UNIVERSITY OF CALIFORNIA
Santa Barbara

Systemic Risk Illustrated

A dissertation submitted in partial satisfaction of the requirements for the degree of Doctor of Philosophy

in Statistics and Applied Probability

by

Li-Hsien Sun

Committee in charge:
Professor Jean-Pierre Fouque, Chair
Professor Michael Ludkovski
Professor Raya Feldman

June 2014

The dissertation of Li-Hsien Sun is approved.

Michael Ludkovski

Raya Feldman

Jean-Pierre Fouque, Committee Chair

May 2014

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Acknowledgments

I would like to express my deepest appreciation and thanks to my advisor Professor Jean-Pierre Fouque who conveyed a spirit of adventure in regard to research and an excitement of teaching. This dissertation would not have been possible without his supervision and constant help. In addition, I would like to thank you for encouraging my research and for allowing me to grow as a research scientist. Your advice on both research as well as on my career have been priceless.

My thanks and appreciation go to my committee members, Professor Raya Feldman and Professor Michael Ludkovski for serving as my committee members even at hardship. I also would like to thank you for advising me on my research and helping me for job application. Your suggestion and good natured support have been greatly helpful for my career.

I must acknowledge as well many friends, colleagues, students, teachers who supported, assisted, advised me on my research and who helped me writing on my dissertation. Thanks also to my department who gave me the great financial support during my studies and provided these great courses of financial mathematics and statistics.

A special thanks to my family. Words cannot express how grateful I am to my mother and father for your support and sacrifices. Your encouragement and prayer were what sustained me thus far.

Curriculum Vitae

Li-Hsien Sun

EDUCATION

- Ph.D, Statistics and Applied Probability, emphasized on financial mathematics and statistics, University of Santa Barbara, Santa Barbara, CA May 2014
- M.S., Statistics, National Central University, Taiwan September 2005
- B.S., Mathematics, National Central University, Taiwan September 2003

PROFESSIONAL EMPLOYMENT

- Visiting scholar, Academia Sinica, Taipei, Taiwan March 2014
- Lecture, University of Santa Barbara, Santa Barbara, CA January 2014
- Invited lecture, National Tsing Hua University December 2012
- Teaching associate,

  University of Santa Barbara, Santa Barbara, CA June 2012 - September 2012
- Teaching assistant,

  University of Santa Barbara, Santa Barbara, CA September 2008 - March 2014

AWARDS

- Graduate Division Dissertation Fellowship, UC Santa Barbara, 2013.
- Abraham Wald Memorial Prize, UC Santa Barbara, 2009.

PUBLICATIONS

- "Mean Field Games and Systemic Risk" (with R. Carmona and J.-P. Fouque), submitted 2013, revised 2014, to appear in Communications in Mathematical Sciences.
- "Systemic Risk Illustrated" (with J.-P. Fouque), to appear in the Hanbook of Systemic Risk, Eds J.-P. Fouque and J. Langsam. Cambridge University Press, 2013.
- "Adjusted Method for Pricing American Option Using Bootstrap" (with Kuang-Fu Cheng), Master thesis, 2005.

Abstract

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by

Li-Hsien Sun

Due to the recent financial crisis, systemic risk is becoming a central research topic. In this study, we propose a simple model of inter-bank borrowing and lending where the evolution of the log-monetary reserves of N banks is described by a system of diffusion processes coupled through their drifts in such a way that stability of the system depends on the rate of inter-bank borrowing and lending. Systemic risk is characterized by the non-negligible probability of a large number of defaults. In order to study the behavior of this coupled system, we discuss the comparison of the coupled diffusions not only coupled through the drift and non-drift but correlated through Brownian motions. In addition, we introduce a game feature in the lending and borrowing system where each bank controls its own rate of borrowing from or lending to the central bank under a quadratic cost. The optimization reflects the desire of each bank to borrow from the central bank when its monetary reserve falls below a critical level or lend if it rises above this critical level which is chosen here as the average monetary reserve. The equilibria with finitely many players are solved explicitly and the financial implication is that the central bank acts as a clearing house, adding liquidity to the system without affecting its systemic risk. We also study the corresponding Mean Field Game in the limit of large number of banks in the presence of a common noise. Finally, we consider two inhomogeneous unsymmetrical grouping problems where banks have strategies using heterogeneous parameters and obtain that the central bank must provide extra cash into the system or keep deposits for banks in order to stabilize this bank system using the heterogeneity framework.

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Chapter 1

Introduction

After the financial crisis of 2007-2008, the term “financial crisis” is becoming an important and critical issue for researchers, policy makers, and governments. This is due to the enormous cost that a financial system crisis way inflict, and 2007-2009 crisis exposed the inability of financial organizations and governments to prevent, and detect such a crisis and minimize its impact. In general, this problem is difficult to explain completely since causes are varied and complicated. In order to approach this problem, we consider one imaginable factor contributing the financial crisis: systemic risk. The temporary definition of systemic risk given by [10] is as follows:

Systemic Risk is the risk of disruption of the market’s ability to facilitate the flows of capital that results in the reduction in the growth of the global GDP.

Systemic risk is now a crucial research topic in financial markets recently. People are seeking the terms to describe, explain, or even prevent probable systemic risk from many points of view (Statistics, Finance, Mathematical Finance, Behavioral Finance, Networks, Counterparty Risk, High Frequency Trading, ...) as seen in [10]. This research tackles this problem by proposing a mathematical model of the economics and financial dynamics of a financial system and demonstrates that complex interactions in the system creates systemic risk in the form of large deviations from a stable state. The evolution of the log-monetary reserves of N banks is described by a system of diffusion processes coupled through their drifts. Note that the drift terms for the log-monetary reserves are not necessarily the interest rate r because the reserves are not tradable assets which gives the flexibility to create the dependence through the drift terms. This type of interaction and the relation *stability–systemic risk* has been recently studied in [11], [14], and [13]. In order to describe systemic risk, we consider the probability of a large number of bank defaults in our coupled system where this probability is non-negligible compared to the independent case. This probability can not be computed explicitly. To find the asymptotic solution, we define "systemic event" which is solved explicitly in our simple setting.

In addition, we introduce a game feature where each bank controls its rate of borrowing/lending to a central bank. The control of each individual bank reflects the desire to borrow from the central bank when its monetary reserve falls below a critical level or lend if it rises above this critical level which is chosen here as the average monetary reserve. Borrowing from or lending to the central bank is also subject to a quadratic cost at a rate which can be fixed by the regulator. As written, our model is an example of a Linear-Quadratic Mean Field Game with finitely many players which can be solved explicitly. We first solve for open-loop equilibria using the Pontryagin stochastic maximum principle. We also solve for closed-loop equilibria using the probabilistic approach based on the Pontryagin stochastic maximum principle leading to the solution of Forward-Backward Stochastic Differential Equations, and the dynamic programming principle leading to the solution of Hamilton-Jacobi-Bellman partial differential equations. We also study the corresponding Mean Field Game in the limit of a large number of banks.

In the model discussed below, the diffusion processes X_t^i for $i = 1, \ldots, N$ represent the log-monetary reserves of N banks lending to and borrowing from each other. The system is driven by N (possibly correlated) standard Brownian motions $\widetilde{W}_t^i, i = 1, \cdots, N$ written as $\widetilde{W}_t^i = \rho W_t^0 + \sqrt{1-\rho^2} W_t^i$ where $W_t^j, j = 0, 1, \cdots, N$ are independent standard Brownian motions, W_t^0 being the *common noise*, and $|\rho| \leq 1$. The system starts at time $t = 0$ from i.i.d. random variables $X_0^i = \xi^i$ independent of the Brownian motions and such that $\mathbb{E}(\xi^i) = 0$. We assume that the diffusion coefficients are constant and identical, denoted by $\sigma > 0$. Our model of lending and borrowing consists in introducing an interaction through drift terms representing the rate at which bank i borrows from or lends to bank j. In this case, the rates are proportional to the difference in log-monetary reserves, and our model is:

$$dX_t^i = \frac{a}{N}\sum_{j=1}^{N}(X_t^j - X_t^i)\,dt + \alpha_t^i dt + \sigma d\widetilde{W}_t^i\,, \quad i = 1, \cdots, N, \tag{1.1}$$

where the overall rate of "mean-reversion" a/N has been normalized by the number of banks with $a \geq 0$. Bank i controls its rate of borrowing/lending to a central bank through the control rate α_t^i. Using the notation

$$\overline{X}_t = \frac{1}{N}\sum_{i=1}^{N} X_t^i,$$

for the empirical mean, the dynamics can be rewritten in the mean field form:

$$dX_t^i = \left[a(\overline{X}_t - X_t^i) + \alpha_t^i\right] dt + \sigma d\widetilde{W}_t^i, \quad i = 1, \cdots, N. \tag{1.2}$$

Bank $i \in \{1, \cdots, N\}$ controls its rate of lending and borrowing at time t by choosing the control α_t^i in order to minimize

$$J^i(\alpha^1, \cdots, \alpha^N) = I\!\!E\left\{\int_0^T f_i(X_t, \alpha_t^i)dt + g_i(X_T^i)\right\}, \tag{1.3}$$

where the running cost function f_i is defined by

$$f_i(x, \alpha^i) = \left[\frac{1}{2}(\alpha^i)^2 - q\alpha^i(\overline{x} - x^i) + \frac{\epsilon}{2}(\overline{x} - x^i)^2\right], \tag{1.4}$$

and the terminal cost function g_i by

$$g_i(x) = \frac{c}{2}\left(\overline{x} - x^i\right)^2. \tag{1.5}$$

Notice that the running quadratic cost $\frac{1}{2}(\alpha^i)^2$ has been normalized and that the effect of the parameter $q > 0$ is to control the *incentive* to borrowing or lending: the bank i

will want to borrow ($\alpha_t^i > 0$) if X_t^i is smaller than the empirical mean ($\overline{X}_t$) and lend ($\alpha_t^i < 0$) if X_t^i is larger than $\overline{X}_t$. Equivalently, after dividing by $q > 0$, this parameter can be thought as a control by the regulator of the cost of borrowing or lending (with q large meaning low fees).

The quadratic terms in $(\overline{x} - x^i)^2$ in the running cost ($\epsilon > 0$) and in the terminal cost ($c > 0$) penalize departure from the average. We assume that

$$q^2 \leq \epsilon, \tag{1.6}$$

so that $f_i(x, \alpha)$ is convex in (x, α).

In the spirit of structural models of defaults, we introduce a default level $D < 0$ and say that bank i defaults by time T if its log-monetary reserve reached the level D before time T. Note that in this simple model, even after reaching the default level, bank i stays in the system until time T and continues to participate in inter-bank and central bank borrowing and lending activities.

This dissertation is organized as follows. In Chapter 2, we comment on the difference between correlated Brownian motions used to model firms in order to analyze credit risk and coupled diffusions considered as the lending and borrowing bank system for systemic risk. The model for systemic risk is given by the coupled system

$$dX_t^{(i)} = \frac{\alpha}{N}\sum_{j=1}^{N}(X_t^{(j)} - X_t^{(i)})\,dt + \sigma dW_t^{(i)}\,, \quad i = 1, \cdots, N,$$

and the model for credit risk is written as

$$dX_t^{(i)} = \sigma\left(\rho dW_t^0 + \sqrt{1-\rho^2}dW_t^i\right) \quad i = 1, \cdots, N,$$

with the common noise (W_t^0) and the independent Brownian motions (W_t^i), $i = 1, \cdots, N$. The difference is shown through the loss distribution generated by both cases. In Chapter 3, we derive the mean-field limit of system (1.1) as the number of banks N becomes large. In this limit, banks become independent of each other and their log-monetary reserves follow Ornstein-Unlenbeck processes. Interestingly, before taking this limit, we observe that each component mean-reverts to a common Brownian motion with a small diffusion of order $1/\sqrt{N}$. We exploit this fact in Section 3.3, to explain systemic risk as the small-probability event where this mean level reaches the default barrier, with a typically large number of components "following" the mean and defaulting. Moreover, this small probability of systemic risk is independent of the mean-reversion rate α so that a large α corresponds to more stability but at the same time of (or "at the price of") a larger systemic event.

Chapter 4 is devoted to the analysis of the stochastic differential game (1.2-1.3). We derive exact Nash equilibria for the open-loop as well as for the closed-loop Markovian models, using both probabilistic and analytic approaches.

Note that stochastic differential games with a large number of players are usually not tractable. But because of the very special nature of our model (homogeneous linear dynamics, homogeneous quadratic costs, and interactions through the empirical mean), that we are able to construct explicitly Nash equilibria for both open and closed loops. For generic models which are not amenable to explicit solutions, Lasry and Lions [21, 22, 23] have recently provided an elegant way to tackle the construction of approximate Nash equilibria for large games with mean field interactions. Their methodology, known as *Mean Field Games* (MFG), has been applied to a wide variety of problems (see [15, 20] for some examples). A similar research program was developed independently by Caines, Huang, and Malhamé with the name of Nash Certainty Equivalent. See for example [17] and [18]. The approach of Lasry and Lions (e.g. [23]) is based on the solution of a system of partial differential equations (PDEs): a Hamilton-Jacobi-Bellman equation evolving backward in time, and a Kolmogorov equation evolving forward in time, these two PDEs being strongly coupled. By its probabilistic nature, it is natural to recast the MFG strategy using appropriate forms of the Pontryagin stochastic maximum principle, leading to the solution of new models. See, for example, [8, 4, 7, 6] or [9] for a probabilistic approach based on the weak formulation of stochastic control. We consider the MFG problem and discuss the existence of approximate Nash equilibria in Chapter 5.

In Chapter 6, the financial implication in terms of liquidity and role of a central bank is discussed through the simple symmetrical mean field type noncooperative games.

Chapter 7 presents two extensions of our toy model. We tackle two heterogeneous group cases. We first study the unsymmetrical groups system using heterogeneous transaction rates a_{ij} from group i to j. Banks use their value functions to obtain the best strategies only depending on the empirical mean of their own groups and their own states. Secondly, we consider a more general extension of the first case. The value function is the function of the empirical mean of all banks in the system with heterogeneous parameters $q^{(i)}$ and $\epsilon^{(i)}$ depending on the group index i. We comment on the change of liquidity and also the role of a central bank. As we desired, the heterogeneity is more realistic since a central bank is more involved in the second case.

Finally, we discuss more possible extensions. One may extend the toy model using varied volatilities and generalized parameters or functions. As credit risk, an intensity based model is useful to describe the number of defaults. The rate using coupled diffusions can create "flocking to default." We also comment on the difficulties of calibrating for systemic risk which may be solved in the near future.

Chapter 2

Systemic Risk and Credit Risk

Before discussing systemic risk, we first introduce another important risk in the financial market called credit risk. It refers to risk of the default from the borrowers who fail to pay the required payment. Since credit risk been studied in the field of financial mathematics for many years, we first briefly discuss credit risk and then systemic risk. The common property that credit risk and systemic risk share is that people need to consider the risk of defaults for multiple firms or organizations. More importantly, those objects are often correlated and established using deterministic or stochastic processes in discrete or continuous time. In our approach, we use continuous time stochastic processes to describe the behavior of each firm. Hence, in order to analyze credit risk or systemic risk, in this framework, it is natural to consider a system with correlated multi-dimensional diffusions.

First, one can consider a system of N diffusions

$$dX_t^i = b_t^i dt + \sigma_t^i dW_t^i, \tag{2.1}$$

driven by Brownian motions (W_t^i), $i = 1, \cdots, N$. Next step, we show three ways to correlate diffusions:

1. Correlated Brownian motion models (to correlate diffusions through Brownian motions):

$$\begin{aligned} b_t^i &= b^i(\text{constant}), \text{for all } i, \\ \sigma_t^i &= \sigma^i(\text{constant}),\ i = 1, \cdots, N, \\ d < W_t^i, W_t^j > &= \rho_{ij} dt,\ \text{for all } i \neq j. \end{aligned}$$

2. Stochastic volatility models (to correlate diffusions through the volatility):

$$(W_t^i):\ \text{independent Brownian motions,}$$

$$b_t^i = \alpha^i(\text{constant}),\ \text{for all } i,$$

$$\sigma_t^i = \sigma(Z_t),$$

where Z_t is a diffusion driven by a Brownian motion B_t correlated to W_t^i.

3. Coupled diffusions (to correlate diffusions through the drift term):

$$(W_t^i):\ \text{independent Brownian motions,}$$

$$b_t^i = f(X_t^1, X_t^2, \ldots X_t^N),\ i = 1, \cdots, N,$$

$$\sigma_t^i = \sigma^i(\text{constant}),\ i = 1, \cdots, N.$$

In Section 2.1, we discuss credit risk and systemic risk using the structural model framework. Credit risk is described using diffusions with drifts imposed by interest rate r and correlated Browian motions. Systemic risk is modelled by coupled diffusions using interacting in the drifts. In addition, Section 2.2 presents the loss distribution which is the number of defaults before the maturity time T with shifting parameters to show the differences between correlated Brownian motions and coupled diffusions using Monte Carlo method.

2.1 From Credit Risk to Systemic Risk

In credit risk analysis, the first important problem is how to describe default for each firm since credit risk is the risk of defaults of multiple firms. In financial mathematics, literature, it appears that there are two popular approaches: structural models and intensity based models. Using structural models to analyze credit risk problems, one can describe firms using stochastic processes with Brownian motions or other jumping processes and then correlate firms through volatilities, Brownian motions, or both. The default is described as the trajectory of the firm passing the default boundary. Therefore, it is also important to determine when the first time to pass the boundary is the same as the first passenger problem.

In intensity based models, instead of modelling the firms directly, one can consider the number of defaults characterized by Poisson processes. The flexibility in this model is its use of an the intensity rate which is the expected defaults in the unit time.

This rate for each firm can be a constant, a deterministic function, or a stochastic process.

Now, for the structural model, we first note that because of discount or tradable condition for tradable assets modelled by (2.1), the drift term b_t^i is imposed by the interest rate r for all i. Therefore, in order to create the correlation for those assets using the no arbitrage condition, one only correlate the dynamics through the volatility σ_t^i or Brownian motions (W_t^i). For simplicity, we choose the interest rate $r = 0$, and so the toy model for credit risk is given by

$$dX_t^i = \sigma dB_t^i, \; i = 1, \cdots, N, \tag{2.2}$$

with the initial point $x_0^i = 0$, $i = 1, \cdots, N$. The B^i's are correlated Brownian motions with common correlation ρ and the first default time is given by

$$\tau_i = \inf(t; \; X_t^i \leq D).$$

On the other hand, for systemic risk, we first assume that all the dynamics are the log-monetary reserves of banks. The drift terms for log-monetary reserves are necessarily the interest rate r since the reserves are not tradable. Using this flexibility in the drift terms, we construct the interaction of the multi-dimensional coupled diffusions through the drift terms. The toy coupled diffusion model for systemic risk is written as

$$dX_t^i = \frac{a}{N} \sum_{i=1}^{N} \left(X_t^j - X_t^i \right) dt + \sigma dW_t^i, \; i = 1, \ldots, N \tag{2.3}$$

with the same initial value $X_0^i = 0$ for all i. The W^i's are independent Brownian motions, and similarly, the first default time for player i is given by

$$\tau_i = \inf(t;\ X_t^i \le D).$$

Using interaction through the drift terms describing lending and borrowing behavior, we obtain that this coupled system with this lending and borrowing is more stable than the independent case, but note that, in the meantime, this transaction also creates systemic risk for the system because the higher possibility of a large amount of defaults than the independent scenario is obtained. This phenomenon can be observed in Section 2.2 and be explained in Chapter 3.

In order to study the difference between coupled diffusions and correlated Brownian motions in the framework of structural models, we study the joint survival probability

$$\mathrm{S}_k = I\!P\left(\tau_i > T,\ \text{for all } i = 1, \cdots, k\right),\ 1 \le k \le N,$$

where $\tau_i = \inf(t; X_t^i \le D)$ with the defaults barrier D and the loss distribution which is the number of default before fixed maturity time T. Note that the solution of the independent case ($\rho = 0$ or $a = 0$) can be solved explicitly. The number of defaults follows a binomial distribution with two parameters: sample size N and probability p given by

$$\begin{aligned} p &= I\!P\left(\min_{0\le t\le T}(\sigma W_t) \le D\right) \\ &= 2\Phi\left(\frac{D}{\sigma\sqrt{T}}\right), \end{aligned} \tag{2.4}$$

where Φ denotes the $N(0,1)$-cdf, and we use the distribution of the minimum of a Brownian motion (see [19] for instance). This gives the uni-modal density curve shown in Figure 2.3 and 2.4. Surprisingly, however, it is difficult to solve the joint survival probability for correlated Brownian motions even when $N = 2$. In systemic risk, the difficulty is the high dimensional diffusions with complicated interacting drift terms. Although we use Girsanov theorem to simply the drifts, the RadonNikodym derivative still creates another obstruction for solving the survival probability. Hence, we first obtain the joint survival probability and the loss distribution using simulations using the Monte Carlo method. The result is shown in Section 2.2.

2.2 Simulation Results

In the following simulation, we simulate 10^4 trajectories to obtain the loss distribution. As mentioned in Chapter 1, we assume that all trajectories keep lending to or borrowing from each other in the game even if they arrive the default boundary. We use the number of trajectory $N = 5$, $D = -1$ and the common volatility $\sigma = 1$.

Figure 2.1 shows one realization for different correlation ρ. When $\rho = 1$, it gives the trajectory of the common noise passing the default bond $D = -1$. Hence, as ρ is large, we obtain that the system is roughly driven by the common noise which forces more defaults. In Figure 2.3, we show the loss distribution with the shifting correlation ρ. When ρ is larger, the possibility of a large number of defaults is increasing. As $\rho = 1$, since the system becomes only one Brownian motion, there are only two peaks in the loss distribution corresponding to either no default, or all defaults and the corresponding probabilities are

$$\mathbb{P}\,(\text{no default}) = \mathbb{P}\left(\tau^i > T\right) = 1 - 2\Phi\left(\frac{D}{\sigma\sqrt{T}}\right),$$

and

$$\mathbb{P}\,(\text{all defaults}) = \mathbb{P}\left(\tau^i < T\right) = 2\Phi\left(\frac{D}{\sigma\sqrt{T}}\right).$$

Note that the shape of the density curve intends to be bi-modal when ρ is increasing. That is, the probability of no default and the probability of a large amount of defaults become more significant as correlation ρ increases.

On the other hand, using the coupled diffusions (2.3), in Figure 2.2, we observe that when the mean-reverting rate a increases, the trajectories are flocking together. Compared to smaller $a = 0$ and $a = 2$, larger a creates stability by imposing flocking trajectories. In Figure 2.4, the probability of no default and the probability of a large amount of defaults are both increasing as a gets larger. The shape of the density curve in (2.3) is similar to (2.2). However, it is important to remark that in the same volatility scale σ, the right-tail generated by (2.3) gives a much smaller probability of all defaults than (2.2), and the left-tail gives the huge probability of no default. The reason is that if the mean-reverting rate a is large, similarly, the system roughly is governed by the ensemble average which has the same distribution as the re-scaled Brownian motion with scaler $\frac{1}{\sqrt{N}}$, shown in Chapter 3. This scaler, depending on N, does not appear in the model (2.2). Since this leading stochastic process (the ensemble average) is scaled by $\frac{1}{\sqrt{N}}$, which gives the a smaller variation for large N, this smaller variation leads to a significantly large probability of no default and a much smaller (but still much larger than the independent case) probability of all defaults considered as a rare event in the systemic risk perspective.

To summarize, through this analysis of correlated Brownian motions (2.2) and coupled diffusions (2.3), we shown that they behave in the same manner in the case of large correlation parameter ρ and large mean reverting rate a. Also, they both yield the bi-modal density curves for the loss distributions. However, the coupled diffusions produce much a smaller probability in the large number of defaults than correlated Browian motions, which is one specific method to describe systemic risk. This interaction considered as borrowing and lending behavior within the drift terms creates the stability of the system by increasing the probability of no default; however the trade-off of this stability is that the interaction also increases the probability of a large amount of defaults compared to the independent case ($a = 0$). Chapter 3 presents the computational details for coupled diffusions (2.3).

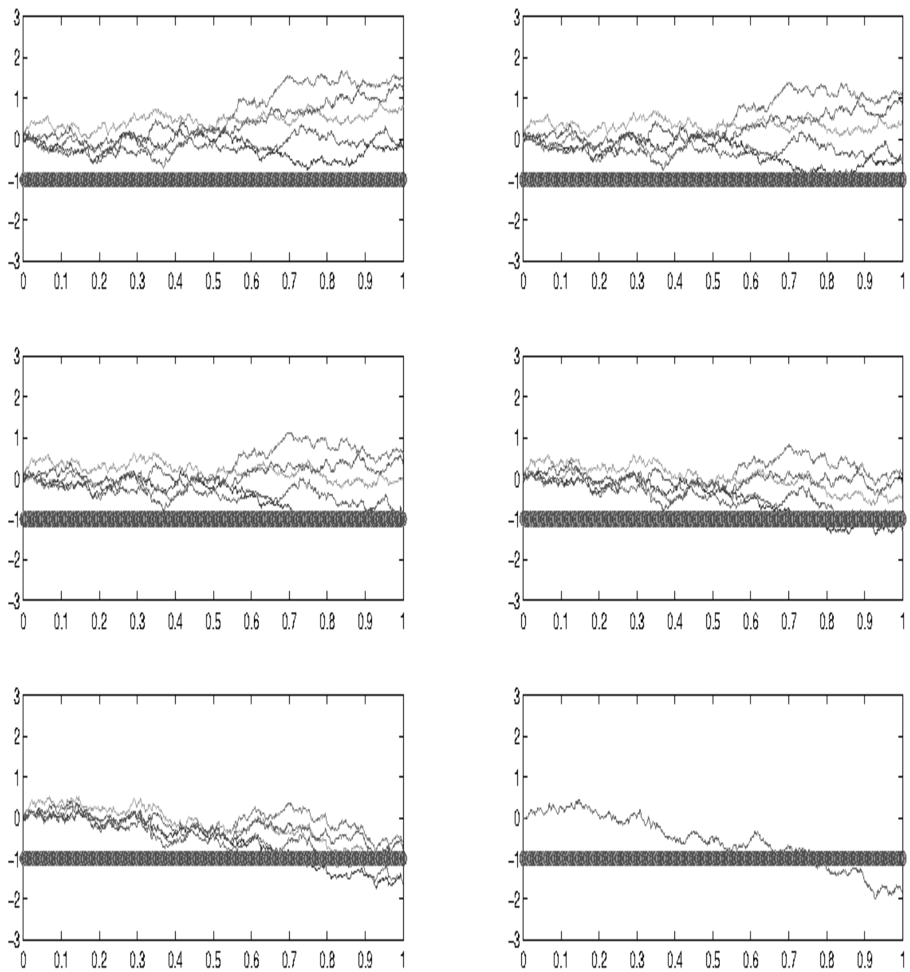

Figure 2.1: One realization of the trajectories of the correlated Brownian motions (2.2) (the first plot on the left column) with $\rho = 0$; one realization of the trajectories of the correlated Brownian motions (2.2) (the first plot on the right column) with $\rho = 0.2$; one realization of the trajectories of the correlated Brownian motions (2.2) (the second plot on the left column) with $\rho = 0.4$; one realization of the trajectories of the correlated Brownian motions (2.2) (the second plot on the right column) with $\rho = 0.6$; one realization of the trajectories of the correlated Brownian motions (2.2) (the third plot on the left column) with $\rho = 0.8$; one realization of the trajectories of the correlated Brownian motions (2.2) (the third plot on the right column) with $\rho = 1$.

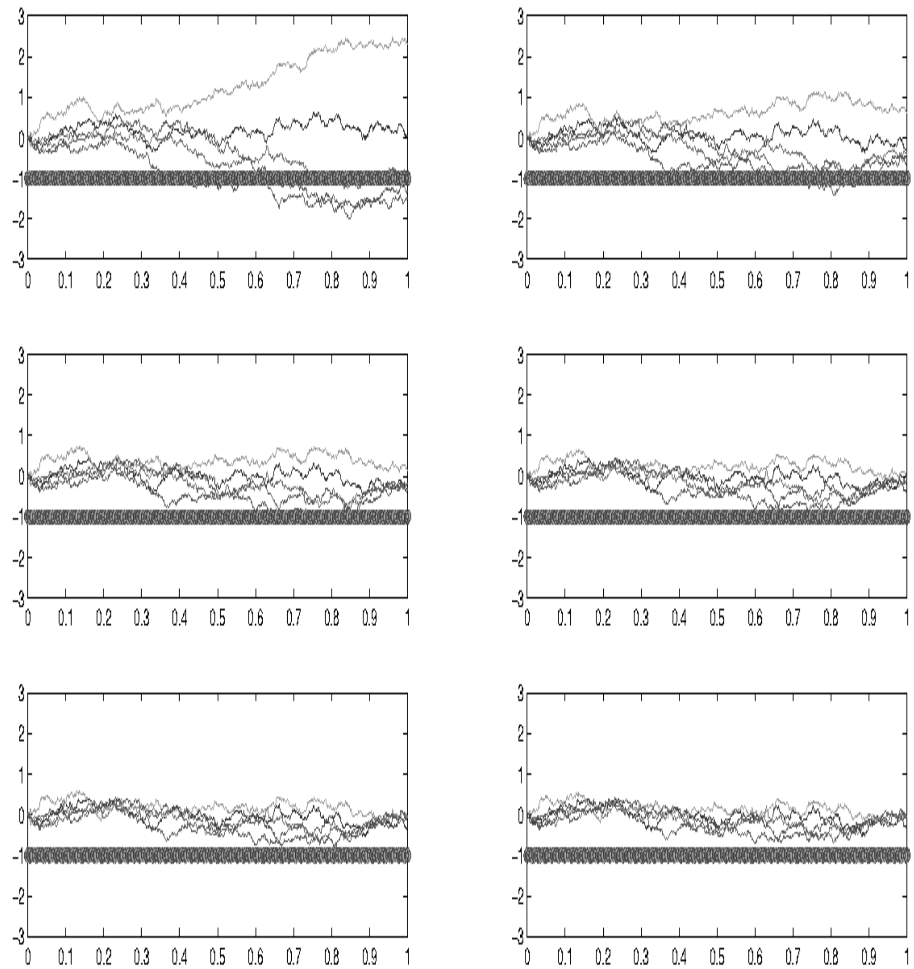

Figure 2.2: One realization of the trajectories of the coupled diffusions (2.3) (the first plot on the left column) with $a = 0$; one realization of the trajectories of the coupled diffusions (2.3) (the first plot on the right column) with $a = 2$; one realization of the trajectories of the coupled diffusions (2.3) (the second plot on the left column) with $a = 4$; one realization of the trajectories of the coupled diffusions (2.3) (the second plot on the right column) with $a = 6$; one realization of the trajectories of the coupled diffusions (2.3) (the third plot on the left column) with $a = 8$; one realization of the trajectories of the coupled diffusions (2.3) (the third plot on the right column) with $a = 10$.

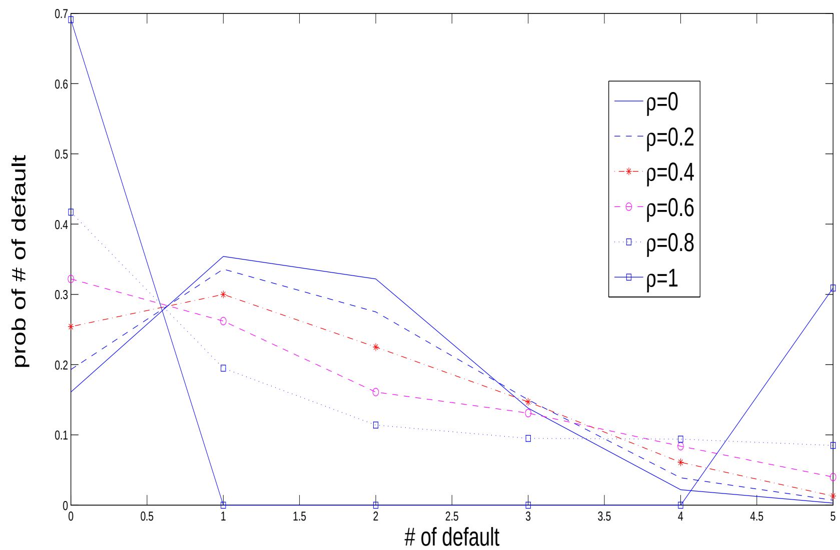

Figure 2.3: The loss distribution of the correlated Brownian motions with shifting correlation ρ.

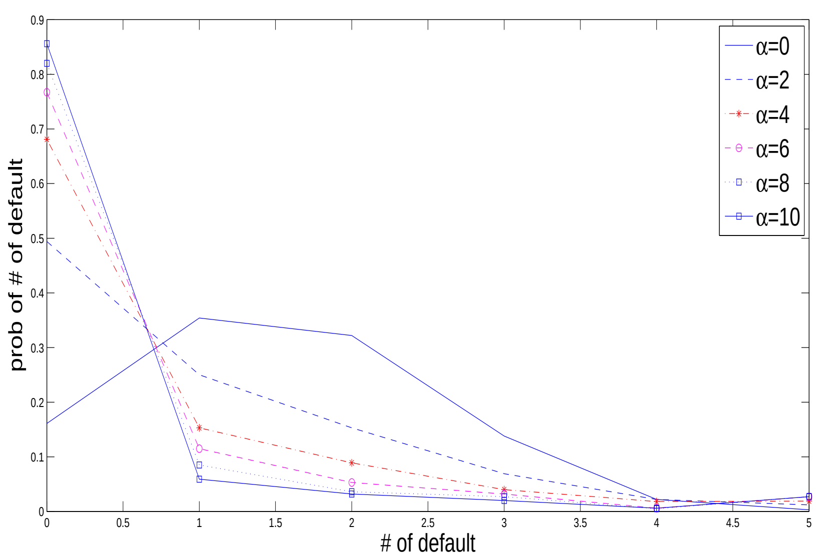

Figure 2.4: The loss distribution of the coupled diffusions with shifting mean-reverting rate a.

Chapter 3

Systemic Risk and Coupled Diffusions

The results to be described in this chapter are related to [12] and [25]. Following Chapter 2, we now focus on systemic risk and the corresponding coupled diffusions (1.1). As previously noted, since we are interested in systemic risk affected by lending and borrowing behavior, we consider this lending and borrowing interaction through the drifts. Systemic risk is described as the possibility of the large amount of defaults before the maturity T.

For simplicity, we first consider a system of N banks without the possibility of borrowing or lending to the central bank; that is (1.1) with $\alpha_t^i = 0$:

$$\begin{aligned} dX_t^i &= \frac{a}{N}\sum_{j=1}^{N}(X_t^j - X_t^i)\,dt + \sigma d\widetilde{W}_t^i \\ &= a(\overline{X}_t - X_t^i)dt + \sigma d\widetilde{W}_t^i, \quad i = 1, \cdots, N. \end{aligned} \tag{3.1}$$

To start with, we assume that the Brownian motions $\widetilde{W}^i$ are independent, that is $\rho = 0$ and $\widetilde{W}^i = W^i$. We also assume the initial value $X_0^i = \xi^i = 0$. In order to see how this lending and borrowing behavior affects the system, Section 3.1 will show the loss distribution of independent Brownian motions given the lending and borrowing rate $a = 0$. Section 3.2 and Section 3.3 provide the computational details to explain the difference between the two models and the asymptotic solution for the possibility of all defaults seen as the systemic risk using the large deviation principle. In Section 3.4, we will consider the correlated (or *common noise*) case.

3.1 Stability and Systemic Risk

We compare the coupled diffusions (3.1) to an independent case which is a system without borrowing from and lending to each other and simply given by driftless Brownian motions:

$$dY_t^i = \sigma dW_t^i \qquad i = 1, \ldots, N. \tag{3.2}$$

Notice that in the case $a = 0$, the system (3.1) reduces to the independent system (3.2). We first compare the coupled system (3.1) to the independent system (3.2) by looking at their trajectories. In our simulation, we choose the initial value $Y_0^i = y_0^i = 0, i = 1, \ldots, N$ and the common parameters $\sigma = 1$, $D = -0.7$, and $N = 10$, and use the Euler scheme with a time-step $\Delta = 10^{-4}$, up to time $T = 1$. In Figures 3.1, 3.2 and 3.3, we show one typical realization in each figure with $a = 1$, $a = 10$ and

$a = 100$, respectively. As we increase the rate a in the case of increasing frequency of borrowing and lending, we can roughly collect all trajectories together, and less trajectories arrive at boundary D. This shows that to avoid bankruptcy, compared to working independently, each bank has to do borrowing and lending more frequently. This is a good sign for this model because it shows that we can create the stability of the system by coupling with each other.

Next, as in Chapter 2, we show the loss distributions of the coupled diffusions (3.1) and the independent system (3.2) in Figures 3.4, 3.5, and 3.6. We compute these loss distributions via the Monte Carlo method using 10^4 simulations with the same parameters. Again, the loss distribution of the dependent case can be computed by a binomial distribution with parameters N and p where p is given by (2.4).

As expected, compared to the independent case (3.2), the coupled system (3.1) gives a much larger probability of all surviving (no default), and a smaller probability of individual default appears by increasing the borrowing and lending rate a. Specifically, when $a = 100$, the probability of all surviving is close to 1. It shows the same information as previous typical trajectories. Coupled and forced lending and borrowing help the system become more stable and durable. However, there is trade-off in the coupled system. in Figures 3.4, 3.5, and 3.6, we obtain that the possibility of large defaults is larger than the independent one. This is one specific term to describe systemic risk. In other words, this lending and borrowing improves the stability of the system by keeping the diffusions near zero (away from default level) most of the

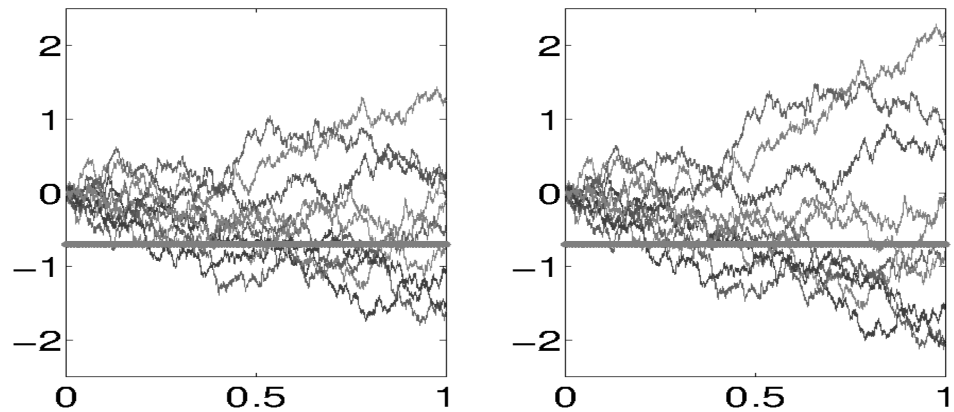

Figure 3.1: One realization of the trajectories of the coupled diffusions (1.1) with $a = 1$ (left plot) and trajectories of the independent Brownian motions (3.2) (right plot) using the same Gaussian increments. The solid horizontal line represents the "default" level $D = -0.7$.

time. However, we also see that there is small but non-negligible probability, that almost all diffusions reach the default level. To summarize, this lending and borrowing system (3.1) not only creates the stability but also systemic risk. In next section, we explain the conclusion by exact computation and show the approximated solution of all defaults by the large deviation principle.

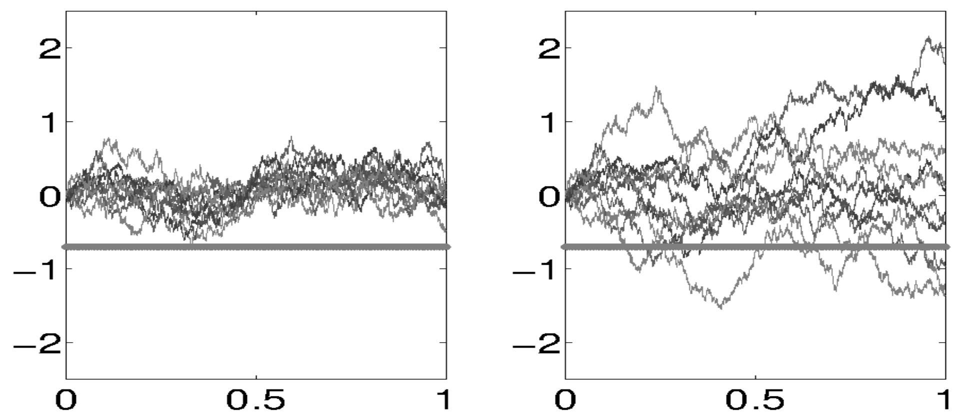

Figure 3.2: One realization of the trajectories of the coupled diffusions (1.1) (left plot) with $a = 10$ and trajectories of the independent Brownian motions (3.2) (right plot) using the same Gaussian increments. The solid horizontal line represents the "default" level $D = -0.7$.

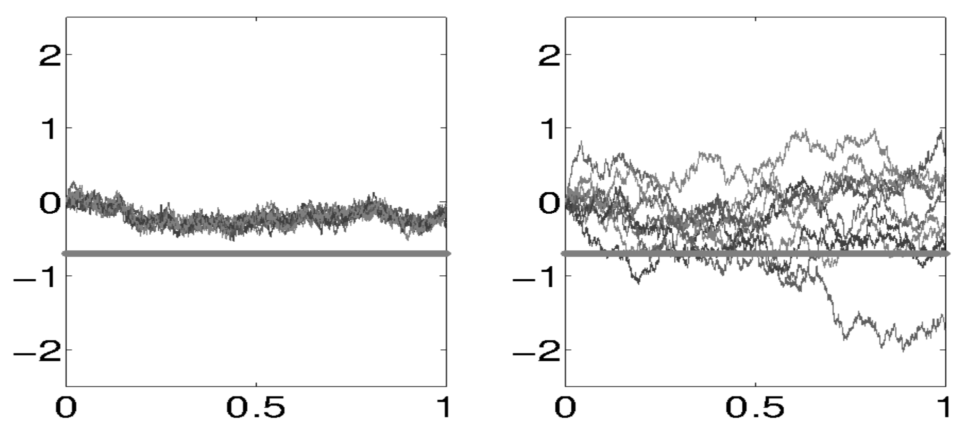

Figure 3.3: One realization of the trajectories of the coupled diffusions (1.1) (left plot) with $a = 100$ and trajectories of the independent Brownian motions (3.2) (right plot) using the same Gaussian increments. The solid horizontal line represents the "default" level $D = -0.7$.

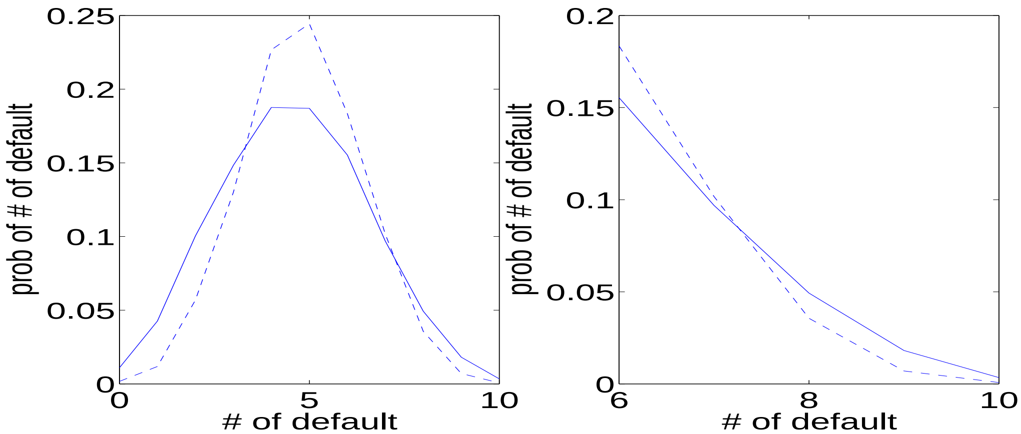

Figure 3.4: On the left, we show plots of the loss distribution for the coupled diffusions with $a = 1$ (solid line) and for the independent Brownian motions (dashed line). The plots on the right show the corresponding tail probabilities.

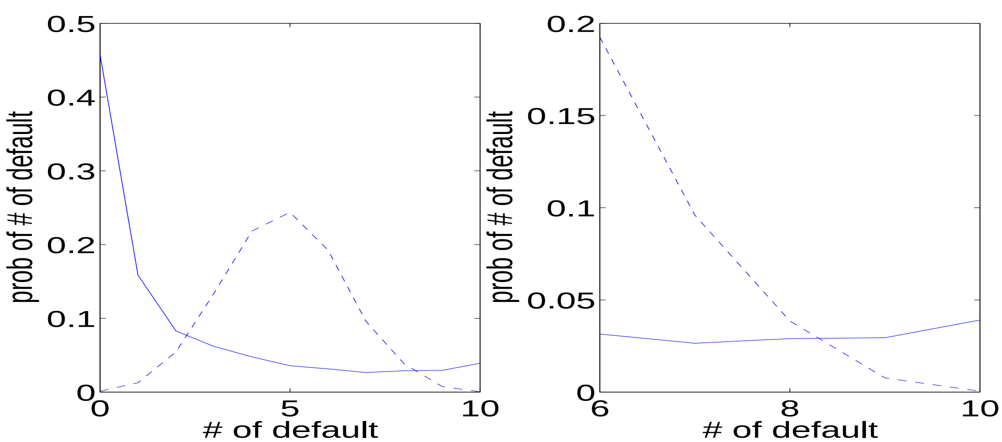

Figure 3.5: On the left, we show plots of the loss distribution for the coupled diffusions with $a = 10$ (solid line) and for the independent Brownian motions (dashed line). The plots on the right show the corresponding tail probabilities.

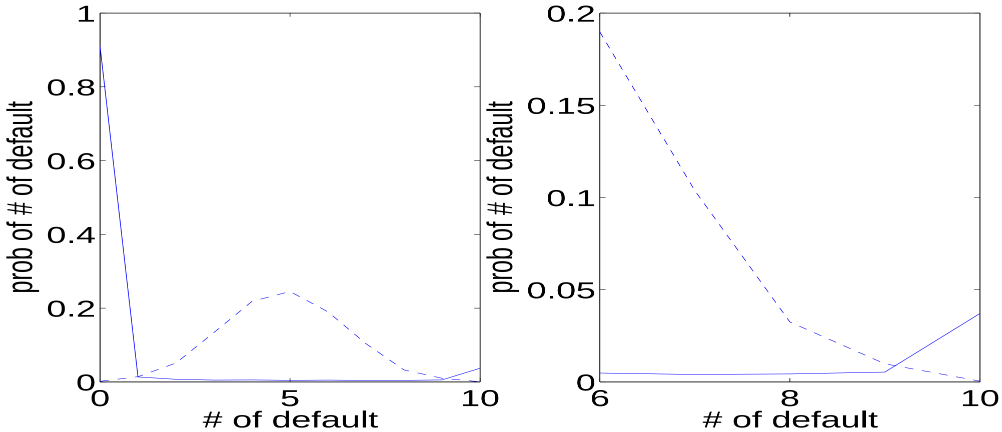

Figure 3.6: On the left, we show plots of the loss distribution for the coupled diffusions with $a = 100$ (solid line) and for the independent Brownian motions (dashed line). The plots on the right show the corresponding tail probabilities.

3.2 Mean-Field Limit

In order to understand the coupled system (3.1) and answer the question in Chapter 2 and Section 3.1, we rewrite the dynamics as

$$\begin{aligned} dX_t^i &= \frac{a}{N}\sum_{j=1}^{N}(X_t^{(j)} - X_t^{(i)})\,dt + \sigma dW_t^i \\ &= a\left[\left(\frac{1}{N}\sum_{j=1}^{N} X_t^j\right) - X_t^{(i)}\right]dt + \sigma dW_t^i. \end{aligned} \tag{3.3}$$

We observe that the diffusions are an OU-type with mean-reverting at their ensemble mean. In other words, the ensemble average roughly leads the whole system. To understand the system, it is important to know the dynamics of ensemble average. By simple algebra, the ensemble average satisfies

$$d\left(\frac{1}{N}\sum_{i=1}^{N} X_t^i\right) = d\left(\frac{\sigma}{N}\sum_{i=1}^{N} W_t^i\right),$$

For simplicity, assuming the initial value $X_0^i = x_0^i = 0$ for all i, we obtain

$$\frac{1}{N}\sum_{i=1}^{N} X_t^{(i)} = \frac{\sigma}{N}\sum_{i=1}^{N} W_t^i, \tag{3.4}$$

and the coupled system

$$dX_t^i = a\left[\left(\frac{\sigma}{N}\sum_{j=1}^{N} W_t^{(j)}\right) - X_t^i\right] dt + \sigma dW_t^i\,. \tag{3.5}$$

The exact solution of X_t^i is

$$X_t^i = \frac{\sigma}{N}\sum_{j=1}^{N} W_t^j + \sigma e^{-at}\int_0^t e^{as} dW_s^i - \frac{\sigma}{N}\sum_{j=1}^{N}\left(e^{-at}\int_0^t e^{\alpha s} dW_s^{(j)}\right). \tag{3.6}$$

In fact, note that the ensemble average is distributed as a Brownian motion with diffusion coefficient $\sigma/\sqrt{N}$.

In the limit $N \to \infty$, the strong law of large numbers gives

$$\frac{1}{N}\sum_{j=1}^{N} W_t^j \to 0 \quad a.s.\,,$$

and, more generally, the processes X_t^i (3.6) converges to the independent OU processes

$$\sigma e^{-at}\int_0^t e^{as} dW_s^i$$

with long-run mean zero. This is in fact a simple example of a mean-field limit and propagation of chaos studied in general in [26].

Note that the distributions of hitting times for OU processes have been studied in [1]. Let us denote

$$p' = I\!\!P\left(\tau \le T\right)\,,$$

τ being the hitting time of the default level for an OU process with long-run mean zero, given by

$$dX_t = -aX_t dt + \sigma dW_t \,.$$

In the interesting regime where $p'N \to \lambda > 0$, obtained as $N \to \infty$ and $D \to -\infty$ appropriately, the loss distribution converges to a Poisson distribution with parameter λ. In this stable regime, the mass is mainly concentrated on a small number of defaults.

3.3 Systemic Risk and Large deviations

In the Section 3.1, our simple toy model (3.1) can be written as OU processes with the mean reverting at the ensemble average and this shows that the coupled system roughly follows the ensemble average. Consequently, the diffusions intend *flocking to default* if the ensemble average arrives at default level D. Therefore, we identify the event

$$\left\{ \min_{0\le t\le T} \left(\frac{\sigma}{N} \sum_{i=1}^{N} X_t^i \right) \le D \right\}$$

as a *systemic event*. In our simple example, this probability can be computed explicitly as follows:

$$\begin{aligned} I\!P \left(\min_{0\le t\le T} \left(\frac{\sigma}{N} \sum_{i=1}^{N} W_t^i \right) \le D \right) &= I\!P \left(\min_{0\le t\le T} \widetilde{W}_t \le \frac{D\sqrt{N}}{\sigma} \right) \\ &= 2\Phi \left(\frac{D\sqrt{N}}{\sigma\sqrt{T}} \right) , \end{aligned} \tag{3.7}$$

where $\widetilde{W}$ is a standard Brownian motion. From (3.7), we observe that probability of this event is small (when N becomes large), and is given by the theory of large deviations. Therefore, using a classical equivalent for the Gaussian cumulative distribution function, we obtain

$$\lim_{N\to\infty} -\frac{1}{N}\log I\!P\left(\min_{0\le t\le T}\left(\frac{\sigma}{N}\sum_{i=1}^{N}W_t^i\right)\le D\right)=\frac{\eta^2}{2\sigma^2 T}\,. \tag{3.8}$$

In other words, for a large number of banks, the probability that the ensemble average reaches the default barrier is of order $\exp(-D^2N/(2\sigma^2 T))$. Observe that this event does not depend on $a > 0$. In Figure 3.6, in the case of $a = 100$, the probability of systemic risk is roughly 3% which can be obtained by using (3.7). We can not prevent systemic risk from increasing α but the individual default can be avoided, and consequently, the probability of a safe system is increased by large α.

To summarize, the simple coupled system (3.1) proves stability but also contributes systemic risk. We identified that this lending and borrowing behavior plays a crucial role and affects the system significantly.

3.4 Systemic Risk and Common Noise

We now consider the coupled diffusions driven by correlated Brownian motions $(\widetilde{W}_t^i)$ and without control ($\alpha^i = 0$). That is, banks not only have the lending and borrowing behavior but have other unexpected correlated noises. For example, banks may invest their money in the same underlying assets. These correlated noises are considered as correlated Brownian motions. We are interested in this common noise creating either a better system (lower probability of a large number of defaults) or a worse system (higher probability of a large number of defaults). The dynamics (1.1) or (3.3) become:

$$dX_t^i = a\left(\frac{1}{N}\sum_{j=1}^N X_t^j - X_t^i\right)dt + \sigma\left(\rho dW_t^0 + \sqrt{1-\rho^2}dW_t^i\right), \quad i = 1, \cdots, N, \quad (3.9)$$

where $(W_t^0, W_t^i, i = 1, \cdots, N)$ are independent standard Brownian motions and W_t^0 is a common noise. As before, we calculate the ensemble average

$$\begin{aligned}\frac{1}{N}\sum_{i=1}^N X_t^i = \frac{\sigma}{N}\sum_{i=1}^N \widetilde{W}_t^i &= \sigma\left(\rho W_t^0 + \frac{\sqrt{1-\rho^2}}{N}\sum_{i=1}^N W_t^i\right) \\ &\overset{D}{=} \sigma\sqrt{\rho^2 + \frac{(1-\rho^2)}{N}}B_t,\end{aligned}$$

where B_t is a standard Brownian motion. Moreover, the explicit solution for X_t^i is

$$X_t^i = \sigma\rho W_t^0 + \sigma\sqrt{1-\rho^2}\left(\frac{1}{N}\sum_{j=1}^N W_t^j + \int_0^t e^{a(s-t)}dW_s^i - \frac{1}{N}\sum_{j=1}^N \int_0^t e^{a(s-t)}dW_s^j\right).$$

The probability of the systemic event (3.7) becomes

$$\begin{aligned}I\!P\left(\min_{0\le s\le T}\frac{1}{N}\sum_{i=1}^N X_s^i < D\right) &= I\!P\left(\min_{0\le s\le T} B_s < \frac{D}{\sigma}\sqrt{\frac{N}{N\rho^2 + (1-\rho^2)}}\right) \\ &= 2\Phi\left(\frac{D}{\sigma\sqrt{T}}\sqrt{\frac{N}{N\rho^2 + (1-\rho^2)}}\right).\end{aligned}$$

From the formula above, we observe that in the correlated case ($\rho \neq 0$), the probability of systemic risk does not vanish as N becomes large, instead it converges to $2\Phi\left(\frac{D}{\sigma|\rho|\sqrt{T}}\right)$. This is in dramatic contrast with the independent case ($\rho = 0$) where the probability of systemic risk is exponentially small in N. We illustrate this

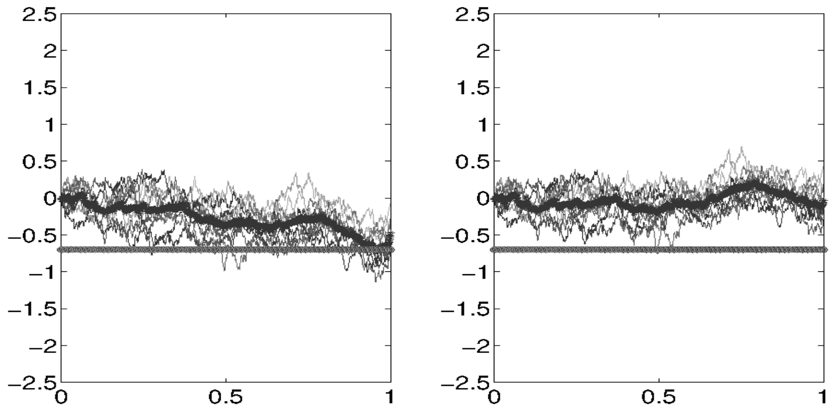

Figure 3.7: One realization of $N = 10$ trajectories of the coupled diffusions with independent Brownian motions (3.1) (left plot) and trajectories of the coupled diffusions with correlated Brownian motions with $\rho = 0.2$ (3.9) (right plot) using the common $a = 10$. The solid horizontal line represents the “default” level $D = -0.7$.

instability created by the common noise in Figures 3.7, 3.8, and 3.9. Furthermore, in Figures 3.10, 3.11, 3.12, compared to independent noises, the much larger probability of all defaults is obtained in the correlated case. This exactly illustrates reality. If banks want to consider risky strategies with higher correlation, they might have a large number of returns together. However, for example, if few banks invest in Lehman bonds, they will crash and consequently contribute systemic risk to the whole banking system through lending and borrowing. Hence, we conclude that in order to help keep the system stable, banks have to reduce the correlation of uncertainty using diverse investments.

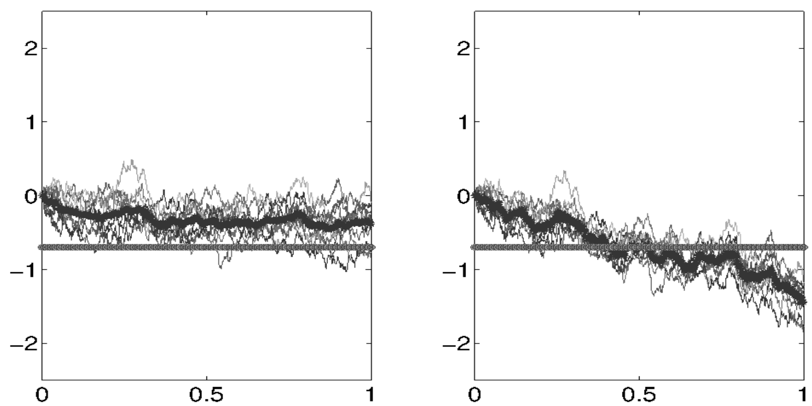

Figure 3.8: One realization of $N = 10$ trajectories of the coupled diffusions with independent Brownian motions (3.1) (left plot) and trajectories of the coupled diffusions with correlated Brownian motions with $\rho = 0.5$ (3.9) (right plot) using the common $a = 10$. The solid horizontal line represents the "default" level $D = -0.7$.

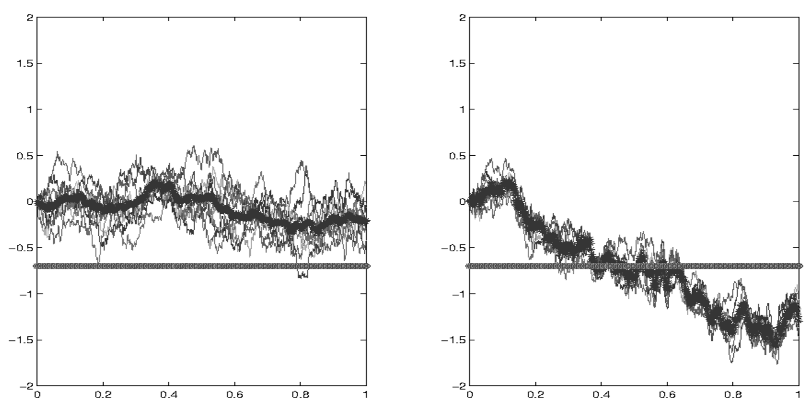

Figure 3.9: One realization of $N = 10$ trajectories of the coupled diffusions with independent Brownian motions (3.1) (left plot) and trajectories of the coupled diffusions with correlated Brownian motions with $\rho = 0.8$ (3.9) (right plot) using the common $a = 10$. The solid horizontal line represents the "default" level $D = -0.7$.

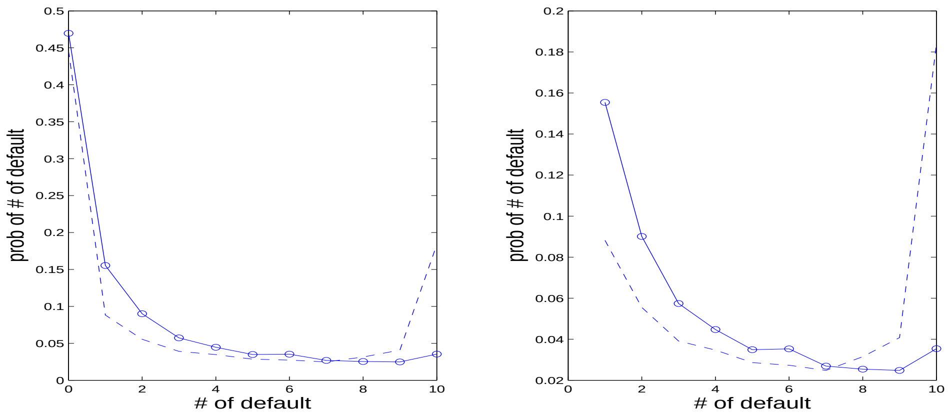

Figure 3.10: On the left, we show plots of the loss distribution for the coupled diffusions with independent Brownian motions (solid line) and with correlated Brownian motions with $\rho = 0.2$(dashed line) with $a = 10$. The plots on the right show the corresponding tail probabilities.

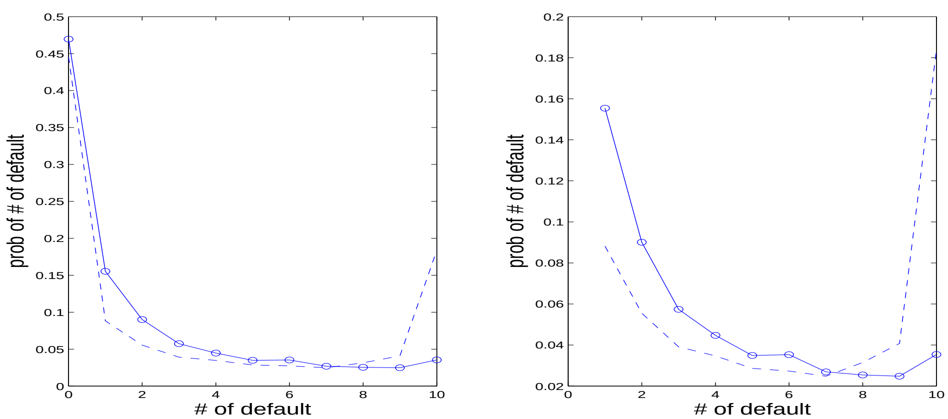

Figure 3.11: On the left, we show plots of the loss distribution for the coupled diffusions with independent Brownian motions (solid line) and with correlated Brownian motions with $\rho = 0.5$(dashed line) with $a = 10$. The plots on the right show the corresponding tail probabilities.

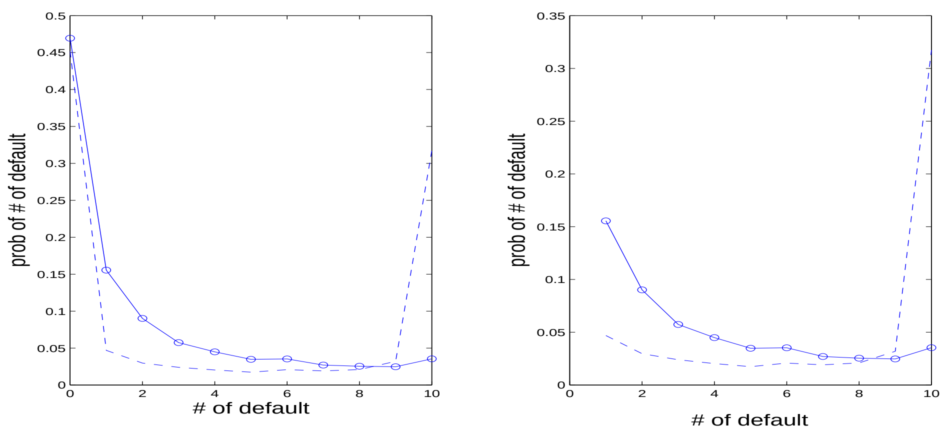

Figure 3.12: On the left, we show plots of the loss distribution for the coupled diffusions with independent Brownian motions (solid line) and with correlated Brownian motions with $\rho = 0.8$(dashed line) with $a = 10$. The plots on the right show the corresponding tail probabilities.

Part II

Systemic Risk and Dynamic Noncooperative Games

Chapter 4

Systemic Risk and Dynamic Noncooperative Games (Mean Field Type)

The noncooperative games analysis in Chapters 4, 5, and 6 is studied in [25]. We return to the original model (1.1). All banks not only make bank-to-bank transactions but also control their rates of borrowing from and lending to the central bank. That is, if banks are under the average, they need to request money from the central bank as much as possible in order to help themselves. Oppositely, if banks are above the average, they consider lending money to the central bank as much as possible for earing interest. However, meanwhile, banks have to pay the corresponding fee for performing these transactions. Therefore, they must find the optimal strategies in order to minimize their own cost functions according to the distance between their states and the ensemble average. We are interested in the influence of this game on systemic risk. Through the optimal strategies, is it possible to reduce systemic risk for this bank system or make this system in danger?

In this chapter, we focus on N player games where N is finite. Unlike optimal portfolio problems for only one player, we need to consider N controls for N players and in fact, each optimal strategy depends on all the other optimal strategies. In other words, we are seeking the equilibria in this game. We discuss two kinds of Nash equilibria: (i) open-loop equilibria and (ii) memoryless closed-loop (feedback) equilibria. We first construct open-loop equilibria using Forward-Backward Stochastic Differential Equations (FBSDE) given by Pontryagin's stochastic minimum principle. Next, we construct memoryless closed-loop (feedback) equilibria using two approaches. The first one is based on the modified version of the Pontryagin principle and FBSDE used in the open-loop case. Another one is based on Hamilton-Jacobi-Bellman (HJB) partial differential equations following from the dynamic programming principle. A comparison of the open-loop and closed-loop equilibria is discussed in the final section. Recall that the log monetary reserves X_t^i for $i = 1, \cdots, N$ are given by

$$dX_t^i = \left[a(\overline{X}_t - X_t^i) + \alpha_t^i\right] dt + \sigma\left(\sqrt{1-\rho^2}dW_t^i + \rho dW_t^0\right), \tag{4.1}$$

where $W_t^i, i = 0, 1, \ldots, N$ are independent Brownian motions, $\sigma > 0$ and $a \geq 0$. Bank $i \in \{1, \cdots, N\}$ is trying to control its rate of lending and borrowing (to a central bank) at time t by choosing the control α_t^i in order to minimize the cost

$$J^i(\alpha^1, \cdots, \alpha^N) = I\!\!E \left\{ \int_0^T f_i(X_t, \alpha_t^i) dt + g_i(X_T^i) \right\}, \tag{4.2}$$

with

$$f_i(x, \alpha^i) = \left[\frac{1}{2}(\alpha^i)^2 - q\alpha^i(\overline{x} - x^i) + \frac{\epsilon}{2}(\overline{x} - x^i)^2 \right], \tag{4.3}$$

$$g_i(x) = \frac{c}{2}\left(\overline{x} - x^i\right)^2, \tag{4.4}$$

and where $f_i(x, \alpha)$ is convex in (x, α) under the assumption $q^2 \leq \epsilon$. This model is one specific case in the class of *Linear-Quadratic (LQ)* noncooperative games with mean field type as discussed in [2] and [5] for instance.

4.1 Open-Loop Equilibria

To construct open-loop equilibria, this problem consists in searching for an equilibrium among strategies. In the deterministic case ($\sigma = 0$), without any information regarding equilibria, we search for an equilibrium among strategies which are (deterministic) functions $\{\alpha_t^i, \ i = 1, \cdots, N\}$ given at the initial time $t = 0$ and from which $\{(X_t^i), \ i = 1, \cdots, N\}$ are deduced by (4.1). See [2] for example. In the stochastic case ($\sigma > 0$), the open-loop problem consists in searching for an equilibrium among strategies $\{\alpha_t^i, i = 1, \cdots, N\}$ which are adapted processes satisfying some integrability property such as $I\!\!E\left(\int_0^T |\alpha_t^i| dt\right) < \infty$, and most importantly. See [5] for example.

Using the Pontryagin principle to obtain open-loop equilibria, the Hamiltonian for bank i is written as

$$H^i(x^1, \cdots, x^N, y^{i,1}, \cdots, y^{i,N}, \alpha^1, \cdots, \alpha^N) \tag{4.5}$$
$$= \sum_{k=1}^{N} \left[a(\overline{x} - x^k) + \alpha^k\right] y^{i,k} + \frac{1}{2}(\alpha^i)^2 - q\alpha^i(\overline{x} - x^i) + \frac{\epsilon}{2}(\overline{x} - x^i)^2.$$

For a given $\alpha = (\alpha^i)_{i=1,\cdots,n}$, the controlled forward dynamics of the states X_t^i are given by (4.1) with initial conditions $X_0^i = x^i$, and based on the Pontryagin principle, the adjoint processes $Y_t^i = (Y_t^{i,j};\, j = 1, \cdots, N)$ and $Z_t^i = (Z_t^{i,j,k};\ j = 1, \cdots, N,\ k = 0, 1, \cdots, N)$ for $i = 1, \cdots, N$ are defined as the solutions to the backward stochastic differential equations (BSDEs):

$$dY_t^{i,j} = -\partial_{x^j} H^i(X_t, Y_t^i, \alpha_t)dt + \sum_{k=0}^{N} Z_t^{i,j,k} dW_t^k, \tag{4.6}$$

with terminal conditions $Y_T^{i,j} = \partial_{x^j} g_i(X_T)$. For each admissible (i.e. satisfying the above integrability condition) strategy profile $\alpha = (\alpha^i)_{i=1,\cdots,n}$, standard existence and uniqueness results for BSDEs apply, and the existence of the adjoint processes is guaranteed. Without plugging any information regarding the strategies α^i, $i = 1, \cdots, N$, the partial derivative Hamiltonian H^i is given by

$$\partial_{x^j} H^i \quad = \quad \frac{a}{N}\sum_{k=1}^{N}(y^{i,k} - y^{i,j}) - q\alpha^i(\frac{1}{N} - \delta_{i,j}) + \epsilon(\overline{x} - x^i)(\frac{1}{N} - \delta_{i,j}). \tag{4.7}$$

To satisfy the necessary condition of the Pontryagin principle, one has to minimize the Hamiltonian with respect to α^i. See, for example, the discussion of the Isaacs conditions in [5]. This leads to the choice of bank i

$$\hat{\alpha}^i = -y^{i,i} + q(\overline{x} - x^i). \tag{4.8}$$

Next step, we assume that all the players are making that choice so as to prove that this choice is a Nash equilibrium. With this choice of control α^i, the forward equations (4.1) become coupled with the backward equation (4.6). Then, by obtaining the solution of (4.6) and identifying each player's own adjoint equation, player i finds the best response. In general, this forward-backward system is difficult to find the solution. In our linear case, we make the ansatz

$$Y_t^{i,j} = \phi_t(\frac{1}{N} - \delta_{i,j})(\overline{X}_t - X_t^i), \tag{4.9}$$

where ϕ_t is a deterministic function satisfying the terminal condition $\phi_T = c$. We now plug the ansatz into the backward equation (4.6); a careful computation gives the backward equation

$$dY_t^{i,j} = (\frac{1}{N} - \delta_{i,j})(\overline{X}_t - X_t^i)\left[(a + (1 - \frac{1}{N})q)\phi_t - (\epsilon - q^2)\right] dt + \sum_{k=0}^{N} Z_t^{i,j,k} dW_t^k. \tag{4.10}$$

On the other hand, by differentiating the ansatz (4.9), the processes $Y_t^{i,j}$ becomes

$$dY_t^{i,j} = \dot{\phi}_t(\frac{1}{N} - \delta_{i,j})(\overline{X}_t - X_t^i)dt + \phi(\frac{1}{N} - \delta_{i,j})d(\overline{X}_t - X_t^i). \tag{4.11}$$

where $\dot{\phi}_t$ denotes the time-derivative of ϕ_t. In order to find the dynamics $Y_t^{i,j}$, it is necessary to show the derivation of $d(\overline{X}_t - X_t^i)$. Using (4.8) and (4.9), the forward equation (4.1) becomes

$$dX_t^i = \left[a + q + (1 - \frac{1}{N})\phi_t\right](\overline{X}_t - X_t^i)dt + \sigma\left(\sqrt{1-\rho^2}dW_t^i + \rho dW_t^0\right), \tag{4.12}$$

which by summation gives

$$d\overline{X}_t = \sigma\rho dW_t^0 + \sigma\sqrt{1-\rho^2}\left(\frac{1}{N}\sum_{i=1}^{N} dW_t^i\right). \tag{4.13}$$

Consequently, one obtains

$$d(\overline{X}_t - X_t^i) = -\left[a + q + (1-\frac{1}{N})\phi_t\right](\overline{X}_t - X_t^i)dt + \sigma\sqrt{1-\rho^2}\left(\frac{1}{N}\sum_{k=1}^{N} dW_t^k - dW_t^i\right). \tag{4.14}$$

Using (4.14), the dynamics of $Y_t^{i,j}$ can be written as

$$\begin{aligned} dY_t^{i,j} &= (\frac{1}{N}-\delta_{i,j})(\overline{X}_t - X_t^i)\left[\dot{\phi}_t - \phi_t\left(a+q+(1-\frac{1}{N})\phi_t\right)\right]dt \\ &\quad +\phi_t(\frac{1}{N}-\delta_{i,j})\sigma\sqrt{1-\rho^2}\left(\frac{1}{N}\sum_{k=1}^{N} dW_t^k - dW_t^i\right). \end{aligned} \tag{4.15}$$

We now compare the two Itô decompositions (4.10) and (4.15). The martingale terms give the processes $Z_t^{i,j,k}$,

$$Z_t^{i,j,0} = 0, \quad Z_t^{i,j,k} = \phi_t\sigma\sqrt{1-\rho^2}(\frac{1}{N}-\delta_{i,j})(\frac{1}{N}-\delta_{i,k}) \text{ for } \; k = 1,\cdots,N,$$

which turn out to be determistic in our case and hence adapted. Identifying the drift terms show that the function ϕ_t must satisfy the scalar Riccati equation

$$\dot{\phi}_t = 2(a+(1-\frac{1}{2N})q)\phi_t + (1-\frac{1}{N})\phi_t^2 - (\epsilon - q^2), \tag{4.16}$$

with the terminal condition $\phi_T = c$. The Riccati equation can be explicitly solved by

$$\phi_t = \frac{-(\epsilon-q^2)\left(e^{(\delta^+-\delta^-)(T-t)}-1\right) - c\left(\delta^+ e^{(\delta^+-\delta^-)(T-t)} - \delta^-\right)}{(\delta^- e^{(\delta^+-\delta^-)(T-t)} - \delta^+) - c(1-\frac{1}{N})\left(e^{(\delta^+-\delta^-)(T-t)}-1\right)}, \tag{4.17}$$

where

$$\delta^{\pm} = -(a+(1-\frac{1}{2N})q) \pm \sqrt{R},$$

with

$$R := (a+(1-\frac{1}{2N})q)^2 + \left(1-\frac{1}{N}\right)(\epsilon - q^2) > 0.$$

In order to check that the solution is well defined, we rewrite the denominator as follows:

$$-\left(e^{(\delta^+-\delta^-)(T-t)}+1\right)\sqrt{R}-\left(a+q+c\left(1-\frac{1}{N}\right)\right)\left(e^{(\delta^+-\delta^-)(T-t)}-1\right).$$

This stays negative for all $t<T$ because $\delta^+-\delta^-=2\sqrt{R}>0$. Furthermore, since $q^2\leq\epsilon$, we observe that ϕ_t for all $t<T$ is positive with $\phi_T=c$ as shown in the Figure 4.1.

Note that in this open-loop equilibrium, each bank i can implement its own strategy by only knowing $\overline{X}_t-X_t^i$ instead of $(X_t)_{0\leq t\leq T}$ which is, in fact, in closed-loop feedback form. The further comparison will be discussed in Section 4.4. It is now natural to search for closed-loop feedback equilibria with the Markovian property as we will discuss in Section 4.2.

4.2 Closed-Loop Equilibria: FBSDE Approach

We now solve for an exact closed-loop Nash equilibrium when players/banks at time t have complete information of states (perfect states) of all other players at time t and use Markovian strategies.

In this context, when all the other players $k\neq i$ have chosen strategies in feedback form given by deterministic functions $\alpha^k(t,x)$ of time and state, player i needs to solve a Markovian control problem to find his best response to these choices, and the

Hamiltonian of his control problem is given by (see [5]):

$$H^i(x, y^{i,1}, \cdots, y^{i,N}, \alpha^1(t,x), \cdots, \alpha^i_t, \cdots, \alpha^N(t,x)) \tag{4.18}$$
$$= \sum_{k \neq i} \left[a(\overline{x} - x^k) + \alpha^k(t,x)\right] y^{i,k} + \left[a(\overline{x} - x^i) + \alpha^i\right] y^{i,i}$$
$$+ \frac{1}{2}(\alpha^i)^2 - q\alpha^i(\overline{x} - x^i) + \frac{\epsilon}{2}(\overline{x} - x^i)^2.$$

The dynamics of the state $X_t = (X^i_t)_{i=1,\cdots,N}$ are again given by (4.1) with initial conditions $X^i_0 = x^i$, and the adjoint processes $Y^i = (Y^{i,j})_{j=1,\cdots,N}$ and $Z^i = (Z^{i,j,k})_{j=1,\cdots,N,\, k=0,1,\cdots,N}$ of player i are the solutions of the same BSDE (4.6) with H^i given in (4.18). Consequently, $\partial_{x^j} H^i$ is computed from (4.18), giving:

$$\begin{aligned} \partial_{x^j} H^i \;&=\; a\sum_{k=1}^{N}(\frac{1}{N} - \delta_{k,j})y^{i,k} + \sum_{k \neq i}(\partial_{x_j}\alpha^k(t,x))y^{i,k} \\ &\quad -q\alpha^i(\frac{1}{N} - \delta_{i,j}) + \epsilon(\overline{x} - x^i)(\frac{1}{N} - \delta_{i,j}). \end{aligned} \tag{4.19}$$

Again, the necessary part of the stochastic maximum principle suggests that one minimizes H^i with respect to α^i, leading again to the choice

$$\hat{\alpha}^i = -y^{i,i} + q(\overline{x} - x^i),$$

which has the same form as (4.8). Again, using Pontryagin's minimum principle and injecting X^i_t and $Y^{i,j}_t$ into the state equations and the adjoint equations, we obtain a large system of forward backward stochastic equations which, if solved, provides a Markovian Nash equilibrium. Similar to the open-loop problem, we make an ansatz of the same form as in

$$Y^{i,j}_t = \eta_t(\frac{1}{N} - \delta_{i,j})(\overline{X}_t - X^i_t), \tag{4.20}$$

for some deterministic scalar function η_t satisfying the terminal condition $\eta_T = c$. With this choice of optimal controls and ansatz (4.20) which ensures that the control α^i is a feedback control , we get

$$\begin{aligned}\alpha^k(t,x) &= \left[q + (1-\frac{1}{N})\eta_t\right](\overline{x} - x^k),\\ \partial_{x^j}\alpha^k(t,x) &= \left[q + (1-\frac{1}{N})\eta_t\right](\frac{1}{N} - \delta_{k,j}),\end{aligned}$$

and the derivation of $\partial_{x^j} H^i$ is written as

$$\begin{aligned}\partial_{x^j} H^i &= a\sum_{k=1}^{N}(\frac{1}{N} - \delta_{k,j})\eta_t(\frac{1}{N} - \delta_{i,k})(\overline{x_t} - x_t^i)\\ &\quad + \sum_{k\neq i}[q + \eta_t(1-\frac{1}{N}](\frac{1}{N} - \delta_{k,j})\eta_t(\frac{1}{N} - \delta_{i,k}(\overline{x_t} - x_t^i)\\ &\quad - q[q + \eta_t(1-\frac{1}{N}](\overline{x_t} - x_t^i)(\frac{1}{N} - \delta_{i,j}) + \epsilon(\overline{x} - x^i)(\frac{1}{N} - \delta_{i,j})\\ &= a\eta_t(\overline{x_t} - x_t^i)\sum_{k=1}^{N}(\frac{1}{N} - \delta_{k,j})(\frac{1}{N} - \delta_{i,k})\\ &\quad + [q + \eta_t(1-\frac{1}{N})]\eta_t(\overline{x_t} - x_t^i)\frac{1}{N}\sum_{k\neq i}(\frac{1}{N} - \delta_{k,j})\\ &\quad - (q^2 - \epsilon)(\overline{x} - x^i)(\frac{1}{N} - \delta_{i,j}) - q\eta_t(1-\frac{1}{N})(\overline{x} - x^i)(\frac{1}{N} - \delta_{i,j})\\ &= (\overline{x} - x^i)(\frac{1}{N} - \delta_{i,j})\left[-(a+q)\eta_t + \frac{1}{N}(\frac{1}{N} - 1)\eta_t^2 - (q^2 - \epsilon)\right]\end{aligned}$$

by using

$$\sum_{k=1}^{N}(\frac{1}{N} - \delta_{k,j})(\frac{1}{N} - \delta_{i,k}) = -(\frac{1}{N} - \delta_{i,j}),\ \sum_{k\neq i}(\frac{1}{N} - \delta_{k,j}) = -(\frac{1}{N} - \delta_{i,j}).$$

Therefore, (4.19) reduces the backward equations to

$$
\begin{aligned}
dY_t^{i,j} &= -\partial_{x^j} H^i dt + \sum_{k=0}^{N} Z_t^{i,j,k} dW_t^k \\
&= (\frac{1}{N} - \delta_{i,j})(\overline{X}_t - X_t^i)\Big[(a+q)\eta_t - \frac{1}{N}(\frac{1}{N} - 1)\eta_t^2 + q^2 - \epsilon\Big] dt \\
&\quad + \sum_{k=0}^{N} Z_t^{i,j,k} dW_t^k, \qquad (4.21)
\end{aligned}
$$

with terminal conditions $Y_T^{i,j} = c(\frac{1}{N} - \delta_{i,j})(\overline{X}_T - X_T^i)$. Using (4.8) and (4.20), the forward equations become

$$
dX_t^i = \Big[a + q + (1 - \frac{1}{N})\eta_t\Big] (\overline{X}_t - X_t^i) dt + \sigma\Big(\sqrt{1-\rho^2} dW_t^i + \rho dW_t^0\Big), \qquad (4.22)
$$

and by summation we deduce that $\overline{X}_t$ satisfies again (4.13). Differentiating the ansatz (4.20), using (4.13) and (4.22) for the forward dynamics and following the same computation in Section 4.1, we obtain that $Y_t^{i,j}$ satisfies equation (4.15) with ϕ_t replaced with η_t:

$$
\begin{aligned}
dY_t^{i,j} = &\Big(\frac{1}{N} - \delta_{i,j}\Big)(\overline{X}_t - X_t^i)\Big[\dot{\eta}_t - \Big(a + q + (1 - \frac{1}{N})\eta_t\Big)\eta_t\Big] dt \\
&+ (\frac{1}{N} - \delta_{i,j})\eta_t \sigma\sqrt{1-\rho^2} \sum_{k=1}^{N} (\frac{1}{N} - \delta_{i,k}) dW_t^k. \qquad (4.23)
\end{aligned}
$$

Again, identifying the martingale terms in the two Itô decompositions (4.21) and (4.23), we get

$$
Z_t^{i,j,0} = 0, \text{ and } Z_t^{i,j,k} = \eta_t \sigma \sqrt{1-\rho^2}(\frac{1}{N} - \delta_{i,j})(\frac{1}{N} - \delta_{i,k}), \text{ for } \ k = 1, \cdots, N,
$$

which satisfy the adapted and square integrable condition. Also by checking the drift terms, we obtain

$$\dot{\eta}_t - \left(a + q + (1 - \frac{1}{N})\eta_t\right)\eta_t = (a+q)\eta_t - \frac{1}{N}(\frac{1}{N} - 1)\eta_t^2 + q^2 - \epsilon.$$

Therefore, η_t must satisfy the scalar Riccati equation

$$\dot{\eta}_t = 2(a+q)\eta_t + (1 - \frac{1}{N^2})\eta_t^2 - (\epsilon - q^2), \tag{4.24}$$

with the terminal condition $\eta_T = c$. Equation (4.24) admits the solution

$$\eta_t = \frac{-(\epsilon - q^2)\left(e^{(\delta^+ - \delta^-)(T-t)} - 1\right) - c\left(\delta^+ e^{(\delta^+ - \delta^-)(T-t)} - \delta^-\right)}{(\delta^- e^{(\delta^+ - \delta^-)(T-t)} - \delta^+) - c(1 - \frac{1}{N^2})\left(e^{(\delta^+ - \delta^-)(T-t)} - 1\right)}, \tag{4.25}$$

where we used the notation

$$\delta^{\pm} = -(a+q) \pm \sqrt{R}, \tag{4.26}$$

with

$$R := (a+q)^2 + \left(1 - \frac{1}{N^2}\right)(\epsilon - q^2) > 0. \tag{4.27}$$

Observe that η_t is well defined for any $t \leq T$ since the denominator in (4.25) can be written as

$$-\left(e^{(\delta^+ - \delta^-)(T-t)} + 1\right)\sqrt{R} - \left(a + q + c\left(1 - \frac{1}{N^2}\right)\right)\left(e^{(\delta^+ - \delta^-)(T-t)} - 1\right),$$

which stays negative because $\delta^+ - \delta^- = 2\sqrt{R} > 0$. In fact, using $q^2 \leq \epsilon$, we see that η_t is positive with $\eta_T = c$ as required and illustrated in Figure 4.1. In Section 4.3, we search for a feedback Nash equlibrium of our N differential games using the HJB approach. In addition, we can verify that the close-loop equilibrium computed by the FBSDE approach is also a feedback Nash equilibrium with strong time consistency.

4.3 Closed-Loop Equilibria: HJB Approach

This section provides an exact feedback Nash equilibrium via HJB approach. Under the Markovian setting, the value function of player i is given by

$$V^i(t,x) = \inf_{\alpha} I\!E_{t,x}\left\{\int_t^T f_i(X_s,\alpha_s^i)ds + g_i(X_T^i)\right\},$$

with the cost functions f_i and g_i given in (4.3) and (4.4), and where the dynamics of X_t is given as before by

$$dX_t^i = \left[a(\overline{X}_t - X_t^i) + \alpha_t^i\right]dt + \sigma\left(\sqrt{1-\rho^2}dW_t^i + \rho dW_t^0\right),\ i = 1,\cdots,N.$$

The dynamic programming principle in search for a closed-loop equilibrium suggests the corresponding HJB equations written as

$$\begin{aligned}
\partial_t V^i \quad &+ \quad \inf_{\alpha^i}\left\{\sum_{j\neq i}\left[a\left(\overline{x}-x^j\right)+\alpha^j(t,x)\right]\partial_{x^j}V^i + \left[a\left(\overline{x}-x^i\right)+\alpha^i\right]\partial_{x^i}V^i\right.\\
&+ \quad \frac{\sigma^2}{2}\sum_{j=1}^N\sum_{k=1}^N\left(\rho^2+\delta_{j,k}(1-\rho^2)\right)\partial_{x^jx^k}V^i\\
&+ \quad \left.\frac{(\alpha^i)^2}{2} - q\alpha^i\left(\overline{x}-x^i\right) + \frac{\epsilon}{2}(\overline{x}-x^i)^2\right\} = 0,
\end{aligned} \tag{4.28}$$

with terminal conditions $V^i(T,x) = \frac{c}{2}(\overline{x}-x^i)^2$. Assuming that $\alpha^j(t,x)$ for $j \neq i$ is chosen, and minimizing (4.28) with respect to α^i, the control for player i is given by

$$\hat{\alpha}^i = q(\overline{x}-x^i) - \partial_{x^i}V^i, \tag{4.29}$$

where V^i is still unknown. Similarly, in order to verify that this α^i is again a feedback Nash equilibrium, one needs to assume that all players follow this choice $\hat{\alpha}^i = q(\overline{x} -$

$x^i) - \partial_{x^i} V^i$. Consequently, the HJB equations (4.28) become

$$\begin{aligned}
\partial_t V^i &+ \sum_{j=1}^{N} \left[(a+q) \left(\overline{x} - x^j \right) - \partial_{x^j} V^j \right] \partial_{x^j} V^i \\
&+ \frac{\sigma^2}{2} \sum_{j=1}^{N} \sum_{k=1}^{N} \left(\rho^2 + \delta_{j,k} (1-\rho^2) \right) \partial_{x^j x^k} V^i \\
&+ \frac{1}{2} (\epsilon - q^2) \left(\overline{x} - x^i \right)^2 + \frac{1}{2} (\partial_{x^i} V^i)^2 = 0. \qquad (4.30)
\end{aligned}$$

We then make the ansatz for the unknown value function V^i:

$$V^i(t,x) = \frac{\tilde{\eta}_t}{2} (\overline{x} - x^i)^2 + \mu_t, \qquad (4.31)$$

where $\tilde{\eta}_t$ and μ_t are deterministic functions satisfying $\tilde{\eta}_T = c$ and $\mu_T = 0$ in order to match the terminal conditions for V^i. It is important to remark that the adjoint variables $y^{i,j}$ introduced in the FBSDE approach correspond to $\partial_{x^j} V^i$, and the ansatz (4.31) corresponds to the ansatz (4.20). The optimal strategies will be

$$\hat{\alpha}^i = q(\overline{X}_t - X_t^i) - \partial_{x^i} V^i = \left(q + (1 - \frac{1}{N}) \tilde{\eta}_t \right) (\overline{X}_t - X_t^i), \qquad (4.32)$$

and the controlled dynamics will become

$$dX_t^i = \left(a + q + (1 - \frac{1}{N}) \tilde{\eta}_t \right) (\overline{X}_t - X_t^i) dt + \sigma \left(\sqrt{1-\rho^2} dW_t^i + \rho dW_t^0 \right). \qquad (4.33)$$

Next, computing

$$\partial_{x^j} V^i = (\frac{1}{N} - \delta_{i,j}) \left(\overline{x} - x^i \right) \tilde{\eta}_t, \text{ and } \partial_{x^j x^k} V^i = (\frac{1}{N} - \delta_{i,j})(\frac{1}{N} - \delta_{i,k}) \tilde{\eta}_t,$$

plugging both into (4.30), and matching the terms in $(\overline{x} - x^i)^2$ and the state-independent terms from both sides, we obtain

$$\begin{aligned} \dot{\tilde{\eta}}_t &= 2(a+q)\tilde{\eta}_t + (1 - \frac{1}{N^2})\tilde{\eta}_t^2 - (\epsilon - q^2), && (4.34) \\ \dot{\mu}_t &= -\frac{1}{2}\sigma^2(1-\rho^2)\left(1 - \frac{1}{N}\right)\tilde{\eta}_t, && (4.35) \end{aligned}$$

with the terminal conditions $\tilde{\eta}_T = c$ and $\mu_T = 0$. Obviously, $\tilde{\eta}_t$ must satisfy the same Riccati equation as (4.24) and the same terminal condition satisfied by η_t. Therefore, by unicity of the solution for this equation, we deduce that $\tilde{\eta}_t = \eta_t$ for all $t \leq T$. Consequently, we verify that the closed-loop equilibrium obtained in Section 4.2 is a feedback Nash equilibrium. The explicit solution for η_t is given by (4.25) and, furthermore, the solution μ_t of (4.35) with the terminal condition $\mu_T = 0$ is given by

$$\mu_t = \frac{1}{2}\sigma^2(1-\rho^2)\left(1 - \frac{1}{N}\right)\int_t^T \tilde{\eta}_s \, ds, \tag{4.36}$$

and the value functions V^i using this exact Nash equilibrium are given by (4.31). It is also interesting to note that the correlation, determined by the parameter ρ does not appear in the control α^i given by (4.32) but in the value function V^i. Namely, the correlation parameter ρ only affects the control α^i through the dynamics of $\overline{X}_t - X_t^i$ since η_t does not depend on ρ. However, it affects the value function V^i given by (4.31) also through the state-independent term μ_t. From (4.36), observing that μ_t is decreasing in ρ, for the fixed initial value, the value function V^i is consequently decreasing in ρ.

4.4 Comparison of the Open-loop and Closed-loop equilibria

Our analysis in Sections 4.1 and 4.2 shows that the two equilibria we obtained are very similar. In fact, the only difference is that in the open-loop case we obtained the Riccati equation (4.16) for the function ϕ_t (with a factor $(1-\frac{1}{N})$ in front of ϕ_t^2 and a factor $(1-\frac{1}{2N})$ in front of q), and in the closed-loop case we obtained the Riccati equation (4.24) for the function η_t (with a factor $(1-\frac{1}{N^2})$ in front of η_t^2 and a factor 1 in front of q).

In the closed-loop case we saw that the optimal strategy is given by

$$\alpha_t^i = \left[q + (1-\frac{1}{N})\eta_t\right](\overline{X}_t - X_t^i),$$

and the forward dynamics are

$$dX_t^i = \left[a + q + (1-\frac{1}{N})\eta_t\right](\overline{X}_t - X_t^i)dt + \sigma\left(\sqrt{1-\rho^2}dW_t^i + \rho dW_t^0\right),$$

with

$$d\overline{X}_t = \sigma\left(\rho dW_t^0 + \sqrt{1-\rho^2}\frac{1}{N}\sum_{k=1}^N dW_t^k\right).$$

These equations are identical in the open-loop case with η_t replaced by ϕ_t. Note that η_t is given explicitly by formula (4.25) and ϕ_t can be obtained similarly by replacing the factor $(1-\frac{1}{N^2})$ with $(1-\frac{1}{N})$ and using

$$\delta^{\pm} = -(a + (1-\frac{1}{2N})q) \pm \sqrt{R},$$

with

$$R := (a + (1 - \frac{1}{2N})q)^2 + \left(1 - \frac{1}{N}\right)(\epsilon - q^2) > 0.$$

In Figure 4.1, we show the functions ϕ_t and η_t involved respectively in the open-loop and closed-loop strategies. As expected, the difference is relatively small for $N = 10$. However, it is enhanced by our choice of $\epsilon = 10$ giving a rather large factor $\epsilon - q^2$ in front of $(1 - \frac{1}{N})$ in the open-loop case or in front of $(1 - \frac{1}{N^2})$ in the closed-loop case. Note that the presence of a terminal cost $c = 1$ in the right panel produces a significant difference.

The individual value functions can be calculated as follows. Considering, for instance, the closed-loop case, we want to calculate

$$V^i(x) = I\!E\left\{\int_0^T \left[\frac{1}{2}(\alpha^i)^2 - q\alpha^i(\overline{X}_t - X_t^i) + \frac{\epsilon}{2}(\overline{X}_t - X_t^i)^2\right] dt + \frac{c}{2}(\overline{X}_T - X_T^i)^2\right\},$$

where x is the initial position of the system and $(\alpha_t^i, X_t^i, \overline{X}_t)$ are given by the equations above. Then, one easily obtains by direct computation

$$V^i(x) = \frac{1}{2}\int_0^T \left[\epsilon - q^2 + (1 - \frac{1}{N})^2\eta_t^2\right] I\!E\left\{(\overline{X}_t - X_t^i)^2\right\} dt + \frac{c}{2} I\!E\left\{(\overline{X}_T - X_T^i)^2\right\},$$

with

$$\begin{aligned} I\!E\left\{(\overline{X}_t - X_t^i)^2\right\} &= (\overline{x} - x^i)^2 e^{-2\int_0^t (a+q+(1-\frac{1}{N})\eta_s)ds} \\ &\quad + (1 - \frac{1}{N})\sigma^2(1 - \rho^2)\int_0^t e^{-2\int_s^t (a+q+(1-\frac{1}{N})\eta_u)du} ds. \end{aligned}$$

The formula in the open-loop case is simply obtained by replacing η_t by ϕ_t.

In Figure 4.2, we compare the value functions V^i in the open-loop and closed-loop equilibria for a particular choice of parameters and as $N \to \infty$.

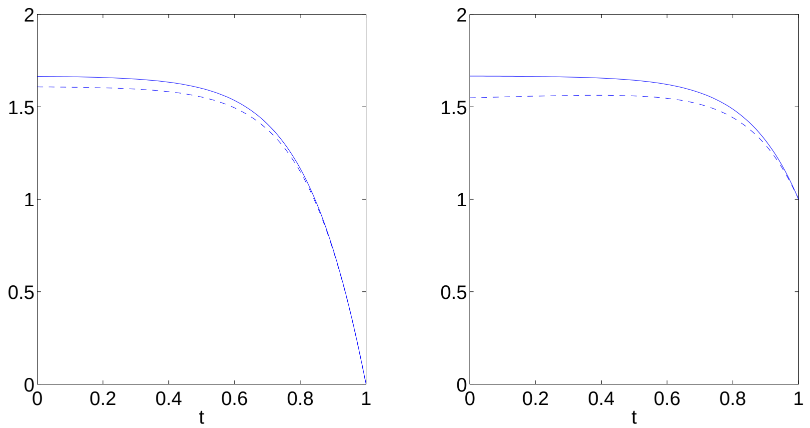

Figure 4.1: Plots of ϕ_t (solid line) and η_t (dashed line) with $N = 10$, $a = 1$, $q = 1$, $\epsilon = 10$, $T = 1$, and $c = 0$ on the left, $c = 1$ on the right.

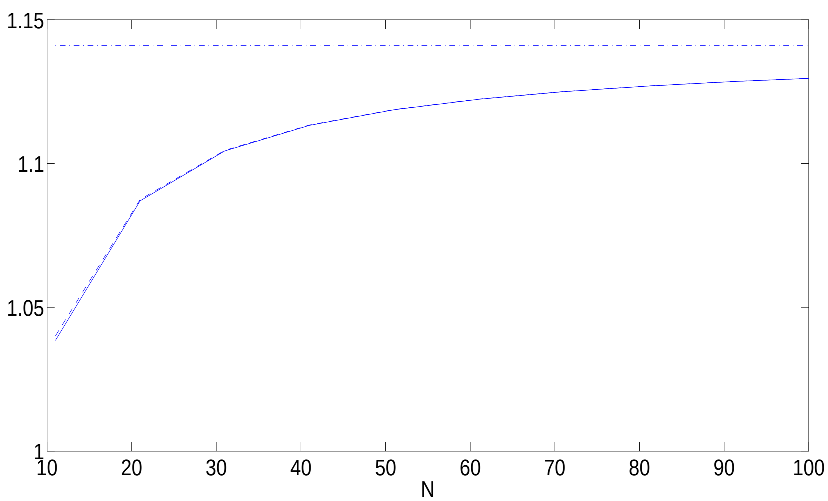

Figure 4.2: Plots of the value function V^i at $t = 0$ and $x^i = 0$ for $i = 1, \cdots, N$, as N increases: open loop (solid line), closed loop (dashed line), and common limit as $N \to \infty$ (dotted line) with $a = 1$, $q = 1$, $\epsilon = 10$, $\rho = 0.2$, $T = 1$, and $c = 10$.

Chapter 5

Systemic Risk and Mean Field Games

This chapter discusses the derivation of an approximate equilibrium in the limit $(N \to \infty)$. First of all, one may ask:

Why would we want an approximate equilibrium when we can compute an exact one?

The game presented in the previous chapter was essentially designed to provide explicit Nash equilibrium, and the fact that it does is already remarkable! However, slight modifications, even minor, dramatically change the equilibrium structures. For instance, the presence in the dynamics of the X_t^i or a nonlinear term in $\overline{X}_t$ in the objective functions J^i renders the computation of exact Nash equilibria hopeless. In the quadratic case, the closed loop equilibrium is expressible in closed form by solving the coupled matrix Riccati equations, and the open loop equilibrium is obtained using matrix differential equations or the coupled matrix Riccati equations as studied in [2] and [16]. In general, it is difficult to find explicit solution in both methods. In our specific homogeneous and symmetrical case, the coupled matrix Riccati equations can be simplified to Riccati equations which can be solved explicitly.

The Mean Field Game strategy is based on the solution of effective equations in the limit as $N \to \infty$, using the theory of the propagation of chaos. We review this strategy in the context of the model presented in the previous section, and compare its output to the exact solutions derived earlier. Furthermore, in the Section 7.2, in order to solve the heterogeneous grouping case, we show the simplification from this mean field strategy and also find the corresponding approximate Nash equilibria.

5.1 The Mean Field Games / FBSDE Approach

If it was not for the presence of a common noise, we could apply the results of [7] to obtain approximate Nash equilibria from the solution of the Mean Field Game (MFG). Notice that linear quadratic MFGs are studied in [4] and [8]. While the latter does not include the cross term $-q\alpha(\overline{x} - x^i)$ in the running cost of player i, the proofs of [8] apply *mutatis mutandis* to the model of the present paper when $\rho = 0$. We review the MFG strategy. It is based on the following three steps:

1. Fix $(m_t)_{t\geq 0}$, which should be thought of as a candidate for the limit of $\overline{X}_t$ as

$N \to \infty$:

$$m_t = \lim_{N\to\infty} \overline{X}_t.$$

Because of the presence of the common noise, $(m_t)_{t\geq 0}$ is a process adapted to the filtration generated by W^0 and one should think of m_t as a function of $(W_s^0)_{s\leq t}$.

2. Solve the one-player standard control problem

$$\inf_{\alpha=(\alpha_t)\in A} I\!E\left\{\int_0^T \left[\frac{\alpha_t^2}{2} - q\alpha_t(m_t - X_t) + \frac{\epsilon}{2}(m_t - X_t)^2\right] dt + \frac{c}{2}(m_T - X_T)^2\right\},$$

subject to the dynamics

$$dX_t = \left[a(m_t - X_t) + \alpha_t\right] dt + \sigma\left(\rho dW_t^0 + \sqrt{1-\rho^2} dW_t\right),$$

where W_t^0 and W_t are independent Brownian motions, independent of the initial value X_0 which may be a square integrable random variable ξ.

3. Solve the fixed point problem: Find m_t so that $m_t = I\!E[X_t \mid (W_s^0)_{s\leq t}]$ for all t.

We treat the above stochastic control problem as a problem of control of non-Markovian dynamics with random coefficients. Note that our control problem is reduced to an one-player problem. The Hamiltonian of the system is given by

$$H(t,x,y,\alpha) = \left[a(m_t - x) + \alpha\right] y + \frac{1}{2}\alpha^2 - q\alpha(m_t - x) + \frac{\epsilon}{2}(m_t - x)^2,$$

which is strictly convex in (x,α) under the condition $q^2 \leq \epsilon$ and attains its minimum at

$$\frac{\partial H}{\partial \alpha} = 0 \longrightarrow \hat{\alpha} = q(m_t - x) - y,$$

where $\hat{\alpha}$ is the optimal strategy suggested by the Pontryagin principle. The corresponding *adjoint forward-backward equations* are given by

$$
\begin{aligned}
dX_t &= [(a+q)(m_t - X_t) - Y_t]\,dt + \sigma\left(\rho dW_t^0 + \sqrt{1-\rho^2}dW_t\right), X_0 = \xi \quad (5.1)\\
dY_t &= -\frac{\partial H}{\partial x}(\hat{\alpha})dt + Z_t^0 dW_t^0 + Z_t dW_t \quad (5.2)\\
&= \left[(a+q)Y_t + (\epsilon - q^2)(m_t - X_t)\right]dt + Z_t^0 dW_t^0 + Z_t dW_t,\\
&\qquad Y_T = c(X_T - m_T),
\end{aligned}
$$

for some adapted square integrable processes (Z_t^0, Z_t). Beware that such linear systems do not always have solutions despite its simple looking structure. The existence of a solution in the present situation is argued in [8] where a solution is shown to exist.

To identify it in the present situation, we first use the notation $m_t^X = \mathbb{E}[X_t \mid (W_s^0)_{s\leq t}]$ and $m_t^Y = \mathbb{E}[Y_t \mid (W_s^0)_{s\leq t}]$. In order to solve (5.1 and (5.2), we first take the conditional expectation given $(W_s^0)_{s\leq t}$ in both sides of (5.2) and (5.2) which gives the dynamics of m_t^X and m_t^X

$$
\begin{aligned}
dm_t^X &= \left[(a+q)(m_t - m_t^X) - m_t^Y\right]dt + \sigma\rho dW_t^0,\ m_0^X = \mathbb{E}[\xi \mid (W_s^0)_{s\leq t}] \quad (5.3)\\
dm_t^Y &= \left[(a+q)m_t^Y + (\epsilon - q^2)(m_t - m_t^X)\right]dt + Z_t^0 dW_t^0 \quad (5.4)\\
&\qquad m_t^Y = c(m_t^Y - m_T).
\end{aligned}
$$

Using the fact that in equilibrium (i.e. after solving for the fixed point), we have $m_t = m_t^X$ for all $t \leq T$ which in turn implies $m_T^Y = c(m_T^X - m_T) = 0$ and

$$dm_t^Y = (a+q)m_t^Y dt + Z_t^0 dW_t^0.$$

Consequently, we obtain

$$m_t^Y = -\int_t^T e^{(a+q)(s-t)} Z_s^0 dW_s^0. \tag{5.5}$$

We deduce that, in equilibrium, we have

$$dm_t^X = -m_t^Y dt + \rho\sigma dW_t^0, \quad m_0^X = \mathbb{E}[\xi \mid (W_s^0)_{s\le t}]. \tag{5.6}$$

Now, we make the (educated) ansatz

$$Y_t = -\eta_t(m_t - X_t), \tag{5.7}$$

for some deterministic function $t \hookrightarrow \eta_t$ to be determined. Differentiating this ansatz and using (5.1) and (5.6) leads to

$$\begin{aligned} dY_t &= -\dot{\eta}_t(m_t - X_t)dt - \eta_t d(m_t - X_t) \qquad (5.8)\\ &= \left[\left(-\dot{\eta}_t + \eta_t(a+q+\eta_t)\right)(m_t - X_t) + \eta_t m_t^Y\right] dt + \eta_t \sigma\sqrt{1-\rho^2} dW_t. \end{aligned}$$

Plugging the ansatz (5.7) into (5.2) gives

$$dY_t = \left[-(a+q)\eta_t + (\epsilon - q^2)\right](m_t - X_t)dt + Z_t^0 dW_t^0 + Z_t dW_t. \tag{5.9}$$

Identifying the two Itô decompositions (5.8) and (5.9), we deduce first from the martingale terms that $Z_t^0 \equiv 0$ and $Z_t = \eta_t\sigma\sqrt{1-\rho^2}$ satisfy the adapted condition because they are deterministic. Equating the drift terms, we see that η_t must be a solution to the Riccati equation

$$\dot{\eta}_t = 2(a+q)\eta_t + \eta_t^2 - (\epsilon - q^2), \tag{5.10}$$

with terminal condition $\eta_T = c$. As expected, the solution for η_t is given explicitly in (4.25) after taking the limit as $N \to \infty$. Next, using $Z_t^0 \equiv 0$ and plugging into (5.5), we deduce that $m_t^Y = 0$ and consequently $m_t^X = I\!\!E[\xi \mid (W_s^0)_{s \le t}] + \rho\sigma W_t^0$ from (5.6) which will plug into the optimal control $(q + \eta_t)(m_t^X - X_t)$.

An important result of the theory of MFGs (see for example [7]) is the fact that, once a solution to the MFG is found, one can use it to construct approximate Nash equilibria for the finitely many players games. Here, if one assumes that each player is given the information $\overline{X}_t$, and if player i uses the strategy

$$\alpha_t^i = (q + \eta_t)(\overline{X}_t - X_t^i),$$

which is the limit as $N \to \infty$ of the strategy used in the finite players game, one sees how solving the limiting MFG problem can provide approximate Nash equilibria. It is interesting to remark that recognizing the strategy α^i in the MFG is similar as the exact Nash equilibria studied in Chapter 4, we can expect that the financial implication are identical. This result is described in Chapter 6.

5.2 The Mean Field Games / HJB Approach

It is interesting to go through the derivation of the MFG solution using the HJB approach since it involves additional difficulties due to the presence of the common noise. In our toy model, it can be handled explicitly through stochastic partial differential equations (SPDE). We will show the full derivation in this section.

For Markovian strategies of the form $\alpha(t,x)$, the dynamics are given, as previously noted, by

$$dX_t = [a(m_t - X_t) + \alpha(t, X_t)]\, dt + \sigma\left(\rho dW_t^0 + \sqrt{1-\rho^2} dW_t\right).$$

As in Section 5.1, we should think as (W_t^0) as given, so that the Kolmogorov forward equation for the conditional density of X_t becomes a SPDE which can be written as

$$dp_t = \left\{-\partial_x \left[(a(m_t - x) + \alpha(t,x))\, p_t\right] + \frac{1}{2}\sigma^2 \partial_{xx} p_t\right\} dt - \rho\sigma(\partial_x p_t) dW_t^0, \tag{5.11}$$

with the initial density p_0 being the density of ξ. The full derivation of the infinitesimal generator for the given common noise (W_t^0) is discussed in Appendix A. Here $\alpha(t,x)$ is given and $m_t = \int x p_t(x) dx$. Consequently, m_t, the conditional mean of X_t given W^0, is a stochastic process which will turn out to be Markovian with its infinitesimal generator denoted by L^m. The HJB equation for the value function $V(t,x,m)$ can be written as

$$\begin{aligned} dV \quad &+ \quad \left[\frac{1}{2}\sigma^2 \partial_{xx} V + L^m V + (\partial_{xm} V)\frac{d\langle m, X\rangle}{dt}\right] \qquad (5.12)\\ &+ \quad \inf_{\alpha}\left\{\left[a(m-x) + \alpha\right]\partial_x V + \frac{\alpha^2}{2} - q\alpha(m-x) + \frac{\epsilon}{2}(m-x)^2\right\} dt = 0. \end{aligned}$$

Next, we minimize with respect to α to get $\hat{\alpha} = q(m-x) - \partial_x V$, and we make the ansatz $V(t,x,m) = \frac{\eta_t}{2}(m-x)^2 + \mu_t$. In order to solve the HJB equation (5.12), we must find the dynamics of the conditonal expectation m_t. Plugging $\hat{\alpha}$ into the forward equation for p_t, the conditional density of X_t becomes

$$dp_t = \left\{-\partial_x\left[(a + q + \eta_t)(m_t - x)p_t\right] + \frac{1}{2}\sigma^2(1-\rho^2)\partial_{xx} p_t\right\} dt - \rho\sigma(\partial_x p_t) dW_t^0,$$

multiplying by x, and integrating with respect to x gives

$$dm_t = \rho\sigma dW_t^0,$$

using

$$\int_{-\infty}^{\infty} x\partial_x p_t = -1, \ \int_{-\infty}^{\infty} x\partial_x x p_t = -m_t, \ \int_{-\infty}^{\infty} x\partial_{xx} x p_t = 0,$$

where we assume that p_t decay rapidly in x. Therefore, conditional on W^0,

$$L^m V = \frac{1}{2}\rho^2\sigma^2\partial_{mm}V = \frac{1}{2}\rho^2\sigma^2\eta_t,$$

and $d\langle m, X\rangle = \rho^2\sigma^2 dt$. Then, verifying that the ansatz satisfies the HJB equation, by canceling terms in $(m-x)^2$ we obtain that η_t must satisfy the Riccati equation (5.10), and canceling state-independent terms leads to $\dot{\mu}_t = -\frac{1}{2}\sigma^2(1-\rho^2)\eta_t$ and therefore

$$\mu_t = \frac{1}{2}\sigma^2(1-\rho^2)\int_t^T \eta_s ds.$$

We observe that the value function $V(t, x, m)$ is the limit as $N \to \infty$ of the value function obtained in the finite players game shown in the conclusion of Section 5.1.

Chapter 6

Financial Implication

In Chapter 4 and Chapter 5, we discussed the derivation of three different strategies: open-loop, closed-loop, and MFG equilibria. Now, it is interesting to explain the financial implication of these strategies. That is, when banks have the ability to control their rate of borrowing from or lending to the central bank, could they reduce possibility of a large number of defaults? Since our toy model is symmetrical and quadratic, the strategies in these cases can be solved explicitly and formed very similarly. The comments below made in the case of the closed-loop equilibrium with the function η_t would be identical in the case of the open-loop equilibrium with η_t replaced by ϕ_t and in the case of the MFG equilibrium by considering the limit $N \to \infty$.

1. Once the function η_t has been obtained in (4.25), bank i implements its strategy by using its control $\hat{\alpha}^i$ given by (4.32). It requires its own log-reserve X_t^i but

also the average reserve $\overline{X}_t$ which may or may not be known to the individual bank i. Here, we assume that the central bank/regulator observes the monetary reserves of all banks and therefore $\overline{X}_t$, and that it gives this aggregated information to each individual banks. Consequently, even though $\overline{X}_t$ is given by (4.13), the banks do not need to know the two parameters σ and ρ in order to implement their optimal strategies. Observe also that the average $\overline{X}_t$ is identical to the average found in Section 3.4. Therefore, systemic risk occurs in the same manner as in the case of uncontrolled dynamics with or without common noise as presented, respectively, in Sections 3.4 and 3.3.

2. However, (4.33) shows that the control affects the rate of borrowing and lending by adding the time-varying component $q+(1-\frac{1}{N})\eta_t$ to the uncontrolled rate a.

3. In fact, from (4.33) rewritten as

$$\begin{aligned} dX_t^i &= \left(a+q+(1-\frac{1}{N})\eta_t\right)\frac{1}{N}\sum_{j=1}^{N}(X_t^j-X_t^i)dt \\ &\quad +\sigma\left(\sqrt{1-\rho^2}dW_t^i+\rho dW_t^0\right), \end{aligned} \tag{6.1}$$

we see that the effect of the banks using their optimal strategies corresponds to inter-bank borrowing and lending at the increased **effective rate**

$$A_t := a+q+(1-\frac{1}{N})\eta_t$$

with no central bank (or a central bank acting as an instantaneous **clearing house**). As a consequence, using this equilibrium, the system is operating as if banks were borrowing from and lending to each other at the rate A_t, and the net effect is **additional liquidity** quantified by the rate of lending/borrowing.

Note that the comments above are valid not only if $a > 0$, in which case the effect of the game is to increase the rate of interbank lending and borrowing, but also if $a = 0$, in which case the effect of the game is to "create" an interbank lending and borrowing activity. In both cases, the central bank acts as a clearing house but needs to provide the information $\overline{X}_t$ so that individual banks can implement their strategies.

4. Observe that the presence of a common noise (quantified by ρ) does not affect the form of the optimal strategies (the function η_t does not depend on ρ). However it affects the value function $V^i(t,x)$ and the dynamics X^i_t, and, as we have seen in Section 3.4, it has a drastic effect on systemic risk.

5. It is also interesting to note that for T large, most of the time ($T-t$ large), η_t is mainly constant. For instance, with $c = 0$,

$$\lim_{T\to\infty} \eta_t = \frac{\epsilon - q^2}{-\delta^-} := \overline{\eta},$$

as illustrated on right panel of Figure 6.1. Therefore, in this infinite-horizon equilibrium, banks are borrowing and lending to each other at the constant rate

$$A := a + q + (1 - \frac{1}{N})\overline{\eta}. \tag{6.2}$$

In Figure 6.2 we show the constant effective rates A (for infinite horizon) for the open-loop and closed-loop equilibria as N increases. Note that liquidity

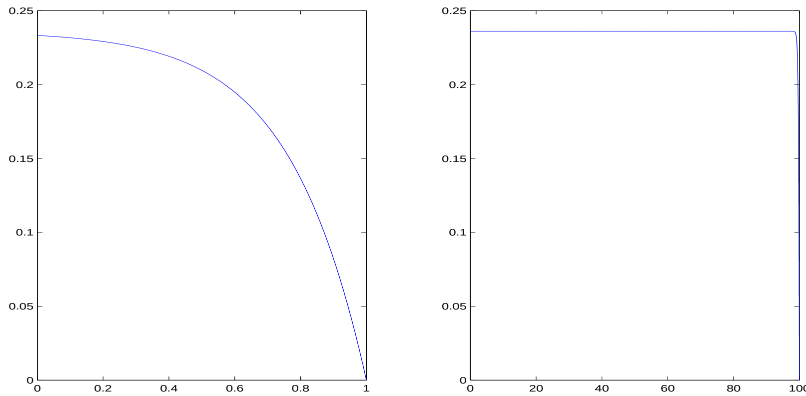

Figure 6.1: Plots of η_t with $c = 0$, $a = 1$, $q = 1$, $\epsilon = 2$ and $T = 1$ on the left, $T = 100$ on the right with $\overline{\eta} \sim 0.24$ (here we used $1/N \equiv 0$).

(quantified by the effective rate of lending/borrowing A) is higher in case of the open-loop equilibrium and increases with N.

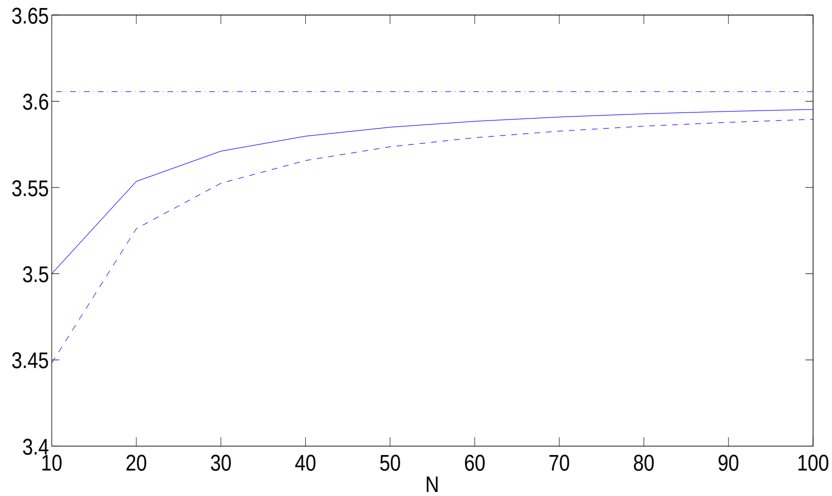

Figure 6.2: Plots of the effective rate A (6.2) for the open-loop equilibrium (solid line) and for the closed-loop equilibrium (dashed line) with $a = 1$, $q = 1$, $\epsilon = 10$, $T = 1$, and as N increases. The dotted line shows the common limit as $N \to \infty$

Chapter 7

Extension: Heterogeneous Grouping

In Chapter 4 and Chapter 5, we obtained that the optimal strategies for banks in the homogeneous system only improve the stability by adding the liquidity rate but do not prevent the system from systemic risk because systemic risk happens in the same manner. Also, surprisingly, the central bank cooperates with other banks as a clearing house. It is natural to break the homogeneity to observe systemic risk being affected by the heterogeneous parameters and the correspoinding diverse optimal strategies. Instead of individuals, we now search the equilibria for d ($d \geq 1$) heterogeneous groups. That is, banks within the same groups consider identical parameters for their object or value functions and dynamics or constraints. The numbers of banks in each group are $N_1, \cdots, N_d$ and $\sum_{k=1}^{d} N_k = N$. We use the index $(k)i$ to indicate the i-th player in the group k and $i = 1, \cdots, N_k$, $k = 1, \cdots, d$. For simplicity, we assume that the Brownian motions $(W_t^{(k)i})$, $i = 1, \cdots, N_k$, $k = 1, \cdots, d$ are all independent. In other words, there is no common noise in our model. Note that the value functions and dynamics are homogeneous within groups with parameters only depending on the group index k.

We propose two different heterogeneous systems. Section 7.1 shows that banks are assumed to use different transaction rates from one group to another group and control their own rates to request money from or send money to the central bank in order to minimize their cost functions only relying on their own group average

$$\overline{X}_t^{(k)} = \frac{1}{N_k} \sum_{i=1}^{N_k} X_t^{(k)i}.$$

Section 7.2 shows that banks are not necessarily imposed to do any transaction but they have to find the optimal strategies to approach the system average

$$\overline{X}_t = \frac{1}{N} \sum_{k=1}^{d} \sum_{i=1}^{N_k} X_t^{(k)i}.$$

The purpose for considering this type of object function is to show that either banks lend more money to get better interests or borrow money to keep themselves more stable by being close to the average.

7.1 Case I

In this section, we break the symmetry using different initial heterogeneous transaction rates a_{kh}. All banks try to minimize their own cost which gives the value functions

$$
\begin{aligned}
V^{(k)i}(t,x) &= \inf_{(\alpha_t)} I\!E\Bigg\{ \int_0^T \left[\frac{(\alpha_t^{(k)i})^2}{2} - q\alpha_t^{(k)i}\left(\overline{X}_t^{(k)} - X_t^{(k)i}\right) + \frac{\epsilon}{2}\left(\overline{X}_t^{(k)} - X_t^{(k)i}\right)^2 \right] dt \\
&\quad + \frac{c}{2}\left(\overline{X}_T^{(k)} - X_T^{(k)i}\right)^2 \Bigg\}, \qquad (7.1)
\end{aligned}
$$

under the constraint

$$
\begin{aligned}
dX_t^{(k)i} &= \left\{ a_{kk}(\overline{X}_t^{(k)} - X_t^{(k)i}) + \sum_{l\neq k} a_{kl}\left(\overline{X}_t^{(l)} - X_t^{(k)i}\right) + \alpha_t^{(k)i} \right\} dt + \sigma dW_t^{(k)i} \\
&= \left\{ \sum_{l=1}^{d} a_{kl}\overline{X}_t^{(l)} - \tilde{a}^{(k)} X_t^{(k)i} + \alpha_t^{(k)i} \right\} + \sigma dW_t^{(k)i} \qquad (7.2)
\end{aligned}
$$

where $\tilde{a}^{(k)} = \sum_{l=1}^{d} a_{kl}$ and $\overline{X}_t^{(k)} = \frac{1}{N_k}\sum_{i=1}^{N_k} X^{(k)i}$. The initial value $X_0^{(k)i}$ may also be a squared integrable random variable $\xi^{(k)}$. Banks are supposed to do the transaction using different rates a_{kh} from group k to group h. In addition, banks have the ability to choose their own rates to borrow money from or lend money to the central bank. From the banks' point of view, they want to be close to the group average $\overline{X}_t^{(k)}$ in order to survive in the group as long as possible. This inclination can be seen in the value function depending on only the group average $\overline{X}_T^{(k)}$. Again, we are interested in differences from the previous homogeneous case discussed in chapters 4 and 5. The following sections provide the explicit solutions for open-loop equilibria and closed-loop equilibria. Finally, Section 7.1.4 gives the discussion for this heterogeneous bank system.

7.1.1 Open-Loop Equilibria: FBSDE Approach

Using Pontryagin's principle to obtain open-loop equilibria, the Hamiltonian for bank $(k)i$ is written as

$$
\begin{aligned}
H^{(k)i} &= \sum_{p=1}^{d}\sum_{r=1}^{N_p}\left\{\left(\sum_{l=1}^{d} a_{pl}\overline{x}^{(l)} - \tilde{a}^{(p)}x^{(p)r}\right) + \alpha^{(p)r}\right\} y^{(k)i,(p)r} \\
&\quad + \frac{(\alpha_t^{(k)i})^2}{2} - q\alpha_t^{(k)i}\left(\overline{x}^{(k)} - x^{(k)i}\right) + \frac{\epsilon}{2}\left(\overline{x}^{(k)} - x^{(k)i}\right)^2.
\end{aligned}
$$

Consequently, the derivative of Hamiltonian $H^{(k)i}$ with respect to $x^{(h)j}$ is given by

$$
\begin{aligned}
\partial_{x^{(h)j}} H^{(k)i} &= \frac{1}{N_h}\sum_{p=1}^{d}\sum_{r=1}^{N_p} a_{ph} y^{(k)i,(p)r} - \tilde{a}^{(h)} y^{(k)i,(h)j} \\
&\quad - q\alpha^{(k)i}\left(\frac{1}{N_k} - \delta_{i,j}\right)\delta_{k,h} + \epsilon\left(\overline{x}_t^{(k)} - x_t^{(k)i}\right)\left(\frac{1}{N_k} - \delta_{i,j}\right)\delta_{k,h},
\end{aligned}
$$

and in order to satisfy the necessary condition of the Pontryagin principle, bank $(k)i$ chooses its strategy as follows:

$$
\hat{\alpha}^{(k)i} = q(\overline{x}^{(k)} - x^{(k)i}) - y^{(k)i,(k)i}.
$$

We then make the ansatz for $Y_t^{(k)i,(h)j}$

$$
Y_t^{(k)i,(h)j} = \eta_t^{(k)}\left(\overline{X}_t^{(k)} - X_t^{(k)i}\right)\left(\frac{1}{N_k} - \delta_{i,j}\right)\delta_{k,h}. \tag{7.3}
$$

Similarly, using the Pontryagin principle and ansatz (7.3), the backward equation is given by

$$
\begin{aligned}
dY_t^{(k)i,(h)j} &= (\overline{X}_t^{(k)} - X_t^{(k)i})\left(\frac{1}{N_k} - \delta_{i,j}\right)\delta_{k,h}\left[(\tilde{a}^{(h)} + (1-\frac{1}{N})q)\eta_t^{(k)} - (\epsilon - q^2)\right]dt \\
&\quad + \sum_{p=1}^{d}\sum_{r=1}^{N_p} Z^{(k)i,(h)j,(p)r} dW_t^{(p)r},
\end{aligned} \tag{7.4}
$$

where $(Z_t^{(k)i,(h)j,(p)r})$ are adapted processes. Differentiating the ansatz (7.3), we obtain

$$
\begin{aligned}
& dY_t^{(k)i,(h)j} \\
= & \left(\overline{X}_t^{(k)} - X_t^{(k)i}\right)\left(\frac{1}{N_k} - \delta_{i,j}\right)\delta_{k,h}\left(\dot{\eta}_t^{(k)} - \eta_t^{(k)}\left(\tilde{a}^{(k)} + q + (1 - \frac{1}{N_k})\eta_t^{(k)}\right)\right)dt \\
& + \eta^{(k)}(\frac{1}{N_k} - \delta_{i,j})\delta_{k,h}\sigma\left(\frac{1}{N_k}\sum_{r=1}^{N_k} dW_t^{(k),r} - dW_t^{(k)i}\right). \qquad (7.5)
\end{aligned}
$$

Comparing (7.4) and (7.5), we observe that if $k \neq h$, $Y_t^{(k)i,(h)j} = 0$. Identifying the drift terms and martingale terms only when $k = h$, we obtain

$$
Z_t^{(k)i,(h)j,(p)r} = \sigma\eta_t^{(k)}(\frac{1}{N_k} - \delta_{i,j})\delta_{k,h}(\frac{1}{N_k} - \delta_{i,r})\delta_{k,p}
$$

which is a deterministic function satisfying the adapted condition, and $\eta_t^{(k)}$ is the solution of the scaled Riccati equation

$$
\dot{\eta}_t^{(k)} = 2(\tilde{a}^{(k)} + (1 - \frac{1}{2N_k})q)\eta_t^{(k)} + (1 - \frac{1}{N_k})(\eta_t^{(k)})^2 - (\epsilon - q^2), \qquad (7.6)
$$

with the terminal condition $\eta_T^{(k)} = c$.

7.1.2 Closed-Loop Equilibria: FBSDE Approach

As we discussed in Section 4.2, in order to find the closed-loop equilibria, the modified Hamiltonian for bank $(k)i$ is given by

$$
\begin{aligned}
H^{(k)i} = & \sum_{p=1}^{d}\sum_{r\neq i}^{N_p}\left\{\left(\sum_{l=1}^{d} a_{pl}\overline{x}^{(l)} - \tilde{a}^{(p)}x^{(p)r}\right) + \alpha^{(p)r}(t,x)\right\} y^{(k)i,(p)r} \\
& + \left\{\sum_{l=1}^{d} a_{kl}\overline{x}_t^{(l)} - \tilde{a}^{(k)}x_t^{(k)i} + \alpha_t^{(k)i}\right\} y^{(k)i,(k)i} \\
& + \frac{(\alpha_t^{(k)i})^2}{2} - q\alpha_t^{(k)i}\left(\overline{x}^{(k)} - x^{(k)i}\right) + \frac{\epsilon}{2}\left(\overline{x}^{(k)} - x^{(k)i}\right)^2.
\end{aligned}
$$

With the choice of the ansatz

$$Y_t^{(k)i,(h)j} = \eta_t^{(k)} \left(\overline{X}_t^{(k)} - X_t^{(k)i}\right) \left(\frac{1}{N_k} - \delta_{i,j}\right) \delta_{k,h}, \tag{7.7}$$

we get

$$\begin{aligned} \alpha^{(k)i} &= \left(q + \eta_t^{(k)}(1 - \frac{1}{N_k})\delta_{k,h}\right) \left(\overline{X}_t^{(k)} - X_t^{(k)i}\right) \\ \partial_{x^{(h)j}} \alpha^{(k)i} &= \left(q + \eta_t^{(k)}(1 - \frac{1}{N_k})\delta_{k,h}\right) \left(\frac{1}{N_k} - \delta_{i,j}\right) \delta_{k,h}. \end{aligned}$$

In order to satisfy the Pontryagin principle, the backward dynamics $Y_t^{(k)i,(h)j}$ is written as

$$\begin{aligned} & dY_t^{(k)i,(h)j} \\ = & \left(\overline{X}_t^{(k)} - X_t^{(k)i}\right) \left(\frac{1}{N_k} - \delta_{i,j}\right) \delta_{k,h} \left((\tilde{a}^{(h)} + q)\eta_t^{(k)} - \frac{1}{N_k}(\frac{1}{N_k} - 1)(\eta_t^{(k)})^2 - (\epsilon - q^2)\right) dt \\ & + \sum_{p=1}^{d} \sum_{r=1}^{N_p} Z^{(k)i,(h)j,(p)r} dW_t^{(p)r} \end{aligned} \tag{7.8}$$

where $(Z_t^{(k)i,(h)j,(p)r})$ are adapted processes. Similarly, differentiating (7.7), we get

$$\begin{aligned} & dY_t^{(k)i,(h)j} \\ = & \left(\overline{X}_t^{(k)} - X_t^{(k)i}\right) \left(\frac{1}{N_k} - \delta_{i,j}\right) \delta_{k,h} \left(\dot{\eta}_t^{(k)} - \eta_t^{(k)} \left(\tilde{a}^{(k)} + q + (1 - \frac{1}{N_k})\eta_t^{(k)}\right)\right) dt \\ & + \eta^{(k)}(\frac{1}{N_k} - \delta_{i,j})\delta_{k,h}\sigma \left(\frac{1}{N_k} \sum_{r=1}^{N_k} dW_t^{(k),r} - dW_t^{(k)i}\right). \end{aligned} \tag{7.9}$$

Now, comparing (7.8) and (7.9), again, we observe that if $k \neq h$, $Y_t^{(k)i,(h)j} = 0$. Identifying the martingale terms, we obtain

$$Z_t^{(k)i,(h)j,(p)r} = \sigma\eta^{(k)}(\frac{1}{N_k} - \delta_{i,j})\delta_{k,h}(\frac{1}{N_k} - \delta_{i,r})\delta_{k,p},$$

which is a deterministic function satisfying the adapted condition. When $k = h$, identifying the drift terms gives that $\eta_t^{(k)}$ must satisfy

$$\dot{\eta}_t^{(k)} - \eta_t^{(k)} \left(\tilde{a}^{(k)} + q + (1 - \frac{1}{N_k})\eta_t^{(k)} \right) = (\tilde{a}^{(k)} + q)\eta_t^{(k)} - \frac{1}{N_k}(\frac{1}{N_k} - 1)(\eta_t^{(k)})^2 - (\epsilon - q^2), \tag{7.10}$$

which is the scalar Riccati equation

$$\dot{\eta}_t^{(k)} = 2(\tilde{a}^{(k)} + q)\eta^{(k)} + (1 - \frac{1}{N_k^2})(\eta_t^{(k)})^2 - (\epsilon - q^2), \tag{7.11}$$

with the terminal condition $\eta_T^{(k)} = c$.

7.1.3 Closed-Loop Equilibria: HJB Approach

In the Markovian setting, we can solve the value function $V^{(k)i}(t, x)$ and find the closed loop optimal control $\alpha^{(k)i}$ using the HJB approach. The HJB equation for $V^{(k)i}(t, x)$ is written as

$$\begin{aligned}
&\partial_t V^{(k)i} + \inf_{\alpha} \left\{ \sum_{p=1}^{d} \sum_{r=1}^{N_p} \left\{ \left(\sum_{l=1}^{d} a_{pl}\overline{x}^{(l)} - \tilde{a}^{(p)} x^{(p)r} \right) + \alpha^{(p)r} \right\} \partial_{x^{(h)j}} V^{(k)i} \right. \\
&\left. + \frac{(\alpha_t^{(k)i})^2}{2} - q\alpha_t^{(k)i} \left(\overline{x}^{(k)} - x^{(k)i} \right) + \frac{\epsilon}{2} \left(\overline{x}^{(k)} - x^{(k)i} \right)^2 \right\} \\
&+ \frac{\sigma^2}{2} \sum_{h=1}^{d} \sum_{j=1}^{N_h} \sum_{p=1}^{d} \sum_{r=1}^{N_p} \partial_{x^{(h)j} x^{(p)r}} V^{(k)i} = 0
\end{aligned} \tag{7.12}$$

with terminal condition $V^{(k)i}(T, x) = \frac{c}{2} \left(\overline{x}^{(k)} - x^{(k)i} \right)$. Again, as in the one group case in Section 4.3, assuming the player $(k)i$ chooses the optimal control $q \left(\overline{x}_t^{(k)} - x_t^{(k)i} \right) -$

$\partial_{x^{(k)i}} V^{(k)i}$ when all other controls are chosen, the HJB (7.12) becomes

$$
\begin{aligned}
&\partial_t V^{(k)i} + \frac{\sigma^2}{2} \sum_{h=1}^{d} \sum_{j=1}^{N_h} \sum_{p=1}^{d} \sum_{r=1}^{N_p} \partial_{x^{(h)j} x^{(p)r}} V^{(k)i} \\
&+ \inf_{\alpha} \left\{ \sum_{p=1}^{d} \sum_{r=1}^{N_p} \left\{ \left(\sum_{l=1}^{d} a_{pl} \overline{x}^{(l)} - \tilde{a}^{(p)} x^{(p)r} \right) + q \left(\overline{x} - x^{(p)r} \right) - \partial_{x^{(p)r}} V^{(p)r} \right\} \partial_{x^{(h)j}} V^{(k)i} \right. \\
&\left. + \frac{\epsilon - q^2}{2} \left(\overline{x}^{(k)} - x^{(k)i} \right)^2 - \left(\partial_{x^{(k)i}} V^{(k)i} \right)^2 \right\} = 0, \qquad (7.13)
\end{aligned}
$$

where $V^{(k)i}(t,x)$ is still unknown. Then, we make the ansatz

$$
V^{(k)i}(t,x) = \frac{\eta_t^{(k)}}{2} \left(\overline{x}^{(k)} - x^{(k)i} \right)^2 + \mu_t^{(k)}. \qquad (7.14)
$$

Consequently, we obtain

$$
\partial_{x^{(h)j}} V^{(k)i} = \eta_t^{(k)} \left(\overline{x}^{(k)} - x^{(k)i} \right) \left(\frac{1}{N_k} - \delta_{i,j} \right) \delta_{k,h}, \qquad (7.15)
$$

and

$$
\partial_{x^{(h)j} x^{(p)r}} V^{(k)i} = \eta_t^{(k)} \left(\frac{1}{N_k} - \delta_{i,j} \right) \delta_{k,h} \left(\frac{1}{N_k} - \delta_{i,r} \right) \delta_{k,p} \qquad (7.16)
$$

where $\eta_t^{(k)}$ and $\mu_t^{(k)}$ are deterministic functions satisfying the terminal values $\eta_T^{(k)} = c$ and $\mu_T^{(k)} = 0$. Plugging (7.14), (7.15) and (7.16) into (7.13), we observe that $\eta_t^{(k)}$ has to satisfy the scalar Riccati equation

$$
\dot{\eta}_t^{(k)} = 2(\tilde{a}^{(k)} + q)\eta^{(k)} + (1 - \frac{1}{N_k^2})(\eta_t^{(k)})^2 - (\epsilon - q^2), \qquad (7.17)
$$

which is the same equation as (7.11), and $\mu_t^{(k)}$ satisfies

$$
\dot{\mu}_t^{(k)} = -\frac{\sigma^2}{2}(1 - \frac{1}{N_k})\eta^{(k)}.
$$

7.1.4 Financial Implications

The results for this model are very similar as the one group case. Bank $(k)i$ only needs to replace a and N with $\tilde{a}$ and N_k, respectively, to obtain the optimal strategy

$$\alpha^{(k)i} = (q + (1 - \frac{1}{N_k})\eta^{(k)}) \left(\overline{x}^{(k)} - x^{(k)i}\right),$$

and the log-monetary reserve of bank $(k)i$ becomes

$$\begin{aligned} & dX_t^{(k)i} \\ = \quad & \left\{ \left(a_{kk} + (q + (1 - \frac{1}{N_k})\eta^{(k)}) \right) (\overline{X}_t^{(k)} - X_t^{(k)i}) + \sum_{l \neq k} a_{kl} \left(\overline{X}_t^{(l)} - X_t^{(k)i} \right) \right\} dt \\ & + \sigma dW_t^{(k)i}. \end{aligned}$$

Bank $(k)i$ only adds liquidity within its own group k. The transaction rates for the other groups are all identical and the central bank acts as a clearing house. We conclude that the system becomes more stable using larger liquidity by adding groups. This result is predictable in this model since the objective functions for all banks in group k are uncorrelated to other groups. In the next section, we assume all banks must consider the system average $\overline{X}_t$ instead of the group average $\overline{X}_t^{(k)}$ in their objective functions. Section 7.2 shows how this heterogeneity affects the strategies for banks and the behavior for the central bank.

7.2 Case II

The second grouping model is as follows: all banks try to minimize their own cost which again gives the value functions

$$\begin{aligned} V^{(k)i}(t,x) &= \inf_{(\alpha_t)} I\!E\left\{ \int_0^T \left[\frac{(\alpha_t^{(k)i})^2}{2} - q_k \alpha_t^{(k)i} \left(\overline{X}_t - X_t^{(k)i} \right) + \frac{\epsilon_k}{2} \left(\overline{X}_t - X_t^{(k)i} \right)^2 \right] dt \right. \\ & \left. + \frac{c_k}{2} \left(\overline{X}_T - X_T^{(k)i} \right)^2 \right\}, \end{aligned} \tag{7.18}$$

under the constraint

$$dX_t^{(k)i} = \alpha_t^{(k)i} dt + \sigma dW_t^{(k)i}, \tag{7.19}$$

with the initial value $X_0^{(k)i}$ which may also be a squared integrable random variable $\xi^{(k)}$ and

$$\overline{X}_t = \frac{1}{N} \sum_{k=1}^{d} \sum_{i=1}^{N_k} X_t^{(k)i}, \; \overline{X}_t^{(k)} = \frac{1}{N_k} \sum_{i=1}^{N_k} X_t^{(k)i}.$$

Note that in this case, due to the ensemble average of all players $\overline{X}_t$ appearing in the value function $V^{(k)i}$, the strategy must include this information from other groups which implies that this heterogeneous problem is much more difficult than the one we discussed in Section 7.1. This multi-dimensional control problem can be represented by the matrix form studied by [2]:

$$dX_t = \sum_{j=1}^{d} b^j \alpha^j dt,$$

and the value function for player i

$$V^i = \inf_{\alpha^1,\cdots,\alpha^N} I\!E \left\{ \int_t^T \left(\frac{X_s' Q^i X_s}{2} + \frac{\alpha_s^i \alpha_s^i}{2} + X_s' S^i \alpha^i \right) ds + \frac{X_T' R^i X_T}{2} \right\}.$$

where X_t is a vector and Q^i, R^i are both symmetrical matrices. The control problem is solved by coupled matrix Riccati equations discussed in [3]. In general, it is difficult to find an explicit solution, especially for closed-loop Nash equilibria. Therefore, we discuss the derivation of approximate equilibria using mean field games in case when $N \to \infty$ and $\frac{N_k}{N} \to \beta_k$ for all k.

7.2.1 Mean Field Games / FBSDE Approach

This section presents the derivation of the approximate Nash equilibria using a mean field game approach. As in the one group case, the strategy is as follows:

1. Fix $(m_t^{(k)})_{t\geq 0}$, which is a candidate for the limit of $\overline{X}_t^{(k)}$ as $N_k \to \infty$:

$$m_t^{(k)} = \lim_{I_k \to \infty} \overline{X}_t^{(k)},$$

for all k, and

$$M_t = \lim_{N, N_1 \ldots, N_k \to \infty} \sum_{k=1}^{d} \frac{N_k}{N} \overline{X}_t^{(k)} = \sum_{k=1}^{d} \beta_k m_t^{(k)}.$$

2. Solve the d-players control problem

$$\inf_{\alpha=(\alpha_t)\in A} \mathbb{E}\Bigg\{ \int_0^T \left[\frac{(\alpha_t^{(k)})^2}{2} - q_k \alpha_t^{(k)} \left(M_t - X_t^{(k)} \right) + \frac{\epsilon_k}{2} \left(M_t - X_t^{(k)} \right)^2 \right] dt$$
$$+ \frac{c_k}{2} \left(M_T - X_T^{(k)} \right)^2 \Bigg\}, \tag{7.20}$$

subject to the dynamics

$$dX_t^{(k)} = \alpha_t^{(k)} dt + \sigma dW_t^{(k)}.$$

3. Like the one-player case, solve the fixed point problem: find $m_t^{(k)} = \mathbb{E}[X_t^{(k)}]$ for all t.

The Hamiltonian for the above system is given by

$$H^{(k)} = \sum_{j=1}^{d} \alpha^{(j)} Y^{(k),j} + \frac{(\alpha^{(k)})^2}{2} - q_k \alpha^{(k)} \left(M - x^{(k)}\right) + \frac{\epsilon_k}{2} \left(M - x^{(k)}\right)^2,$$

which is strictly convex in $(x^{(k)}, \alpha^{(k)})$ under the condition $q_k^2 < \epsilon_k$ for all k and, again, attains its minimum at

$$\frac{\partial H^{(k)}}{\partial \alpha^{(k)}} = 0 \rightarrow \alpha^{(k)} = q_k \left(M - x^{(k)}\right) - y^{(k),k}.$$

The adjoint forward-backward equations are given by

$$\begin{aligned} dX_t^{(k)} &= \left(q_k \left(M_t - X_t^{(k)}\right) - y^{(k),k}\right) dt + \sigma dW_t^{(k)}, \qquad (7.21) \\ & \qquad\qquad X_0^{(k)} = \xi^{(k)}; \\ dY_t^{(k),j} &= -\frac{\partial H^{(k)}}{\partial x^{(j)}} dt + \sum_{h=1}^{d} Z_t^{(k),j,h} dW_t^h \qquad (7.22) \\ &= \left(q_k Y^{(k),k} + (\epsilon_k - q_k^2) \left(M_t - X_t^{(k)}\right)\right) \delta_{k,j} + \sum_{h=1}^{d} Z_t^{(k),j,h} dW_t^h, \\ & \qquad\qquad Y_T^{(k),j} = c_k \left(X_T^{(k)} - M_T\right) \delta_{k,j}. \end{aligned}$$

Taking the expectation for both forward and backward equations give

$$\begin{aligned} dm_t^{(k)} &= \left(q_k \left(M_t - m_t^{(k)}\right) - m_t^{Y^{(k)}}\right) dt, \; m_0^{(k)} = \mathbb{E}(\xi^{(K)}) \\ dm_t^{Y^{(k),k}} &= \left(q_k m_t^{Y^{(k),k}} + (\epsilon_k - q_k^2) \left(M_t - m_t^{X^{(k)}}\right)\right) dt, \qquad (7.23) \\ & \qquad\qquad m_T^{Y^{(k),k}} = c_k (m_T^{X^{(k)}} - M_T). \end{aligned}$$

The dynamics of $M_t = \sum_{k=1}^d \beta_k m_t^{(k)}$ becomes

$$dM_t = \left(\sum_{k=1}^d \beta_k q_k (M_t - m_t^{(k)}) - \sum_{k=1}^d \beta_k m_t^{Y^{(k),k}}\right) dt, \quad M_0 = \sum_{j=1}^d \beta_j m_0^{(j)} \tag{7.24}$$

Now, we make the ansatz

$$Y_t^{(k),j} = -\left(\eta_t^{(k)}(M_t - X_t^{(k)}) + \phi_t^{(k)}\right)\delta_{k,j}, \tag{7.25}$$

and consequently,

$$m_t^{Y^{(k),j}} = -\left(\eta_t^{(k)}(M_t - m_t^{(k)}) + \phi_t^{(k)}\right)\delta_{k,j} \tag{7.26}$$

where $\eta_t^{(k)}$ and $\phi_t^{(k)}$ are both deterministic functions. Again, differentiating (7.25), we obtain

$$\begin{aligned} dY_t^{(k),j} = & \Bigg\{ \left(-\dot{\eta}_t^{(k)} + \eta_t^{(k)}(q_k + \eta_t^{(k)})\right)(M_t - X_t^{(k)}) \\ & -\eta_t^{(k)}\left(\sum_{h=1}^d \beta_h (q_h + \eta_t^{(h)})(M_t - m_t^{(h)}) + \sum_{h=1}^d \beta_k \phi_t^{(h)}\right) \\ & +\eta_t^{(k)}\phi_t^{(k)} - \dot{\phi}_t^{(k)}\Bigg\}\delta_{k,j}dt + \eta_t^{(k)}\sigma\delta_{k,j}dW_t^k. \end{aligned} \tag{7.27}$$

Plugging (7.25) into (7.23), the backward equation becomes

$$\begin{aligned} dY_t^{(k),j} &= \left(q_k Y_t^{(k),k} + (\epsilon_k - q_k^2)(M_t - X_t^{(k)})\right)\delta_{k,j}dt + \sum_{h=1}^d Z_t^{(k),j,h}dW_t^h \\ &= \left\{\left(-q_k\eta_t^{((k)} + \epsilon_k - q_k^2\right)(M_t - X_t^{(k)}) - q_k\phi_t^{(k)}\right\}\delta_{k,j}dt + \sum_{h=1}^d Z_t^{(k),j,h}dW_t^h. \end{aligned} \tag{7.28}$$

Identifying the martingale and drift terms in (7.27) and (7.28), we get $Z_t^{(k),j,k} = \eta_t^{(k)}\sigma\delta_{k,j}$, and $Z^{(k),j,d} = 0$ if $d \neq k$, and $\eta_t^{(k)}$ and $\phi_t^{(k)}$ must satisfy

$$\dot{\eta}_t^{(k)} = 2q_k\eta_t^{(k)} + (\eta_t^{(k)})^2 - (\epsilon_k - q_k^2), \tag{7.29}$$

$$\dot{\phi}_t^{(k)} = (q_k + \eta_t^{(k)})\phi_t^{(k)} + \eta_s^{(k)} \left(\sum_{h=1}^{d} \beta_h (q_h + \eta_t^{(h)})(M_t - m_t^{(h)}) - \sum_{h=1}^{d} \beta_k \phi_t^{Y^{(h)},h} \right), \quad (7.30)$$

with terminal condition $\eta_T^{(k)} = c_k$ and $\phi_T^{(k)} = 0$. Next, in order to find the solution for control $\alpha^{(k)}$, we need to solve the system $\phi_t^{(k)}$ and $(M_t - m_t^{(k)})$ for all k and t. The dynamics of $(M_t - m_t^{(k)})$ is written as

$$\begin{aligned} d(M_t - m_t^{(k)}) &= \Bigg\{ \left(\sum_{h=1}^{d} \beta_h (q_h + \eta_t^{(h)})(M_t - m_t^{(h)}) + \sum_{h=1}^{d} \beta_k \phi_t^{(h)} \right) \\ &\quad - \left((q_h + \eta_t^{(k)})(M_t - m_t^{(k)}) - \phi_t^{(k)} \right) \Bigg\} dt, \\ M_0 - m_0^{(k)} &= \sum_{h=1}^{d} \beta_h m_0^{(h)} - m_0^{(k)}, \end{aligned} \quad (7.31)$$

$$\begin{aligned} d\phi_t^{(k)} &= \left\{ (q_k + \eta_t^{(k)})\phi_t^{(k)} + \eta_s^{(k)} \left(\sum_{h=1}^{d} \beta_h (q_h + \eta_t^{(h)})(M_t - m_t^{(h)}) - \sum_{h=1}^{d} \beta_h \phi_t^{(h)} \right) \right\} dt, \\ \phi_T^{(k)} &= 0. \end{aligned} \quad (7.32)$$

In fact, the system (7.31) and (7.32) can be written as the linear differential equation using the matrix form. We first introduce the vector

$$\Phi_t = \left(M_t - m_t^{(1)}, \cdots, M_t - m_t^{(d)}, \phi_t^{(1)}, \cdots, \phi_t^{(d)} \right)'$$

satisfying the system of linear differential equation with the time dependent coefficients

$$\dot{\Phi}_t = G_t \Phi_t,$$

and

$$P\Phi_0 + Q\Phi_T = (M_0 - m_0^{(1)}, \cdots, M_0 - m_0^{(d)}, 0, \cdots, 0)', \quad (7.33)$$

with matrices G_t, P, and Q. From (7.33), the initial condition Φ_0 satisfies the equation

$$(Pe^{\int_0^T G_s ds} + Q)e^{-\int_0^T G_s ds}\Phi_0 = (M_0 - m_0^{(1)}, \cdots, M_0 - m_0^{(d)}, 0, \cdots, 0)',$$

where $\int_0^T G_s ds$ can be approximated using Magnus expansion [24]. The solution of Φ_t is given by

$$\Phi_t = \Phi_0 e^{\int_0^t G_s ds}$$

which gives the solution of $\phi_t^{(k)}$.

7.2.2 Mean Field Games / HJB Approach

It is also interesting to go through the derivation of the MFG solution using the HJB approach. For Markovian strategies of the form $\alpha^{(k)}(t, x^{(k)})$, the dynamics are given by

$$dX_t^{(k)} = \alpha^{(k)}(t, X^{(k)})dt + \sigma dW_t^{(k)}.$$

The Kolmogorov forward equation for the density of $X_t^{(k)}$ can be written as

$$dp_t^{(k)} = \left\{ -\partial_x^{(k)} \left[\alpha^{(k)}(t, x^{(k)}) p_t^{(k)} \right] + \frac{1}{2}\sigma^2 \partial_{x^{(k)}x^{(k)}} p_t^{(k)} \right\} dt, \tag{7.34}$$

with the initial density $p_0^{(k)}$ being the density of $\xi^{(k)}$. Here $\alpha^{(k)}(t, x)$ is given and $m_t^{(k)} = \int x p_t^{(k)}(x)dx$. The infinitesimal generator of $m_t^{(k)}$ is denoted by $L^{m^{(k)}}$. The

HJB equation for the value function $V^{(k)}(t,x,m^{(1)},\cdots,m^{(d)})$ can be written as

$$
\begin{aligned}
\partial_t V^{(k)} \quad &+ \quad \inf_{\alpha^{(k)}} \Bigg\{ \sum_{j\neq k} \alpha^{(j)}(t,x)\partial_{x^{(j)}} V^{(k)} + \alpha^{(k)}\partial_{x^{(k)}} V^{(k)} + \sum_{j=1}^{d} L^{m^{(j)}} V^{(k)} \\
&+ \quad \frac{\sigma^2}{2}\sum_{j=1}^{d}\sum_{h=1}^{d} \partial_{x^{(j)}x^{(h)}} V^{(k)} + \sum_{j=1}^{d}\sum_{h=1}^{d} (\partial_{x^{(j)}m^{(h)}} V^{(k)}) \frac{d\langle m^{(h)}, X^{(j)}\rangle}{dt} \\
&+ \quad \frac{(\alpha^{(k)})^2}{2} - q_k\alpha^{(k)}(M - x^{(k)}) + \frac{\epsilon_k}{2}(M - x^{(k)})^2 \Bigg\} = 0, \qquad (7.35)
\end{aligned}
$$

Next, we minimize with respect to $\alpha^{(k)}$ to get $\hat{\alpha}^{(k)} = q_k(M - x^{(k)}) - \partial_{x^{(k)}} V^{(k)}$, and we make the ansatz

$$
V^{(k)}(t,x,m^{(1)},\cdots,m^{(d)}) = \frac{\eta_t^{(k)}}{2}(M - x^{(k)})^2 + \phi_t^{(k)}(M - x^{(k)}) + \mu_t^{(k)},
$$

and consequently

$$
\hat{\alpha}_t = (q_k + \eta_t^{(k)})(M_t - x_t^{(k)}) + \phi_t^{(k)}.
$$

Plugging $\hat{\alpha}$ into the forward equation for $p_t^{(k)}$, the density of $X_t^{(k)}$ becomes

$$
dp_t^{(k)} = \left\{ -\partial_{x^{(k)}} \left[\left((q_k + \eta_t^{(k)})(M_t - x^{(k)}) + \phi_t^{(k)} \right) p_t^{(k)} \right] + \frac{1}{2}\sigma^2 \partial_{x^{(k)}x^{(k)}} p_t^{(k)} \right\} dt,
$$

multiplying by $x^{(k)}$, and integrating with respect to $x^{(k)}$ gives

$$
dm_t^{(k)} = \left((q_k + \eta_t^{(k)})(M_t - m_t^{(k)}) + \phi_t^{(k)} \right) dt.
$$

Therefore, the infinitesimal generator $L^{m^{(j)}} V^{(k)}$ is given by

$$
L^{m^{(j)}} V^{(k)} = \left((q_j + \eta_t^{(j)})(M_t - m_t^{(j)}) + \phi_t^{(j)} \right) \partial_{m^{(j)}} V^{(k)},
$$

and $d\langle m^{(h)}, X^{(j)}\rangle = 0$ for all $h = 1,\cdots,d$ and $j = 1,\cdots,d$. Then, verifying that the ansatz satisfies the HJB equation, by canceling terms in $(M - x^{(k)})^2$ and $(M -$

$x^{(k)}$), we obtain that $\eta_t^{(k)}$ and $\phi_t^{(k)}$ must satisfy the Riccati equation (7.29) and (7.30) respectively. Canceling state-independent terms leads to

$$\dot{\mu}_t^{(k)} = -\frac{1}{2}\sigma^2\eta_t^{(k)} - (\phi_t^{(k)})^2 + \phi_t^{(k)}\sum_{j=1}^{d}\beta_j\phi_t^{(j)},$$

and therefore

$$\mu_t = \int_t^T \left(\frac{1}{2}\sigma^2\eta_s^{(k)} + (\phi_s^{(k)})^2 - \phi_s^{(k)}\sum_{j=1}^{d}\beta_j\phi_s^{(j)}\right) ds.$$

The financial implications for this heterogeneous grouping model is discussed in Section 7.2.3.

7.2.3 Financial Implications

The equilibrium in the mean field limit can be written as

$$\alpha_t^{(k)} = \left(q_k + \eta_t^{(k)}\right)\left(M_t - X_t^{(k)}\right) + \phi_t^{(k)}.$$

Hence the approximate Nash equilibrium of player i in group k in the finite player game is given by

$$\alpha^{(k)i} = \left(q_k + \eta_t^{(k)}\right)\left(\overline{X}_t - X_t^{(k)i}\right) + \phi_t^{(k)},$$

where

$$\overline{X}_t = \frac{1}{N}\sum_{h=1}^{d}\sum_{j=1}^{N_h} X_t^{(h)j},$$

Plugging $\alpha^{(k)i}$ into the system, the forward equation $X^{(k)i}$ can be rewritten as

$$dX_t^{(k)i} = \left\{\frac{q_k + \eta_t^{(k)}}{N}\sum_{h=1}^{d}\sum_{j=1}^{N_h}\left(X_t^{(h)j} - X_t^{(k)i}\right) + \phi_t^{(k)}\right\}dt + \sigma dW_t^{(k)i}.$$

Notice that in this specific heterogeneous case, using the MFG approach, we observe the adjustment term $\phi_t^{(k)}$ containing all heterogeneous parameters. In order to obtain the optimal strategies, banks need all parameters in all heterogeneous object functions and initial mean values. However, in reality, banks do not have access to obtain initial mean values from their competitors. Hence, for simplicity, we assume that all parameters and initial means are all given by the regulator. It is also important to remark that as we desired, the adjusted rate function $\phi_t^{(k)}$, $k = 1, \cdots, d$ forces the central bank to provide extra cash flows into the system or keep deposits for banks instead of acting as a clearing house.

In particular, if $m_0^{(k)} = \mathbb{E}(\xi^{(k)})$ are identical in group index k, then $\phi_t^{(k)} = 0$ for all k. Therefore, the control $\alpha^{(k)i}$ degenerates into $\left(q_k + \eta_t^{(k)}\right)\left(\overline{X}_t - X_t^{(k)i}\right)$. It shows that players in the other groups only affect the control $\alpha^{(k)i}$ through $\overline{X}_t$. The parameters and deterministic functions in control $\alpha^{(k)i}$ are determined only by its own group.

Chapter 8

Conclusion

Finding and understanding why the financial crisis happened is becoming a critical research topic. In this article, we study one possible factor: systemic risk. Here, in order to describe systemic risk properly, we propose a simple toy model established from coupled diffusions and homogeneous dynamics noncooperative games. We also introduce a mean field games approach which gives an easier way to find approximate Nash equilibria. The most important message from this simple model is that this lending and borrowing behavior creates stability but also systemic risk. In addition, regarding the role of the central bank, if banks use the homogeneous value functions to create their strategies, the central bank works as clearing house. However, if banks consider heterogeneous value functions, the central bank must provide additional cash flows to the system or accept deposits from other banks in order to stabilize this bank system.

For the time being, our analysis relies heavily on the symmetry of dynamics and cost functions. It is interesting to consider a few extension of this toy model. In fact, the results can be generalized to the case where each bank has its own constant volatility σ^i. In Chapter 3 with a finite number N of banks there is no restriction on these σ_i which will simply enter in the averaging of the Brownian motions in (4.13). In the analysis when $N \to \infty$, one needs to impose a condition of the form

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \sigma_i^2 = \sigma^2,$$

with $0 < \sigma < \infty$, and that is the effective volatility which will appear in the limiting formulas. Treating the case of non constant volatilities such as $\sigma_i(t, X_t^i)$ is much more involved and no explicit solution can be expected in general. In addition, in order to stabilize the bank system, banks may want to not only control their own drifts but volatilities affected by their optimal strategies. It is interesting to observe whether this modification reduces the probability of a large number of defaults.

One way to depart from symmetry is that we can consider homogeneity in the same group but heterogeneity between groups. Here, one would have to deal with coupled matrix Riccati equations. In Chapter 7, we tackled this problem and provided the approximate Nash equilibria using the MFG approach. The explicit solution for this problem is a work in progress. In addition, in the case of our model, one may question whether this model is too simple to describe a realistic situation. One possible extension could be adding more constraints in this borrowing and lending system. For instance, the borrower may need to pay some money back to the central bank before some fixed maturity time T or pay money back in installments. The delayed stochastic processes could be applied for this case.

Since we explained systemic risk using the survival probability and the loss distribution, it is straightforward to consider intensity based models. In order to create the contagion in the system, one may use the coupled diffusions for the intensity rate. Note that we need to create few contagious factors in the model. That is, one default will force more defaults. The benefit for this model is to compute the loss distribution more efficiently; however, it is not straightforward to explain systemic risk through this model because the interaction is in the rate function. This explanation needs to be considered carefully.

In this discussion, we only consider describing or explaining the lending and borrowing behavior as causes for systemic risk using a toy model. However, the verification or calibration of our model is questionable. First, what kind of data is appropriate for calibration or verification? Second, useful data for the analysis may be difficult to obtain because it is very confidential. These problems have been studied in [10]. Through this study, we expect to learn the connection between models and data and consequently to analyze systemic risk efficiently.

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Appendix A

Infinitesimal Generator for Common Noise

We discuss the derivation of the infinitesimal generator for the given common noise (W_t^0). For simplicity, we define

$$dX_t = dW_t^0,$$

and the condition probability density

$$p_t = P_{(X_t|(W_s^0)_{s\leq t}}(x) = \delta_{\{W_t^0\}}(x).$$

Define a test function ϕ and $< p_t, \phi >= \int p_t\phi = \phi(W_t^0)$. Consequently,

$$\begin{aligned} d < p_t, \phi > &= \frac{1}{2}\partial_{xx}\phi(W_t^0)dt + \partial_x\phi(W_t^0)dW_t^0, \\ &= < \frac{1}{2}\partial_{xx}p_t dt - \partial_x p_t dW_t^0, \phi > . \end{aligned}$$

Since $d < p_t, \phi > = < dp_t, \phi >$, we obtain

$$dp_t = \frac{1}{2} \partial_{xx} p_t dt - \partial_x p_t dW_t^0.$$