A}_H(r))'.
$$

Then we may rewrite the above inequality as

$$\ln\left(\frac{\mathcal{A}_f(r_2,\theta)}{\mathcal{A}_f(r_1,\theta)}\right) \leq \ln\left(\frac{\mathcal{A}_H(r_2)}{\mathcal{A}_H(r_1)}\right) + 2k[\cosh(2\sqrt{-H}r_2)+1].$$

Hence

$$\frac{\mathcal{A}_f(r_2,\theta)}{\mathcal{A}_f(r_1,\theta)} \leq \frac{\mathcal{A}_H(r_2)}{\mathcal{A}_H(r_1)} e^{2k[\cosh(2\sqrt{-H}r_2)+1].}$$

Then

$$\mathcal{A}_f(r_2,\theta)\mathcal{A}_H(r_1) \leq \mathcal{A}_H(r_2)\mathcal{A}_f(r_1,\theta)e^{2k[\cosh(2\sqrt{-H}r_2)+1]}.$$

Integrating both sides of the inequality over $\mathbb{S}^{n-1}$ with respect to θ yields

$$\mathcal{A}_H(r_1)\int_{\mathbb{S}^{n-1}} \mathcal{A}_f(r_2,\theta)d\theta \leq \mathcal{A}_H(r_2)e^{2k[\cosh(2\sqrt{-H}r_2)+1]}\int_{\mathbb{S}^{n-1}} \mathcal{A}_f(r_1,\theta)d\theta.$$

Then we integrate both sides of the inequality with respect to r_1 from $r_1 = 0$ to $r_1 = R_1$:

$$Vol_H(B(R_1))\int_{\mathbb{S}^{n-1}} \mathcal{A}_f(r_2,\theta)d\theta \leq Vol_f(B(p,R_1))\mathcal{A}_H(r_2)e^{2k[\cosh(2\sqrt{-H}r_2)+1]}.$$

Finally, we integrate both sides of the inequality with respect to r_2 from $r_2 = 0$ to $r_2 = R_2$:

$$Vol_H(B(R_1))Vol_f(B(p,R_2)) \leq Vol_f(B(p,R_1))\int_0^{R_2} \mathcal{A}_H(r_2)e^{2k[\cosh(2\sqrt{-H}r_2)+1]}dr_2,$$

thus yielding a new volume inequality:

$$\frac{Vol_H(B(R_1))}{Vol_f(B(p,R_1)} \leq \frac{\int_0^{R_2} \mathcal{A}_H(r_2)e^{2k[\cosh(2\sqrt{-H}r_2)+1]}dr}{Vol_f(B(p,R_2))}.$$

Note that the left hand side of the inequality tends to $\frac{1}{f(p)}$ as $R_1 \to 0$. Then

$$Vol_f(B(p, R_2)) \leq f(p) \int_0^{R_2} \mathcal{A}_H(r_2) e^{2k[\cosh(2\sqrt{-H}r_2)+1]} dr_2. \qquad \square$$

4.2 The Finiteness Theorem

Recall that Anderson's finiteness result, Theorem 2.7, gives that for the class of n-dimensional Riemannian manifolds satisfying $\mathrm{Ric}(M) \geq (n-1)H$, $\mathrm{V}ol(M) \geq v$, and $\mathrm{diam}\,(M) \leq D$, there are only finitely many isomorphism classes of $\pi_1(M)$. This finiteness result is essential in the proof of Theorem 2.8. Since we seek to extend Theorem 2.8 to the smooth metric measure space setting, it would be useful to have a smooth metric measure space version of Theorem 2.7. We note that such a result is stated in Wei-Wylie [WW09, Theorem 4.7] without proof. For completeness, we include the statement with proof at the end of this section. The proof we include utilizes the absolute volume comparison, Proposition 4.1.

Related to this result, and used explicitly in the proof showing that the fundamental group is almost abelian, is the fact that for M in the same class as above, the length of noncontractible curves in M can be controlled [And90, Theorem 2.1].

Below, we show that we can use the volume comparison, Proposition 4.1 to control the length of noncontractible curves in smooth metric measure spaces in the following manner.

Proposition 4.2. *Let $(M^n, g, e^{-f}dvol_g)$ be a smooth metric measure space with $|f|\leq k$ satisfying the bounds $Ric_f \geq (n-1)H$ where $H < 0$, $diam_M \leq D$ and $Vol_f(M) \geq v$. If γ is a loop in M such that $[\gamma]^p \neq 0$ for $p \leq N = \frac{k}{v}\int_0^{2D} \mathcal{A}_H e^{2k[\cosh(2\sqrt{-H}r)+1]}dr$, then*

$$l(\gamma) \geq \frac{D}{N}.$$

The proof of Proposition 4.2 follows Anderson's proof of [And90, Theorem 2.1], where in place of the absolute volume comparison of Bishop and Gromov, we use Proposition 4.1. A sketch of the proof is provided below.

Proof. Consider the subgroup $\Gamma = \langle\gamma\rangle$ of $\pi_1(M) = \pi_1(M, x_0)$ where elements act as deck transformations on the universal cover $\widetilde{M}$ of M. Choose $\widetilde{x}_0 \in \widetilde{M}$ such that $\tilde{x_0} \to x_0$ under the covering map. Then, choose $F \subseteq \tilde{M}$ be a fundamental domain of $\pi_1(M)$ containing $\tilde{x_0}$.

Let $U(r) = \{g \in \Gamma : g = \gamma^i, |i|\leq r\}$. Since $[\gamma]^p \neq 0$ in $\pi_1(M)$ for $p \leq N$, we have $|\Gamma|\geq N$ and we may choose the smallest $r = r_0$ such $\#U(r_0) > N$. Note now that

$$\bigcup_{g\in U(r_0)} g(B(\tilde{x_0}, D) \cap F) \subseteq B(\tilde{x_0}, Nl(\gamma) + D).$$

Then, by (4.1), we have

$$N \cdot \mathrm{V}ol_f M \leq \mathrm{V}ol_f(B(\tilde{x_0}, Nl(\gamma) + D)) \leq k\int_0^{Nl(\gamma)+D} \mathcal{A}_H(r)e^{2k[\cosh(2\sqrt{-H}r)+1]}dr. \tag{4.4}$$

Seeking contradiction, suppose that $l(\gamma) \leq \frac{D}{N}$. Then by equation (4.4), we have

$$N < \frac{k}{v} \int_0^{2D} \mathcal{A}_H(r) e^{2k[\cosh(2\sqrt{-H}r)+1]} dr,$$

contradicting the definition of N. $\square$

As indicated in the proof of [And90, Theorem 2.2], this control on the length of the loops can be used to show that the fundamental group has finitely many isomorphism types. In particular, Anderson uses this lower bound on length of noncontractible loops and the Bishop-Gromov volume comparison to argue that there is a bound on the number of generators. Essential in Anderson's proof is the theorem of Gromov [Gro01, Proposition 5.28] which guarantees a set of generators $g_1, \ldots, g_l$ of $\pi_1(M)$ such that $d(g_i(\tilde{x}_0), \tilde{x}_0) \leq 2D$ and every relation is of the form $g_i g_j = g_k$. Finding a bound on the number of generators is then sufficient, as the group will then be determined by the relations on those generators.

Below, we extend Anderson's finiteness theorem to smooth metric measure spaces, using our absolute volume comparison, Proposition 4.1.

Proposition 4.3. *For the class of manifolds M^n with $Ric_f \geq (n-1)H$, $Vol_f \geq v$, $diam(M) \leq D$ and $|f| \leq k$, there are only finitely many isomorphism types of $\pi_1(M)$.*

Proof. Choose generators $g_1, \ldots, g_l$ of $\pi_1(M, x_0)$ as guaranteed by Gromov's theorem. Again, it will be sufficient to show that the number of generators of $\pi_1(M, x_0)$

is bounded. Choose $\tilde{x_0} \in \tilde{M}$ such that $\tilde{x_0} \to x_0$ under the covering map. Let F be a fundamental domain in $\tilde{M}$ containing $\tilde{x_0}$. Then

$$\bigcup_{i=1}^{l} g_i(F) \subset B(\tilde{x_0}, 3D).$$

This implies

$$l \leq \frac{\mathrm{Vol}_f(B(\tilde{x_0}, 3D))}{\mathrm{Vol} M} \leq \frac{k}{v} \int_0^{3D} \mathcal{A}_H(r) e^{2k[\cosh(2\sqrt{-H}r)+1]} dr. \qquad \square$$

4.3 Polynomial Growth of the Fundamental Group

Now, with Proposition 4.1 and Proposition 4.3, we may extend Wei's theorem about polynomial growth of the fundamental group [Wei90] to smooth metric measure spaces.

Theorem 4.4. *For any constant $v \geq 0$, there exists $\epsilon = \epsilon(n, v, k, H, D) > 0$ such that if a smooth metric measure space $(M^n, g, e^{-f} dvol_g)$ with $|f| \leq k$ satisfies the conditions (1.1) - (1.3), then the fundamental group of M is of polynomial growth of degree $\leq n$.*

Proof. Let $\Gamma(s)$ denote the growth function of $\pi_1(M)$, as in definition 2.2. Let us assume by means of contradiction that $\pi_1(M)$ is not of polynomial growth with degree $\leq n$. It follows that for any set of generators of $\pi_1(M)$, we can find real numbers s_i for all i, such that

$$\Gamma(s_i) > i s_i^n. \tag{4.5}$$

Choose a base point $\tilde{x}_0$ in the universal covering $p : \tilde{M} \to M$, and let $x_0 = p(\tilde{x}_0)$. By Proposition 4.3, there are only finitely many isomorphism types of $\pi_1(M)$. For each isomorphism type, choose a set of generators $g_1, \ldots, g_N$ of $\pi_1(M)$ such that $d(g_i(\tilde{x}_0), \tilde{x}_0) \leq 3D$ and every relation is of the form $g_i g_j = g_k$. Again, such a set of generators is guaranteed by a theorem of Gromov [Gro01, Proposition 5.28]. By the proof of Proposition 4.3, we know that N is uniformly bounded. Having chosen generators in this manner, we are guaranteed that (4.5) is independent of ϵ. View this set of generators of the fundamental group $\pi_1(M)$ as deck transformations in the isometry group of $\tilde{M}$.

Now, choose a fundamental domain F of $\pi_1(M)$ containing $\tilde{x}_0$. Then

$$\bigcup_{g \in \Gamma(s)} g(F) \subseteq B(\tilde{x}_0, D(3s+1))),$$

which implies

$$\Gamma(s) \leq \frac{1}{v} \mathrm{V}ol_f(B(\tilde{x}_0, D(3s+1))).$$

Then, by Proposition 4.1, it follows that

$$\Gamma(s) \leq \frac{k}{v} \int_0^{D(3s+1)} \frac{\sinh \sqrt{\epsilon} r}{\epsilon} e^{2k[\cosh(2\sqrt{\epsilon} r)+1]} dr.$$

For any fixed, sufficiently large s_0, there exists $\epsilon_0 = \epsilon(s_0)$ such that for all $\epsilon \leq \epsilon_0$, we have

$$\Gamma(s) \leq \frac{2^{3n} e^4 k}{nv} s^n. \tag{4.6}$$

Let $i_0 > \frac{2^{3n}e^4k}{nv}$. Then $\epsilon < \epsilon(s_{i_0})$ together with (4.5) and (4.6) yields a contradiction. $\square$

4.4 Proof of Theorem 1.4

With Theorems 1.6, 4.4, and Proposition 4.2, we may now generalize the arguments in [Yun97] to the smooth metric measure space setting.

Proof of Theorem 1.4. By Theorem 4.4, there exists $\epsilon_0 = \epsilon_0(n, v, k, H, D) > 0$ such that if a smooth metric measure space $(M, g, e^{-f}\mathrm{dvol}_g)$ with $|f|< k$, satisfies (1.1) - (1.3), then $\pi_1(M)$ is a finitely generated group of polynomial growth of order $\leq n$.

Assume Theorem 1.4 is not true. Then there exists a contradicting sequence of smooth metric measure spaces $(M_i, g_i, e^{-f_i}\mathrm{dvol}_{g_i})$ with $|f_i|\leq k$ and

$$\mathrm{Ric}_{f_i}(M_i) \geq -\epsilon_i \to 0, \quad \epsilon_i \leq \epsilon_0, \quad \mathrm{V}ol_{f_i}(M_i) \geq v, \quad \mathrm{diam}\,(M_i) \leq D,$$

such that $\pi_1(M_i)$ is not almost abelian for each i. Note, however, $\pi_1(M_i)$ is of polynomial growth for each i.

Since $\pi_1(M_i)$ is of polynomial growth, [Yun97, Lemma 1.3] implies it contains a torsion free nilpotent subgroup Γ_i of finite index. Since Γ_i has finite index in $\pi_1(M)$, it must be nontrivial. Furthermore Γ_i cannot be almost abelian.

Consider the action Γ_i on the universal cover $\widetilde{M_i}$. For $p_i \in \widetilde{M_i}$ consider the sequence $(\widetilde{M_i}, \Gamma_i, p_i)$. By Theorem 2.26 there exists a length space (Y, q) and a closed subgroup G of $Isom(Y)$ such that $(\widetilde{M_i}, \Gamma_i, p_i)$ subconverges to a triple (Y, G, q) with respect to the pointed equivariant Gromov-Hausdorff distance.

Using the Almost Splitting Theorem 1.6, we know Y splits as an isometric product $Y = \mathbb{R}^k \times Y_0$ for some k and length space Y_0 which contains no lines. By Proposition 4.3 it follows that $[\pi_1(M_i) : \Gamma_i]$ is uniformly bounded, say $[\pi_1(M_i) : \Gamma_i] \leq m$. Hence $\operatorname{diam}(\widetilde{M_i}/\Gamma_i) \leq Dm$. Then by Theorem 2.25, $(\widetilde{M_i}, \Gamma_i, p_i) \to (\mathbb{R}^k \times Y_0, G, q)$ implies that $(\widetilde{M_i}/\Gamma_i, \overline{p_i}) \to ((\mathbb{R}^k \times Y_0)/G, \overline{q})$ in the pointed Gromov-Hausdorff sense. Then it follows that $\operatorname{diam}(\mathbb{R}^k \times Y_0/G) \leq Dm$. Then Y_0 must be compact. Otherwise, it would contain a line. Thus we may consider the projection

$$\phi : G \to Isom(\mathbb{R}^k).$$

By [FY92, Theorem 6.1], for every $\delta > 0$ there exists a normal subgroup G_δ of G such that G/G_δ contains a finite index, free abelian group of rank not greater than $\dim(\mathbb{R}^k/\phi(G))$. Since Γ_i is torsion free, Proposition 4.2 gives that for all nontrivial $\gamma \in \Gamma_i$, we have $l(\gamma) \geq \frac{D}{N}$ where $N = \frac{k}{v}\int_0^{2D} \mathcal{A}_{\epsilon_0} e^{2k[\cosh(2\sqrt{\epsilon_0}r)+1]}dr$. Choose $\delta = \frac{D}{N}$ and set $\delta_0 = \delta/2$.

Define

$$\Gamma_i(\delta) = \{\gamma \in \Gamma_i : d(p_i, \gamma(p_i)) < \delta\}.$$

Similarly, define

$$G(\delta) = \{\gamma \in G : d(q, \gamma(q)) < \delta\}.$$

Then

$$\Gamma_i(\delta) = \{1\}.$$

Since $(\widetilde{M}_i, \gamma_i, p_i) \to (\mathbb{R}^k \times Y_0, G, q)$, it follows that

$$G(\delta_0) = \{1\}.$$

Let K denote the kernel of ϕ. Since $\delta_0 > 0$ was chosen so that $G(\delta_0) = \{1\}$, it follows that

$$\{\gamma \in K | d(\gamma(x), x) < \delta_0 \text{ for all } x \in Y\} = \{1\}.$$

Thus the subgroup generated by this set is trivial. That is,

$$K_{\delta_0} = \langle\{\gamma \in K | d(\gamma(x), x) < \delta \text{ for all } x \in Y\}\rangle = \{1\}.$$

Then, the quotient map

$$\pi : G \to G/K_{\delta_0}$$

is simply the identity map. The subgroup G_{δ_0} of G which has the properties we seek is defined by

$$G_{\delta_0} = \pi^{-1}([1]),$$

where [1] denotes the coset containing the identity element of G/K_{δ_0}. But since K_{δ_0} is trivial and π is the identity map, it follows that $G_{\delta_0} = \{1\}$. Thus by [FY92, Lemma 6.1], $G/G_{\delta_0} = G$ contains a finite index free abelian subgroup of rank $\leq k$; that is, G is almost abelian. Moreover, [FY92, Theorem 3.10] we have that Γ_i is isomorphic to G for i sufficiently large. But this contradicts the fact that Γ_i is not almost abelian for each i. $\square$

Chapter 5

Bound on Number of Generators of the Fundamental Group

The question of how curvature affects generating sets of fundamental groups can be an interesting one. Under the stronger assumption of a complete manifold with nonnegative sectional curvature, Gromov has given an upper bound on the number of generators of the fundamental group [Gro78, Theorem 1.5]. As previously mentioned, the analogous statement for complete manifolds with nonnegative Ricci curvature is a conjecture of Milnor which still remains open. Though we know that compact manifolds have finitely generated fundamental groups, one may still ask about bounds for the number of generators. In particular, one can ask if there exists a uniform bound on the number of generators of the fundamental group of compact manifolds for a certain class. Wei gave a such a bound for the class of compact manifolds with Ricci curvature and conjugate radius bounded from below and diameter bounded from above [Wei97, Theorem 1.3]. Kapovitch and Wilking show that a uniform bound exists without the lower bound on conjugate radius (see Theorem 2.10). In this chapter, we show that Theorem 2.10 extends to the smooth metric measure space setting.

We begin this chapter with discussion of two results related to the splitting theorem, which we will utilize in generalizing Theorem 2.10. We also review some of the essential lemmas from [KW11]. We conclude the chapter with the proof of a more general version of Theorem 1.5 (see Theorem 5.9).

5.1 Results Related to the Splitting Theorem

In order to obtain a uniform bound on the number of generators of the fundamental group, Kapovitch and Wilking require two results closely related to the Cheeger-Colding Splitting Theorem. The first of these results is due to Cheeger and Colding [CC00, see Section 1], see also [Che07, Theorem 9.29].

Theorem 5.1. *Given $R > 0$ and $L > 2R + 1$, let $Ric(M^n) \geq -(n-1)\delta$ and $d_{GH}(B(p, L), B(0, L)) \leq \delta$, where $B(0, L) \subset \mathbb{R}^n$. Then there exist harmonic functions $\bar{b}_1, \ldots, \bar{b}_n$ on $B(p, R)$ such that in the Gromov-Hausdorff sense $d(e_i, \bar{b}_i) \leq \Psi$,* *where* $\{e_i\}$ *denote the standard coordinate functions on* $\mathbb{R}^n$ *and*

$$\fint_{B(p,R)} \sum_i |\nabla \bar{b}_i - 1|^2 + \sum_{i \neq j} |\langle \nabla \bar{b}_i, \nabla \bar{b}_j \rangle| + \sum_i |Hess \bar{b}_i|^2 \leq \Psi. \tag{5.1}$$

Here, Ψ is a nonnegative function as defined in equation (3.10). In the smooth metric measure space setting, a similar statement may be made:

Theorem 5.2. *Given* $R > 0$ *and* $L > 2R + 1$, *let* $Ric_f \geq -(n-1)\delta$, *with* $|f| \leq k$ *and* $d_{GH}(B(p,L), B(0,L)) \leq \delta$, *where* $B(0,L) \subset \mathbb{R}^n$. *Then there exist* f*-harmonic functions* $\bar{b}_1, \ldots, \bar{b}_n$ *on* $B(p,R)$ *such that in the Gromov-Hausdorff sense* $d(e_i, \bar{b}_i) \leq \Psi$, *where* $\{e_i\}$ *denote the standard coordinate functions on* $\mathbb{R}^n$ *and*

$$\fint_{B(p,R)} \left(\sum_i |\nabla \bar{b}_i - 1|^2 + \sum_{i \neq j} |\langle \nabla \bar{b}_i, \nabla \bar{b}_j \rangle| + \sum_i |Hess \bar{b}_i|^2 \right) e^{-f} dvol_g \leq \Psi. \tag{5.2}$$

Proof. The manner in which the harmonic functions $\bar{b}_i$ are constructed for Theorems 5.1 and 5.2 is similar to the manner in which the harmonic functions $\bar{b}_{\pm}$ are constructed in the proof of the Almost Splitting Theorem in both the Riemannian and smooth metric measure space settings. Since the two L-balls are δ-close in the Gromov-Hausdorff sense, there exists a δ-Gromov-Hausdorff approximation

$$F : B(0,L) \to B(p,L).$$

For each $i = 1, \ldots, n$, set

$$q_i = F(Le_i)$$

and define $b_i : M \to \mathbb{R}$ by

$$b_i(x) = d(x, q_i) - d(p, q_i).$$

For the smooth metric measure space version, let $\bar{b}_i$ be the f-harmonic function such that $\bar{b}_i|_{\partial B(p,R)} = b_i|_{\partial B(p,R)}$. Integrating each term of equation (5.2) separately, we see that the first term can be controlled by (3.17) and the third term by (3.18). One can show a Ψ-upper bound for the middle term of the integrand (5.2) by noting that

$$\begin{aligned}\langle \nabla \bar{b}_i, \nabla \bar{b}_j \rangle &= \langle \nabla \bar{b}_i - \nabla b_i + \nabla b_i, \nabla \bar{b}_j - \nabla b_j + \nabla b_j \rangle \\ &= \langle \nabla \bar{b}_i - \nabla b_i, \nabla \bar{b}_j \rangle + \langle \nabla \bar{b}_j - \nabla b_j, \nabla \bar{b}_i \rangle + \langle \nabla b_i, \nabla b_j \rangle\end{aligned}$$

Using integration by parts and (3.16), one can show that the average value of each of the first two terms of the summand is bounded from above by Ψ. Moreover, $\langle \nabla b_i, \nabla b_j \rangle \to 0$ as $L \to \infty$. $\square$

The Product Lemma of Kapovitch and Wilking, stated below, can be viewed as another type of splitting result.

Theorem 5.3. *[KW11, Lemma 2.1] Let M_i be a sequence of manifolds with $Ric_{M_i} > -\epsilon_i \to 0$ satisfying*

- *$\overline{B_{r_i}(p_i)}$ is compact for all i with $r_i \to \infty$, $p_i \in M_i$,*
- *for all i and $j = 1, \dots, k$ there exist harmonic functions $\overline{b}_j^i : B(p_i, r_i) \to \mathbb{R}$ which are L-Lipschitz and fulfill*

$$\fint_{B(p_i,R)} \left(\sum_{j,l=1}^{k} |\langle \nabla \overline{b}_j^i, \nabla \overline{b}_l^i \rangle - \delta_{jl}| + \sum_{j=1}^{k} |Hess \overline{b}_j^i|^2 \right) d\mu_i \to 0 \qquad \text{for all } R > 0,$$

then $(B(p_i, r_i), p_i)$ subconverges in the pointed Gromov-Hausdorff topology to a metric product $(\mathbb{R}^k \times X, p_\infty)$ for some metric space X.

Let (Y, p_∞) be the subsequential limit of $(B(p_i, r_i), p_i)$. Without the assumption that a line exists in the limit space, Kapovitch and Wilking instead show that each of the functions $\overline{b}_j^i$, as in the hypothesis of Theorem 5.3, limit to a submetry $\overline{b}_j^\infty$ from Y to $\mathbb{R}$ as $i \to \infty$. Then one uses the fact that submetries lift lines to lines in order to apply the Cheeger-Colding Splitting Theorem to show that Y indeed splits. Their argument may be modified to the smooth metric measure space setting by using the volume comparison, Theorem 2.14, the Segment Inequality, Proposition 3.3, the Splitting Theorem 1.6, and the fact that gradient flow of an f-harmonic function is measure preserving with respect to the weighted measure $e^{-f}\text{dvol}_g$. Augmenting their arguments in this manner yields the following extension.

Theorem 5.4. *Let $(M_i, g_i, e^{-f_i}dvol_{g_i})$ be a sequence of smooth metric measure spaces with $|f_i|\leq k$ and $Ric_{f_i} > -\epsilon_i \to 0$. Suppose that $r_i \to \infty$ and for every i and $j = 1, \ldots, m$, there are harmonic functions $\overline{b}^i_j : B(p_i, r_i) \to \mathbb{R}$ which are L-Lipschitz and fulfill*

$$\fint_{B(p_i,R)} \left(\sum_{j,l=1}^{m} |\langle \nabla \overline{b}^i_j, \nabla \overline{b}^i_l \rangle - \delta_{jl}| + \sum_{j=1}^{m} |Hess \overline{b}^i_j|^2 \right) e^{-f_i} dvol_{g_i} \to 0 \text{ for all } R > 0.$$

Then $(B(p_i, r_i), p_i)$ subconverges in the pointed Gromov-Hausdorff topology to a metric product $(\mathbb{R}^m \times X, p_\infty)$ for some metric space X.

5.2 Essential Lemmas

In order to prove Theorem 5.9 in the final section of this chapter we review and extend, when necessary, lemmas used by Kapovitch-Wilking to prove Theorem 2.10. The following lemma of Kapovitch and Wilking requires only an inner metric space structure and hence may be applied to smooth metric measure spaces.

Lemma 5.5. *[KW11, Lemma 2.2] Let $(Y_i, \tilde{p}_i)$ be an inner metric space endowed with an action of a closed subgroup G_i of its isometry group, $i \in \mathbb{N} \cup \{\infty\}$. Suppose $(Y_i, G_i, \tilde{p}_i) \to (Y_\infty, G_\infty, \tilde{p}_\infty)$ in the equivariant Gromov-Hausdorff topology. Let $G_i(r)$ denote the subgroup generated by those elements that displace $\tilde{p}_i$ by at most r, $i \in \mathbf{N} \cup \{\infty\}$. Suppose there are $0 \leq a < b$ with $G_\infty(r) = G_\infty(\frac{a+b}{2})$ for all*

$r \in (a,b)$. Then there is some sequence $\epsilon_i \to 0$ such that $G_i(r) = G_i(\frac{a+b}{2})$ for all $r \in (a+\epsilon_i, b-\epsilon_i)$.

Lemma 5.5 and the Almost Splitting Theorem 1.6 allow us to modify arguments of the proof of [KW11, Lemma 2.3] to show that the following holds for smooth metric measure spaces.

Lemma 5.6. *Suppose (M_i^n, q_i) is a pointed sequence of smooth metric measure spaces where $(M_i^n, g_i, e^{-f_i} dvol_{g_i})$ has $|f_i| \leq k$ and $Ric_{f_i}(M_i) \geq -1/i$. Moreover, assume $(M_i^n, q_i) \to (\mathbb{R}^m \times K, q_\infty)$ where K is compact, and the action of $\pi_1(M_i)$ on the universal cover $(\widetilde{M}_i, \tilde{q}_i)$ converges to a limit action of a group G on some limit space $(Y, \tilde{q}_\infty)$. Then $G(r) = G(r')$ for all $r, r' > 2diam(K)$.*

We will also need the following result on the dimension of the limit space.

Lemma 5.7. *Let $(M_i^n, g_i, e^{-f_i} dvol_g)$ be a sequence of smooth metric measure spaces such that $|f_i| \leq k$, $diam(M_i^n) \leq D$, and $Ric_f \geq -(n-1)H$, $H > 0$. If M_i^n converges to the length space Y^m in the Gromov-Hausdorff sense, then for the Hausdorff dimension we have $m \leq n + 4k$.*

Proof. Begin by noting that for any $(M^n, g, e^{-f}\mathrm{dvol}_g)$ with $\mathrm{Ric}_f \geq -(n-1)H$, $H > 0$, and fixed $x \in M$ and $R > 0$, the f-volume comparison, Theorem 2.14, gives a bound on the number of disjoint ϵ-balls contained in $B(x,R)$: Let $B(x_1, \epsilon), \ldots, B(x_l, \epsilon) \subset B(x,R)$ be disjoint. Let $B(x_i, \epsilon)$ denote the ball with the smallest f-volume. Then

$$l \leq \frac{\mathrm{V}ol_f B(x,R)}{\mathrm{V}ol_f B(x_i,\epsilon)} \leq \frac{\mathrm{V}ol_f B(x_i,2R)}{\mathrm{V}ol_f B(x_i,\epsilon)} \leq \frac{\mathrm{V}ol_H^{n+4k} B(2R)}{\mathrm{V}ol_H^{n+4k} B(\epsilon)} = C(n+4k,H,R,\epsilon).$$

Thus $\mathrm{Cap}_{M_i}(\epsilon)$, the maximum number of disjoint $\epsilon/2$-balls which can be contained in M_i^n, is bounded above by $C = C(n+4k,H,D,\frac{\epsilon}{2})$ for each i. Moreover $\mathrm{Cov}_{M_i}(\epsilon)$, the minimum number of ϵ-balls covering M_i^n less than or equal to $\mathrm{Cap}_{M_i}(\epsilon)$, so $\mathrm{Cov}_{M_i} \leq C$.

Since $M_i^n \to Y$ in the Gromov-Hausdorff sense, there exists a sequence $\delta_i > 0$ such that $d_{GH}(M_i,Y) < \delta_i \to 0$ as $i \to \infty$. Then $\mathrm{Cov}_Y(\epsilon) \leq \mathrm{Cov}_{M_i}(\epsilon - 2\delta_i) \leq C$. As $i \to \infty$, we have $\mathrm{Cov}_Y(\epsilon) \leq C$.

To see that the Hausdorff dimension is bounded above by $n+4k$, recall that the d-dimension Hausdorff measure of Y is defined by

$$H^d(Y) = \lim_{\epsilon\to 0} H^d_\epsilon(Y),$$

where

$$H^d_\epsilon(Y) = \inf\left\{ \sum_{i=1}^\infty (\operatorname{diam} U_i)^d \,\middle|\, \bigcup_{i=1}^\infty U_i \supset Y, \operatorname{diam} U_i \leq \epsilon \right\}.$$

Since $\mathrm{Cov}_Y(\epsilon) \leq C$, it follows that $H^d_\epsilon(Y) \leq \sum_{i=1}^C (2\epsilon)^d$. Notice

$$C = \frac{\mathrm{V}ol_H^{n+4k} B(D)}{\mathrm{V}ol_H^{n+4k} B(\epsilon/2)} \sim (\epsilon/2)^{-(n+4k)}$$

as $\epsilon \to 0$. Thus as $\epsilon \to 0$

$$\sum_{i=1}^{C} (2\epsilon)^d = C(2\epsilon)^d \to 0$$

for all $d > n + 4k$. Thus the Hausdorff dimension of Y, defined by $\dim_H(Y) = \inf\{d \geq 0 | H^d(Y) = 0\}$ is at most $n + 4k$. $\square$

The final tool we will use to extend [KW11, Theorem 2.5] to smooth metric measure spaces is a type of Hardy-Littlewood maximal inequality for smooth metric measure spaces.

Proposition 5.8 (Weak 1-1 Inequality). *Suppose $(M^n, g, e^{-f}\,dvol_g)$ with $|f| < k$ has $Ric_f \geq -(n-1)H$ and $h : M \to \mathbb{R}$ is a nonnegative function. Define $Mx_\rho h(p) = \sup_{r \leq \rho} ⨍_{B(p,r)} he^{-f}\,dvol_g$ for $\rho \in (0, 1]$. Then if $h \in L^1(M)$, we have*

$$Vol_f\{x | Mx_\rho h(x) > c\} \leq \frac{C(n+4k, H)}{c} \int_M he^{-f}\,dvol_g$$

for any $c > 0$.

As in the proof of the Hardy-Littlewood maximal inequality for Euclidean spaces, one utilizes the Vitali Covering Lemma which states that for an arbitrary collection of balls $\{B(x_j, r_j) : j \in J\}$ in a metric space, there exists a subcollection of balls $\{B(x_j, r_j) : j \in J'\}$ with $J' \subseteq J$ from the original collection which are disjoint and satisfy

$$\bigcup_{j \in J} B(x_j, r_j) \subseteq \bigcup_{j \in J'} B(x_j, 5r_j).$$

We also note that the f-Volume Comparison [WW09, Theorem 1.2] gives a type of doubling estimate. In particular, for all $r \leq 1$, we have

$$\mathrm{V}ol_f(B(x,5r)) \leq C(n+4k,H)\mathrm{V}ol_f(B(x,r)).$$

Proof. Let $J = \{x|\mathrm{Mx}_\rho h(x) > c\}$. For all $x \in J$ there exists a ball $B(x, r_x)$ centered at x with radius $r_x \leq 1$ such that

$$\int_{B(x,r_x)} he^{-f}\mathrm{dvol}_g \geq c\mathrm{V}ol_f B(x,r_x). \tag{5.3}$$

Then by the Vitali Covering Lemma, we have

$$J \subseteq \bigcup_{x\in J} B(x,r_x) \subseteq \bigcup_{x\in J'} B(x,5r_x)$$

where $J' \subseteq J$. Then

$$\mathrm{V}ol_f\{x|\mathrm{Mx}_\rho h(x) > c\} \leq \mathrm{V}ol_f\left(\bigcup_{x\in J'} B(x,5r_x)\right) \leq C(n+4k,H)\sum_{x\in J'}\mathrm{V}ol_f B(x,r_x). \tag{5.4}$$

Combining (5.3) and (5.4) yields the desired result. □

5.3 Proof of Theorem 5.9

Before continuing to the proof of the theorem, we take a moment to recall Gromov's short generator system and the notion of a regular point. As in Gromov

[Gro78, 2.1], to construct Gromov short generators of the fundamental group $\pi_1(p, M)$, we represent each element of $\pi_1(p, M)$ by a shortest geodesic loop γ in that homotopy class. A minimal γ_1 is chosen so that it represents a nontrivial homotopy class of $\pi_1(M)$. If $\langle\gamma_1\rangle = \pi_1(M)$, then $\{\gamma_1\}$ is a Gromov short generator system of $\pi_1(M)$. If not, consider $\pi_1(M) \setminus \langle\gamma_1\rangle$. Choose $\gamma_2 \in \pi_1(M) \setminus \langle\gamma_1\rangle$ to be of minimal length. If $\langle\gamma_1, \gamma_2\rangle = \pi_1(M)$, then $\{\gamma_1, \gamma_2\}$ is a Gromov short generator system of $\pi_1(M)$. If not, choose $\gamma_3 \in \pi_1(M) \setminus \langle\gamma_1, \gamma_2\rangle$ such that γ_3 is of minimal length. Continue in this manner until $\pi_1(M)$ is generated. By this construction, we obtain a sequence of generators $\{\gamma_1, \gamma_2, \ldots\}$ such that $|\gamma_i| \leq |\gamma_{i+1}|$ for all i. The short generators have the property $|\gamma_i| \leq |\gamma_j^{-1}\gamma_i|$ for $i > j$. Although this sequence of generators is not unique, the sequence of lengths of generators $\{|\gamma_1|, |\gamma_2|, \ldots\}$ is unique.

To review the notion of a regular point, we first recall that for a Riemannian manifold (M^n, g) a tangent cone C_pM at $p \in M$ is a pointed Gromov-Hausdorff limit of rescaled spaces $(M, p, r_i g)$ for $r_i \to \infty$. Note that tangent cones may depend on the choice of convergent subsequence and hence may not be unique. As defined in Cheeger-Colding [CC97, Definition 0.1], a point $p \in M$ is regular if for some k, every tangent cone at p is isometric to $\mathbb{R}^k$. We note that in the case of Ricci curvature bounded from below, Cheeger and Colding have shown that the set of regular points has full measure [CC97, Theorem 2.1].

Now we have the necessary tools and concepts which will allow us to modify the argument of Kapovitch and Wilking to obtain a bound for the number of generators of $\pi_1(M)$ in the smooth metric measure space setting. As in [KW11], we prove a more general statement from which Theorem 1.5 is a consequence. This general statement, as well as its proof, is parallel to the statement and proof of [KW11, Theorem 2.5]. The argument is included in its entirety below for completeness.

Theorem 5.9. *Given n, k, and R, there is a constant C such that the following holds. Suppose $(M^n, g, e^{-f}dvol_g)$ is a smooth metric measure space with $|f|\leq k$, $p \in M$ and $Ric_f \geq -(n-1)$ on $B(p, 2R)$. Suppose also that $\pi_1(M, p)$ is generated by loops of length $\leq R$. Then $\pi_1(M, p)$ can be generated by C loops of length $\leq R$.*

Proof of Theorem 5.9. In order to prove Theorem 5.9 we begin, as in Kapovitch and Wilking's argument, by showing that there is a point $q \in B(p, \frac{R}{4})$ such that any Gromov short generator system of $\pi_1(M, q)$ has at most C elements.

For $q \in B(p, \frac{R}{4})$ consider a Gromov short generator system $\{\gamma_1, \gamma_2, \dots\}$ of $\pi_1(M, q)$. By assumption, $\pi_1(M, p)$ is generated by loops of length $\leq R$. In choosing generators for any Gromov short generator system of $\pi_1(M, q)$, loops of the form $\sigma \circ g \circ \sigma^{-1}$, where σ is a minimal geodesic from q to p and g is a generator of length $\leq R$ of $\pi_1(M, p)$, are contained in each of the homotopy classes of $\pi_1(M, q)$. Such a loop has length $\leq \frac{3R}{2}$ and hence the minimal length representative of that class, γ_i must have the property that $|\gamma_i|\leq \frac{3R}{2}$. Moreover, there are a priori bounds on the number of short generators of length $\geq r$. To see this, let us only consider the short generators such that $|\gamma_i| \geq r$. In the universal cover $\widetilde{M}$ of M, if $\tilde{q} \in \pi_1^{-1}(q)$, we have

$$r \leq d(\gamma_i\tilde{q}, \tilde{q}) \leq d(\gamma_j^{-1}\gamma_i\tilde{q}, \tilde{q}) = d(\gamma_i\tilde{q}, \gamma_j\tilde{q})$$

for $i > j$. Thus the balls $B(\gamma_i\tilde{q}, r/2)$ are pairwise disjoint for all γ_i such that $|\gamma_i| \geq r$. Then,

$$\bigcup_{\{\gamma_i : |\gamma_i| \geq r\}} B(\gamma_i\tilde{q}, \frac{r}{2}) \subset B(\tilde{q}, 2R + \frac{r}{2})$$

implies that

$$\#\{\gamma_i : |\gamma_i| \geq r\} \mathrm{V}ol_f B(q, \frac{r}{2}) \leq \mathrm{V}ol_f B(q, 2R + \frac{r}{2}).$$

And hence by the volume comparison [WW09, Theorem 1.2(a)], it follows that $\#\{\gamma_i : |\gamma_i| \geq r\} \leq C(n, k, r, R)$. Since one can control the number of short generators of length between r and $\frac{3R}{2}$ for $r < R$, one needs only show that the number of short generators of $\pi_1(M, q)$ with length $< r$ can also be controlled. This argument proceeds by contradiction. We assume the existence of a contradicting pointed sequence of smooth metric measure spaces (M_i, p_i) such that $(M_i, g_i, e^{-f_i}\mathrm{dvol}_g)$ has the property that

- $|f_i| \leq k$

- $\mathrm{Ric}_{f_i} \geq -(n-1)$ on $B(p_i, 3)$
- for all $q_i \in B(p_i, 1)$ the number of short generators of $\pi_1(M_i, q_i)$ of length ≤ 4 is larger than 2^i.

By Gromov precompactness, Theorem 2.23, we may assume that $(B(p_i, 3), p_i)$ converges to a limit space (X, p_∞). Set

$$\dim(X) = \max\{k : \text{ there is a regular } x \in B(p_\infty, 1/4) \text{ with } C_x X \simeq \mathbb{R}^k\}$$

where $C_x X$ denotes a tangent cone of X at x.

We prove that there is no such contradicting sequence by reverse induction on $\dim(X)$. For the base case, let $m > n+4k+1$. By Lemma 5.7, $\dim(X) \leq n+4k+1$, so there is nothing to prove here. Suppose then that there is no contradicting sequence with $\dim(X) = j$ where $j \in \{m+1, \ldots, n+4k\}$ but that there exists a contradicting sequence with $\dim(X) = m$. The induction step is divided into two substeps.

Step 1 For any contradicting sequence (M_i, p_i) converging to (X, p_∞) there is a new contradicting sequence converging to $(\mathbb{R}^{\dim X}, 0)$.

Suppose (M_i, p_i) is a contradicting sequence converging to (X, p_∞). By definition of $\dim(X)$, there exists $q_\infty \in B(p_\infty, \frac{1}{4})$ such that $C_{q_\infty} X \simeq \mathbb{R}^m$. Let $q_i \in B(p_i, \frac{1}{2})$ such that $q_i \to q_\infty$ as $i \to \infty$. Since this is a contradicting sequence, it follows that the Gromov short generator systems of $\pi_1(M_i, x_i)$ for all $x_i \in B(q_i, \frac{1}{4})$ contain at least 2^i generators of length ≤ 4. As noted earlier, for each fixed $\epsilon < 4$, the number of short generators of $\pi_1(M_i, x)$ of length $\in [\epsilon, 4]$ is bounded by a constant $C(n, k, \epsilon, 4)$. Then we can find a rescaling $\lambda_i \to \infty$ such that for every $x_i \in B(q_i, \frac{1}{\lambda_i})$, the number of generators of $\pi_1(M_i, x)$ of length $\leq 4/\lambda_i$ is at least 2^i. Moreover, $(\lambda_i M_i, q_i) \to (\mathbb{R}^m, 0)$, where $\lambda_i M_i$ denotes the smooth metric measure space $(M_i, \lambda_i g_i, e^{-f_i}\mathrm{dvol}_{\lambda_i g_i})$. Thus the sequence $(\lambda_i M_i, q_i)$ is the new contradicting sequence desired.

Step 2 If there is a contradicting sequence converging to $(\mathbb{R}^m, 0)$, then we can find a contradicting sequence converging to a space whose dimension is larger than m.

Let (M_i, q_i) denote the contradicting sequence converging to $(\mathbb{R}^m, 0)$ as obtained in Step 1 above. Without loss of generality, assume that for some $r_i \to \infty$ and $\epsilon_i \to 0$, $\mathrm{Ric}_f \geq -\epsilon_i$ on $B(p_i, r_i)$. By Theorem 5.2 there exist f-harmonic functions $(\overline{b}^i_1, \cdots, \overline{b}^i_m) : B(q_i, 1) \to \mathbb{R}^m$ such that

$$\fint_{B(q_i,1)} \left(\sum_{j,l=1}^{m} |\langle \nabla \overline{b}^i_l, \nabla \overline{b}^i_j \rangle - \delta_{lj}| + ||\mathrm{Hess}\,(\overline{b}^i_l)||^2 \right) e^{-f} \mathrm{dvol}_g < \delta_i \to 0.$$

Claim There exists $z_i \in B(q_i, \frac{1}{2})$, $c > 0$ such that for any $r \leq \frac{1}{4}$,

$$\fint_{B(z_i,r)} \left(\sum_{j,l=1}^{m} |\langle \nabla \overline{b}^i_l, \nabla \overline{b}^i_j \rangle - \delta_{lj}| + ||\mathrm{Hess}\,(\overline{b}^i_l)||^2 \right) e^{-f} \mathrm{dvol}_g \leq c\delta_i \to 0.$$

Let $h(x)$ denote $\sum_{j,l=1}^{m} |\langle \nabla \overline{b}_l^i, \nabla \overline{b}_j^i \rangle - \delta_{lj}| + ||\mathrm{Hess}\,(\overline{b}_l^i)||^2$ evaluated at x. Seeking contradiction, suppose that for all $c > 0$, $r \leq 1/2$, and $z \in B(q_i, \frac{1}{2})$

$$\fint_{B(z,r)} h e^{-f} \mathrm{dvol}_g > c\delta_i,$$

then it follows that $\mathrm{Mx}_{1/2} h(z) = \sup_{r \leq 1/2} \fint_{B(z,r)} h e^{-f} \mathrm{dvol}_g \geq c\delta_i$. Hence

$$\mathrm{V}ol_f\{x | \mathrm{Mx}_{1/2} h(x) \geq c\delta_i\} \geq \mathrm{V}ol_f(B(q_i, \frac{1}{2})). \tag{5.5}$$

By Proposition 5.8, we also have that for all $c \geq 0$,

$$\mathrm{V}ol_f\{x | \mathrm{Mx}_{1/2} h(x) \geq c\delta_i\} \leq \frac{C(n+4k,-1)}{c}. \tag{5.6}$$

Combining (5.5) and (5.6), we have

$$1 \leq \frac{\mathrm{V}ol_f\{x | \mathrm{Mx}_{1/2} h(x) \geq c\delta_i\}}{\mathrm{V}ol_f(B(q_i, \frac{1}{2}))} \leq \frac{C(n+4k,-1)}{c \cdot \mathrm{V}ol_f(B(q_i, \frac{1}{2}))}.$$

Choosing $c > C(n+4k,-1)/\mathrm{V}ol_f(B(q_i, \frac{1}{2}))$ yields a contradiction and hence the claim is proven.

By Lemmas 5.5 and 5.6, there exists a sequence $\delta_i \to 0$ such that for all $z_i \in B(p_i, 2)$ the Gromov short generator system of $\pi_1(M_i, z_i)$ does not contain any elements of length in $[\delta_i, 4]$. Choose $r_i \leq 1$ maximal with the property that there is $y_i \in B(z_i, r_i)$ such that the short generators of $\pi_1(M_i, y_i)$ contains a generator of length r_i. Then $r_i < \delta_i \to 0$.

Rescaling by $\frac{1}{r_i}$ gives that $\pi_1(\frac{1}{r_i}M_i, y_i)$ has at least 2^i short generators of length ≤ 1 for all $y_i \in B(z_i, 1)$. By the choice of rescaling, there is at least one $y_i \in B(z_i, r_i)$ such that the Gromov short generator system at that y_i contains a generator of length 1. Moreover, the above claim together with the Product Lemma 5.4 give $(\frac{1}{r_i}M_i, z_i) \to (\mathbb{R}^k \times Z, z_\infty)$. Moreover, by Lemmas 5.5 and 5.6, Z is nontrivial and thus $\dim(\mathbb{R}^m \times Z) \geq m + 1$, a contradiction. So, we have completed the induction step.

Thus there exists $q \in B(p, \frac{R}{4})$ such that number of generators of $\pi_1(M, q)$ has at most C elements. Thus the subgroup of $\pi_1(M, p)$ generated by loops of length $< 3R/5$ can be generated by C elements. Moreover, the number of short generators of $\pi_1(M, p)$ with length in $[3R/5, R]$ is bounded by some a priori constant. $\square$

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