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A Walk Through Quaternionic Structures

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Abstract

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In 1980, Murray Marshall proved that the category of Quaternionic Structures is naturally equivalent to the category of abstract Witt rings. This paper develops a combinatorial theory for finite Quaternionic Structures in the case where $1 = -1$, by demonstrating an equivalence between finite quaternionic structures and Steiner Triple Systems (STSs) with suitable block colorings. Associated to these STSs are Block Intersection Graphs (BIGs) with induced vertex colorings. This equivalence allows for a classification of BIGs corresponding to the basic indecomposable Witt rings via their associated quaternionic structures. Further, this paper classifies the BIGs associated to the Witt rings of so-called elementary type, by providing necessary and sufficient conditions for a BIG associated to a product or group extension.

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Chapter 1

Introduction

The algebraic theory of quadratic forms over fields is a prevalent area of modern mathematics. Ernst Witt introduced this topic to the mathematics community in 1937 [7], where it has undergone substantial growth and development, continuing to this day. The Witt ring of a field (not of characteristic 2), denoted WF, is arguably Witt's most significant contribution, and has been thoroughly studied. Murray Marshall proved that all finitely generated torsion-free Witt rings are realizable as the Witt ring of a pythagorean field in [4]. When the torsion of WF is non-trivial, there are many remaining open problems. These open problems about Witt rings of a field form a subset of a larger set of open problems surrounding what are known as abstract Witt rings.

In 1972 Knebusch, Rosenberg, and Ware introduced the notion of an abstract Witt Ring in [2]. The axiomatic approach to abstract Witt rings is attributed to Yucas, who defined and developed the a theory of Quaternionic Structures, which are ordered triples of an elementary abelian group of exponent 2, a pointed set, and a map $q : G \times G \longrightarrow Q$ satisfying four axioms. In 1980 Murray Marshall produced the paper abstract Witt rings [4], which catalogued and exposed much of the recent theory. In this paper Marshall proved that the category of abstract Witt rings is naturally equivalent to the category of

Quaternionic Structures, and provided a full classification of finitely generated torsion-free abstract Witt rings, which were demonstrated to be realizable as Witt rings of pythagorean fields.

Every Witt ring of a field is an abstract Witt ring; however, it is still an open problem if every abstract Witt ring is realized so. In the case where the abstract Witt ring is finitely generated, the problem is posed by Marshall as the Elementary Type Conjecture in Quadratic Form Theory [5]. Significant effort has been put toward the resolution of this conjecture; however, it remains open, along with many other problems in this field, such as the Arason-Pfister property, denoted $AP(k)$ for $k \in \mathbb{N}$.

The definition of an abstract Witt ring as presented in Marshall's text [4] requires only 3 axioms, the second axiom requires that $AP(1)$ and $AP(2)$ are true for an abstract Witt ring. This leads to another question, "does $AP(3)$ hold for arbitrary Witt rings?" The answer is unknown, again, except for the case where the abstract Witt ring is torsion-free or when there are sufficiently many constraints on the quaternionic structures themselves.

Whether or not $AP(3)$ is true for an abstract Witt ring is of potentially deep importance. Given any quaternionic structure, (G, Q, q), there is an associated abstract Witt ring call it R. Let $I \lhd R$ be the unique maximal ideal so that $R/I \cong \mathbb{Z}/2\mathbb{Z}$. Marshall showed that there is a natural isomorphism $I/I^2 \cong G$, and that there is a natural map, $Q \longrightarrow I^2/I^3$. The significance of $AP(3)$ is then captured by the following diagram.

$$\begin{array}{ccc} G \times G & \xrightarrow{\quad q \quad} & Q \\ \cong \big\downarrow & & \big\downarrow s \\ I/I^2 \times I/I^2 & \xrightarrow{\quad \otimes \quad} & I^2/I^3. \end{array}$$

The dashed arrow highlights the uncertainty involved in the natural map s which is injective if and only if $AP(3)$ is true for the Witt ring R. To approach this problem, we have developed combinatorial methods for any abstract Witt ring with characteristic 2, and to avoid the potential obstructions set forth by $AP(3)$, we have avoided using the natural map, in favor of the less studied quaternionic structure.

Chapter 2

Quaternionic Structures

The following definition is from Marshall [4]. Let G be any elementary abelian group of exponent 2, that is, any group isomorphic to a direct sum of copies of $\mathbb{Z}/2\mathbb{Z}$. Declare a distinguished element $-1 \in G$. Let Q be a pointed set with distinguished element $0 \in Q$. Denote $-a = -1 \cdot a \in G$, and define

$$q : G \times G \longrightarrow Q$$

to be surjective map with the following properties:

Q_1 **Symmetry:** $q(a,b) = q(b,a)$ for every $a, b \in G$.

Q_2 **Excision:** $q(a,-a) = 0$ for every $a \in G$.

Q_3 **Weak Bilinearity:** For any $a,b,c \in G$ $q(a,b) = q(a,c) \iff q(a,bc) = 0$.

Q_4 **Linkage:** For any $a,b,c,d \in G$ $q(a,b) = q(c,d) \implies \exists x \in G$ s.t. $q(a,b) = q(a,x)$ and $q(c,d) = q(c,x)$.

Definition 2.0.0.1 *If q satisfies Q_1 through Q_4, we call the triple (G, Q, q) a* ***quaternionic structure.***

All of the axiom names are provided by Marshall's text [4] with the exception of Q_2, which we call the excision axiom for reasons that will become clear once we have motivated our combinatorial approach. The name "quaternionic structure" originates from the study of quaternion algebras over a field, and in particular the Brauer group of a field, which is intimately connected to the Witt ring.

In the case where F is a field not of characteristic 2, there is a natural construction of G, and (G, Q, q), typically denoted $(G(F), Q(F), q)$. Let $\dot{F}$ denote the multiplicative group of units of F, then $\dot{F}/\dot{F}^2 = G(F)$ is a group of exponent 2. The distinguished element $-1 \in G(F)$, may be regarded as $-1\dot{F}^2 \in \dot{F}/\dot{F}^2$. Whenever we declare $1 = -1 \in G(F)$, it is to say that the cosets $1\dot{F}^2 = -1\dot{F}^2$, that is, -1 is a square in F.

If we consider the Witt ring of F, denoted WF, the fundamental ideal is the unique maximal ideal so that $WF/IF \cong \mathbb{Z}/2\mathbb{Z}$. Of interest are the powers of the fundamental ideal I^2F, and I^3F. The nomenclature of these so-called quaternionic structures originates in the study of the the Brauer group of F denoted $Br(F)$. Denote the 2-torsion of the Brauer group as $Br_2(F)$. A classical result is that

$$H^2(Gal(F), \mathbb{Z}/2\mathbb{Z}) \cong Br_2(F),$$

and Murkeryjev's Theorem [6] provides the natural isomorphism

$$I^2/I^3 \cong H^2(\mathrm{Gal}(F), \mathbb{Z}/2\mathbb{Z}),$$

giving a natural isomorphism

$$I^2/I^3 \cong Br_2(F). \tag{2.1}$$

This natural isomorphism and by defining the map q as $q(a, b) = \ll a, b \gg \in I^2F$, led to calling $Q(F) = \mathrm{range}(q)$ the set of quaternions, and $(G(F), Q(F), q)$ as the quaternionic structure associated to WF. For a thorough treatment of quadratic forms over fields, the Witt ring of a field, the Brauer group, and the proof of the natural isomorphism in (2.1), see [3]. This leaves us with the following definition.

Definition 2.0.0.2 *For any $x \in Q$, we may refer to x a **quaternion**.*

Let W be any abstract Witt ring corresponding to some quaternionic structure (G, Q, q), and I the fundamental ideal. As mentioned in the introduction, there is a natural way one might attempt to embed $Q \hookrightarrow I^2/I^3$. Define $Q^* = Q \setminus \{0\}$. Denote the free vector space over the field with two elements $GF(2) = \mathbb{F}_2$ with basis Q^* as $\mathbb{F}_2 \cdot Q^*$. There is a natural map:

$$s : \mathbb{F}_2 \cdot Q^* \longrightarrow I^2/I^3, \quad \text{by} \quad q(a,b) \overset{s}{\longmapsto} \ll a, b \gg + I^3.$$

However, it is precisely this approach which captures how essential the resolution of $AP(3)$ is. If $AP(3)$ is not true, then the 2-fold Pfister form $\ll a, b \gg$ may be a non-trivial element of I^3, giving this map non-trivial kernel. To avoid this exact problem, we will not involve the Witt ring directly, but remain grounded in these quaternionic structures.

In the case where (G, Q, q) is an "abstract" quaternionic structure, we will primarily regard G as a vector space over $\mathbb{F}_2$; however, we will maintain our binary operation as multiplication to stay consistent with the definitions and notation set forth in Marshall's work [4]. If the need arises we will specify a spanning set and define

$$\text{span}_{\mathbb{F}_2}\{x_1, \ldots, x_k\} = \langle x_1, \ldots, x_k \rangle = G.$$

Where for any $v \in G$, we write the linear combinations multiplicatively; that is

$$v = \prod_{i=1}^{k} x_i^{\alpha_i}, \quad \alpha_i \in \mathbb{F}_2.$$

Now that we have established the fundamental definition, that of a quaternionic structure, we begin our deviation from Marshall's text [4].

Definition 2.0.0.3 *Denote the identity of any group H as $1 \in H$, and define $H^* = H \setminus \{1\}$. For any set of quaternions Q, define $Q^* = Q \setminus \{0\}$. Let (G, Q, q) be any quaternionic structure. If $|G| < \infty$, then we call (G, Q, q) a* ***finite*** *quaternionic structure. We will refer to any (x, y) as a* **pairing***. If $q(x, y) = 0$ then we call (x, y) a* **trivial pairing***. If $q(x, y) \neq 0$, we call (x, y) a* **non-trivial pairing***, and we call (G, Q, q)* ***non-trivial*** *if $|Q| > 1$.*

Observation 2.0.0.4 (Marshall [4]) *For any $a, b \in G$, the following are immediate from the axioms:*

(i) $q(a, 1) = 0$, *since* $q(a, 1) = q(a, 1) \iff q(a, 1^2) = q(a, 1) = 0$.

(ii) $q(a, -ab) = q(a, b) \iff q(a, -ab^2) = q(a, -a) = 0$.

Definition 2.0.0.5 *We define the notion of radical elements here, together with the notation and terminology involving radicals. Let $a \in G$, and suppose*

$$q(a, b) = 0, \quad \forall b \in G,$$

then we say that a is ***radical****. For any fixed $a \in G$, define the* ***relative radical at*** *a*

$$\mathrm{Rad}(a) = \{b \in G \mid q(a, b) = 0\}.$$

For any (G,Q,q) *define the* ***radical of*** (G,Q,q) *as*

$$\operatorname{Rad}(q) = \{a \in G \mid q(a,b) = 0,\ \forall\, b \in G\}.$$

If $\operatorname{Rad}(q) = G$*, then we call* (G,Q,q) ***purely radical****.*

Observation 2.0.0.6 *For any* $a \in G$, $\operatorname{Rad}(a)$ *and* $\operatorname{Rad}(q)$ *are subgroups of* G*.*

Proof: Let $a \in G$, and suppose $x, y \in \operatorname{Rad}(a)$. It suffices to show that $xy \in \operatorname{Rad}(a)$, since $g^{-1} = g$ for all $g \in G$. By assumption $q(a,x) = q(a,y) = 0$, then $q(a,xy) = 0$, by Q_3, so $xy \in \operatorname{Rad}(a)$, have shown that $\operatorname{Rad}(a)$ is a subgroup of G. It follows that

$$\operatorname{Rad}(q) = \bigcap_{a \in G} \operatorname{Rad}(a).$$

Since the arbitrary intersection of subgroups is a subgroup, then $\operatorname{Rad}(q)$ is a subgroup of G. ■

Definition 2.0.0.7 *Let* $a \in G$*, and suppose for any* $b \in G$*,*

$$q(a,b) = 0 \implies b \in \{1, -a\},$$

then we call a ***rigid****.*

It is clear that if $a \in G$ is rigid, then $\operatorname{Rad}(a) = \{1, -a\}$. We will use this fact repeatedly without references, as it is clear from the definition of the radical at a. Dual to the notion of being rigid is that of being **basic**, which we now define.

Definition 2.0.0.8 *Suppose* $a \in G$*, and* a *or* $-a$ *is not rigid, then* a *is called* ***basic****. If every* $a \in G$ *is basic, then we call* (G,Q,q) *a* ***basic structure****.*

The following nomenclature may seem peculiar; however, the Witt ring associated to any quaternionic structure with $1 = -1$ is characteristic 2, so the following definition stems from this fact. Other names and notations will be adopted from the Witt ring moving forward, and will be described in a similar fashion. The information previous to this point is primarily from Marshall's paper, with the exception of some of the definitions, which we use to simplify communication. The reader interested in abstract Witt rings should refer to [4] for more information.

Definition 2.0.0.9 *If $1 = -1 \in G$, then we call (G, Q, q) a* ***characteristic 2*** *quaternionic structure.*

Observation 2.0.0.10 *Let $-1 = 1 \in G$, and let $a, b \in G$ be distinct. Since $q(a, b) = q(a, ab)$, by Observation 2.0.0.4, then $q(b, ab) = q(a, ab) \iff q(ab, ab) = 0$, is true by Q_2. Thus*

$$q(a, b) = q(a, ab) = q(b, ab)$$

for all $a, b \in G$.

Observation 2.0.0.11 *Given any characteristic 2 quaternionic structure (G, Q, q), let $x, y \in G$ be arbitrary, and let $\tau \in \mathrm{Rad}(x) \cap \mathrm{Rad}(y)$, then*

$$q(x, y) = q(x\tau, y\tau) = q(x, \tau y),$$

for all $x, y \in G$.

Proof: Since $q(x\tau, y\tau) = q(xy, x\tau)$, and $q(x, y) = q(xy, x)$, by Observation 2.0.0.10, then weak bilinearity gives us

$$q(xy, x) = q(xy, x\tau) \iff q(xy, \tau) = 0,$$

which is true by the assumption of $\tau \in \mathrm{Rad}(x) \cap \mathrm{Rad}(y)$, then we conclude $q(x\tau, y\tau) = q(x, y)$. Again, weak bilinearity gives

$$q(x, \tau y) = q(x, y) \iff q(x, \tau) = 0,$$

again by the assumption that $\tau \in \mathrm{Rad}(x) \cap \mathrm{Rad}(y)$. So we may conclude

$$q(x, y) = q(x\tau, y\tau) = q(x, \tau y),$$

for all $x, y \in G$, $\tau \in \mathrm{Rad}(x) \cap \mathrm{Rad}(y)$. ■

We see from Observation 2.0.0.11, that there are typically many pairings which are assigned the same quaternion in Q. This motivates the following definition, as the pairings which are mapped to some fixed quaternion value in Q will be of importance.

Definition 2.0.0.12 *Let (G, Q, q) be some quaternionic structure. Then for any $k \in Q$, define the **replication number** of k as $|q^{-1}(\{k\})|/6$.*

Observation 2.0.0.10 tells us that there are three pairings, up to symmetry which are assigned the same quaternion. With symmetry that gives us 6 total, thus when defining the replication number, Observation 2.0.0.10 and Q_2 forces us to divide by 6 when counting distinct pairings.

2.1 Terminology

Let G be any elementary abelian group of exponent 2. It is clear that $(G, \{0\}, q)$, where $q(a, b) = 0$, for all $a, b \in G$, is a quaternionic structure.

Definition 2.1.0.1 *Let $G = \{1\}$, then we define $(G, \{0\}, q)$ to be the **trivial quaternionic structure**, and refer to $\{1\}$ alone as the trivial quaternionic structure. For any* *choice of G, we call $(G, \{0\}, q)$, with q the constant 0 map a* ***purely radical*** *quaternionic structure, and when there is no confusion we will simply write G rather than $(G, \{0\}, q)$.*

Recall, we will often regard $G = \langle g_1, \ldots, g_n \rangle$ as a vector space over $\mathbb{F}_2$ while writing linear combinations multiplicatively. If the dimension of $\dim G$ is even, and q is any anti-symmetric (here this is trivial if $-1 = 1$), non-degenerate bilinear form, then there exists a basis so that $G = \langle x_1, y_1, \ldots, x_k, y_k \rangle$ and

$$q(x_i, y_j) = \delta_{ij},$$

where δ_{ij} is the Kronecker delta, and all other pairings of basis vectors are assigned the 0 quaternion. Marshall provides a proof that this is a quaternionic structure, for a proof of this see lemma 5.4 [4].

Definition 2.1.0.2 *Suppose $(G, \mathbb{F}_2, q)$ is as stated above, then we say that G is the* ***symplectic quaternionic structure*** *of dimension $2k$.*

Lemma 2.1.0.3 *Let $(G, \mathbb{F}_2, q)$ be the symplectic quaternionic structure of dimension $2n$, let*

$$\beta = \{x_1, y_1, \ldots, x_n, y_n\},$$

and let $G = \langle \beta \rangle$. Then $\mathrm{Rad}(x_i) = \langle \beta \setminus \{y_i\} \rangle$.

Proof: Fix $x_i \in G$, then $q(x_i, -) : G \longrightarrow \mathbb{F}_2$, is a linear map, which vanishes on $\beta \setminus \{y_i\}$, by definition. So

$$\ker(q(x_i, -)) = \langle \beta \setminus \{y_i\} \rangle = \mathrm{Rad}(x_i),$$

and thus we have proved the claim. ■

2.2 Quaternionic Morphisms

A natural consideration is that of a morphism in the category of quaternionic structures. The definition of a morphism in this category will appear too weak at first; however, it is clear that the definition we set forth preserves the first three of the quaternionic structure axioms, but the fourth and most essential axiom, linkage, requires some proof.

Definition 2.2.0.1 *A morphism of quaternionic structures is defined as follows. Suppose (G, Q, q) and (G', Q', q') are both quaternionic structures. Then if $\alpha : G \longrightarrow G'$, is any group homomorphism satisfying the following:*

$$\alpha(-1) = -1, \text{ and } q(a, b) = 0 \implies q'(\alpha(a), \alpha(b)) = 0, \quad \forall a, b \in G,$$

then we call α a quaternionic momorphism.

Lemma 2.2.0.2 *If $\alpha : (G, Q, q) \longrightarrow (G', Q', q')$, then*

$$q(a, b) = q(c, d) \implies q'(\alpha(a), \alpha(b)) = q'(\alpha(c), \alpha(d)).$$

That is, α preserves the linkage axiom.

Proof: By linkage, $q(a, b) = q(c, d) \implies \exists\, x \in G$ so that

$$q(a, b) = q(a, x) = q(c, x) = q(c, d) \iff q(a, bx) = q(c, dx) = q(ac, x) = 0$$

by weak bilinearity. So assuming α is a quaternionic morphism, then

$$q'(\alpha(a), \alpha(bx)) = q'(\alpha(c), \alpha(dx)) = q'(\alpha(ac), \alpha(x)) = 0$$

and since α is a group homomorphism by assumption, then weak bilinearity of q' implies

$$q'(\alpha(a), \alpha(b)) = q'(\alpha(a), \alpha(x)) = q'(\alpha(c), \alpha(x)) = q'(\alpha(c), \alpha(d))$$

and thus we have the desired implication,

$$q(a, b) = q(c, d) \implies q'(\alpha(a), \alpha(b)) = q'(\alpha(c), \alpha(d)).$$

■

By the preceding lemma, we may conclude that all of the axioms of a quaternionic structure are preserved.

Definition 2.2.0.3 *Let (G, Q, q), and (H, Q', q') quaternionic structures, and suppose there exists an injective quaternionic homomorphism*

$$\alpha : (G, Q, q) \longrightarrow (H, Q', q').$$

Then write

$$(G, Q, q) \hookrightarrow (H, Q', q'),$$

and say that (G, Q, q), is a ***quaternionic substructure*** *of (H, Q', q'), or more generally we will simply refer to this as a substructure, when there is no confusion.*

2.2.1 The Product of Quaternionic Structures

Let $\{(G_i, Q_i, q_i)\}_{i=1}^{m}$, be a family of quaternionic structures.

Definition 2.2.1.1 *The* ***product of quaternionic structures*** *is given as follows. Let $G = G_1 \oplus \cdots \oplus G_m$, $Q = Q_1 \times \cdots \times Q_m$, and for any $g = (g_1, \ldots, g_m)$, $h = (h_1, \ldots, h_m)$*

elements of G*, then define the map*

$$q : G \times G \longrightarrow Q, \quad q(g, h) = (q_1(g_1, h_1), \ldots, q_m(g_m, h_m)),$$

and we write

$$\prod_{i=1}^{m}(G_i, Q_i, q_i) = (G, Q, q).$$

Where the distinguished element is $0 = (0, \ldots, 0) \in Q$*. The verification of the axioms follows immediately, as they are true for each coordinate function. Furthermore, there are canonical projections*

$$\pi_i : (G, Q) \longrightarrow (G_i, Q_i),$$

so that for any quaternionic structure (H, Q_H, q_H) *and any family of quaternionic morphisms*

$$\alpha_i : (H, Q_H, q_H) \longrightarrow (G_i, Q_i, q_i)$$

then there exists a unique

$$\alpha : (H, Q_H, q_H) \longrightarrow (G, Q, q)$$

so that the following diagrams commute for all i :

$$\begin{array}{ccc} H & \xrightarrow{\alpha} & G \\ & \searrow^{\alpha_i} & \downarrow \pi_i \\ & & G_i \end{array} \qquad\qquad \begin{array}{ccc} Q_H & \xrightarrow{q \circ \alpha} & Q \\ & \searrow^{q_H \circ (\alpha_i \times \alpha_i)} & \downarrow q_i \circ (\pi_i \times \pi_i) \\ & & Q_i. \end{array}$$

Lemma 2.2.1.2 *Let* $\prod_{i=1}^{m}(G_i, Q_i, q_i) = (G, Q, q)$*, be any product of characteristic 2* *quaternionic structures. Then for each i, there exists a canonical injection*

$$\eta_i : (G_i, Q_i, q_i) \hookrightarrow (G, Q, q).$$

Proof: For any $g \in G_i$, define $\eta_i : G_i \longrightarrow G$ by

$$\eta_i(g) = (1, \ldots, 1, \underset{i^{\text{th}}}{g}, 1, \ldots, 1).$$

Then η_i extends naturally to a morphism of quaternionic structures. Since $1 = -1 \in G_i$ for all i by assumption, then $\eta_i(-1) = \eta_i(1) = (1, \ldots, 1) \in G$, so the first condition is met trivially. If $g, h \in G_i$ are so that $q_i(g, h) = 0$, then

$$q(\eta_i(g), \eta_i(h)) = (q_1(1,1), \ldots, q_{i-1}(1,1), q_i(g,h), q_{i+1}(1,1), \ldots, q_m(1,1)) = (0, \ldots, 0) \in Q,$$

and thus $q_i(g, h) = 0$ implies that $q(\eta_i(g), \eta_i(h)) = 0$. Since i was arbitrary, we have shown that there exist canonical injections $\eta_i : G_i \hookrightarrow G$, and thus we have proved the claim.

■

We conclude this section by examining how the quaternionic substructures of a product "interact" with respect to their product structure.

Lemma 2.2.1.3 *Let (G_1, Q_1, q_1) and (G_2, Q_2, q_2) be non-trivial finite characteristic 2 quaternionic structures, and let $(G, Q, q) = (G_1, Q_1, q_1) \times (G_2, Q_2, q_2)$. Then for all $(g, 1) \in G_1$, $\{1\} \times G_2 \subset \text{Rad}((g, 1))$, and for all $(1, h) \in G_2$, $G_1 \times \{1\} \subset \text{Rad}((1, h))$.*

Proof: Let (G, Q, q) be as defined above. Let $(1, h) \in G_2$, then for any $(g, 1) \in G_1$, we have

$$q((g, 1), (1, h)) = (q_1(g, 1), q_2(1, h)) = (0, 0),$$

by Observation 2.0.0.4, since $(g,1) \in G_1$, was arbitrary then we have shown $G_1 \times \{1\} \subset \text{Rad}((1,h))$, and a symmetric argument demonstrates that $\{1\} \times G_2 \subset \text{Rad}((g,1))$. ■

By the previous lemma, it is clear that these quaternion maps q_1 and q_2 don't interact in a direct product, that is, the substructures (G_1, Q_1, q_1), and (G_2, Q_2, q_2) always yield trivial pairings. So these substructures may be referred to as relatively radical. Before we conclude this section, we explore a result regarding the nature of certain quaternionic structures where $1 \neq 1$.

Theorem 2.2.1.4 *Let* (G, Q, q) *be any quaternionic structure, with* $1 \neq 1$. *If* $q(g,g) = 0$ *for all* $g \in G$, *then* $\{1, -1\} \subset \text{Rad}(q)$, *and*

$$(G, Q, q) \cong (G', Q', q') \times \langle -1 \rangle.$$

Proof: Suppose $q(g,g) = 0$ for all $g \in G$. By excision we have $q(g,-g) = 0$ for all $g \in G$, so by weak bilinearity we get $q(g,-g) = q(g,g) \iff q(g,-1) = 0$, for all $g \in G$. So $-1 \in \text{Rad}(q)$. Let $\pi_1 : G \longrightarrow G'$ be so that $\ker(\pi_1) = \{1, -1\}$. Since $-1 \in \ker(\pi_1)$, $\pi_1(-1) = 1$ satisfies the first condition of a quaternionic morphism. It is certainly the case that if $q(g,h) = 0$ for some $g, h \in G$, then $q'(\pi_1(g), \pi_1(h)) = q'(\pi_1(g), \pi_1(h))$ if and only if $q'(\pi_1(g), 1) = 0$ by weak bilinearity of q', but this is true by Observation 2.0.0.4. Furthermore, this map is surjective, by Observation 2.0.0.11. Since $-1 \in \text{Rad}(q)$, we have $q(g,-h) = q(g,h)$ for all $g, h \in G$. Thus we have shown that π_1 is a canonical projection. Similarly, take $\pi_2 : G \longrightarrow \langle -1 \rangle$. Then this map is certainly a quaternionic morphism, since we equip $\langle -1 \rangle$ with the set of quaternions $\{0\}$, so the axioms are trivially satisfied. Thus we have shown that $(G, Q, q) \cong (G', Q', q') \times \langle -1 \rangle$, and thus we have proved the claim. ■

We have now motivated enough to describe the fundamental factors from which we will construct our main body of quaternionic structures, the so-called basic indecomposable structures.

Definition 2.2.1.5 *Let (G, Q, q) be any quaternionic structure so that every element of G is basic, as in Definition 2.0.0.8. We say (G, Q, q) is* ***basic indecomposable*** *if*

$$(G, Q, q) \cong (G_1, Q_1, q_1) \times (G_2, Q_2, q_2)$$

then either (G_1, Q_1, q_1) or (G_2, Q_2, q_2) is the trivial quaternionic structure, as in Definition 2.1.0.1.

2.2.2 Group Extensions of Quaternionic Structures

We saw in the previous section that product factors "interact" by trivially pairing to the zero quaternion. In this section we explore a dual notion to that of a product of quaternionic structures, where factors in a product of groups will not be radical relative to one another, but rather we will induce a rigid "interaction." For an arbitrary characteristic 2 quaternionic structure, (G, Q, q), we will equip a group of rigid elements, call it W, and we will describe the most free interaction of G and of W, where we obtain a new structure (G', Q', q'), with $(G, Q, q) \hookrightarrow (G', Q', q')$, together with $W \hookrightarrow G$, in such a way that $G' \cong G \oplus W$, and q', when restricted to pairings from $G \times G$, will be the identity, and when the two structures interact under this map q', rather than yielding the zero quaternion, we will always yield a non-zero quaternion, making W rigid relative to G.

Definition 2.2.2.1 *Let (G, Q, q) be any finite characteristic 2 quaternionic structure, and let W be any elementary abelian group of exponent 2. Then we form the direct sum $G[W] = G \oplus W$, and the associated set*

$$Q[W] \subset Q \vee (G \otimes_{\mathbb{F}_2} W) \oplus (W \wedge W),$$

where $\vee$ *denotes the union of pointed sets with distinguished point* 0*, and* $Q[W]$ *is the range of the map defined as follows: for any* $(g_1, w_1), (g_2, w_2) \in G \oplus W$*,*

$$q[W]((g_1, w_1), (g_2, w_2)) = (q(g_1, g_2), g_1 \otimes w_2 + g_2 \otimes w_1, w_1 \wedge w_2).$$

When $Q = \{0\}$ *we denote* $\{0\}[W]$*, as simply* $[W]$*, when there is no confusion.*

The choice of language, and notation for these "group extensions" has been appropriated from the case of a ring, say R and a group, W, where the group ring is typically denoted $R[W]$. We do this because these quaternionic structures, which are group extensions in our language, correspond precisely to the group extensions of Witt rings. Certain special and significant group extensions are called "purely rigid." To simplify the proof that our definition of a group extension is a quaternionic structure, we will construct a canonical purely rigid structure.

Definition 2.2.2.2 *We will call a quaternionic structure* (G, Q, q) ***purely rigid*** *if every* $a \in G$ *is rigid, that is* $\mathrm{Rad}(a) = \{1, a\}$.

Take G as a vector space over $\mathbb{F}_2$, and define $Q \subseteq \Lambda^2_{\mathbb{F}_2}(G)$, the second exterior power of G, and $q(a, b) = a \wedge b$, the canonical anti-symmetric bilinear map for all $a, b \in G$. For simplicity, denote $G \wedge G = \{a \wedge b \mid a, b \in G\}$ when there is no ambiguity. It is important to keep in mind that $1 \in G$ is the zero vector.

Proposition 2.2.2.3 *The triple* $(G, G \wedge G, \wedge)$ *is a purely rigid quaternionic structure when* $-1 = 1 \in G$.

Proof: Let $G = \langle x_1, \ldots, x_k \rangle$. Define $q(a, b) = a \wedge b$, for all $a, b \in G$. Since every element $a \wedge b$ is a linear combination of the pairings $x_i \wedge x_j$, then it is clear that $q : G \times G \longrightarrow G \wedge G$ is onto. As for the axioms:

$Q_1:$ By definition $a \wedge b = b \wedge a$, since $1 = -1 \in G$.

$Q_2:$ Again, since $1 = -1$, we have $a \wedge a = 0 \in G \wedge G$.

$Q_3:$ Suppose $a \wedge b = a \wedge c$. Since $G \wedge G$ is a vector space over $\mathbb{F}_2$, then this implies $a \wedge b + a \wedge c = 0$, but this is equivalent to $a \wedge bc = 0$. Conversely, if $a \wedge bc = 0$, then $a \wedge b = a \wedge c$, and thus we have proved weak bilinearity.

$Q_4:$ Suppose $a \wedge b = c \wedge d$, then $a \wedge b + c \wedge d = 0$, and since the additive inverse of $a \wedge b$ is unique and $a \wedge b + a \wedge b = 0$, then $c \in \langle a, b\rangle^*$, or $d \in \langle a, b\rangle^*$, without loss of generality, let $c = a$, then

$$a \wedge b = a \wedge d \implies a \wedge b + a \wedge d = a \wedge bd = 0.$$

Thus $\{a, bd\}$ is linearly dependent, so $bd = a$, or $bd = 1$. If $bd = a$, then $d = ab$, so

$$\begin{aligned} a \wedge b &= a \wedge ab \\ &= a \wedge a + a \wedge b \\ &= a \wedge b \end{aligned}$$

If on the other hand $bd = 1$, then $d = b$, and we have shown that there is a unique quaternion $a \wedge b$. We can give a similar argument for $d \in \langle a, b\rangle^*$ by symmetry. So we may conclude that linkage holds, since the quaternion $a \wedge b$ is unique.

The uniqueness of this quaternionic structure is immediate from Q being maximal, since all quaternions $q(a, b)$ are distinct, up to Observation 2.0.0.10. ■

The following proposition motivates a property of rigid elements in a general quaternionic structure. We see that the existence of a rigid element in (G, Q, q) forces a lower bound for $|Q|$.

Proposition 2.2.2.4 *Let (G, Q, q) be any finite characteristic 2 quaternionic structure, and $|G| \geq 4$. If $a \in G$ is rigid, then $|Q| \geq |G|/2$.*

Proof: Let $a \in G$ be rigid. Since $|G| \geq 4$, there are at least two elements $b, c \in G \setminus \{1, a\}$. Then $q(a, b) = q(a, c) \iff q(a, bc) = 0$. So $bc \in \{1, a\}$ by assumption of a being rigid. If $bc = 1$, then $b = c$, but this contradicts our assumption that b, c are distinct. So consider $bc = a$, then $c = ab$, and we have $q(a, b) = q(a, ab)$, but this is trivial due to Observation 2.0.0.10. Thus, for any non-trivial representatives $b'\langle a\rangle, c'\langle a\rangle \in G/\langle a\rangle$, we have that $q(a, b') \neq q(a, c')$ so $|Q| \geq |G/\langle a\rangle| = |G|/2$, as desired. ■

Recall that if a is basic in (G, Q, q) if a is not rigid. By the previous proposition, we have the following useful upper bound on $|Q|$ a characteristic 2 quaternionic structure to be basic.

Theorem 2.2.2.5 *Suppose (G, Q, q) is a characteristic 2 quaternionic structure with $|G| \geq 4$, and $|Q| < |G|/2$, then (G, Q, q) is basic.*

Proof: By the contrapositive of Proposition 2.2.2.4, we have that $|Q| < |G|/2$, implies that $a \in G$ is not rigid, and therefore basic. Since $a \in G$ was arbitrary then we conclude that (G, Q, q) is basic. ■

It is still an open question if there exists a basic indecomposable quaternionic structure with $|Q| > 2$. Thus, the previous theorem gives an upper bound on $|Q|$ for (G, Q, q) to be a basic characteristic 2 quaternionic structure.

Proposition 2.2.2.6 *The group extension of (G, Q, q) by W is a quaternionic structure.*

Proof: Since every map is surjective onto its range, we need only check the quaternionic structure axioms. Let $g_1, g_2 \in G$, and $w_1, w_2 \in W$ and consider the following computations.

$Q_1:$

$$\begin{aligned} q[W]((g_1,w_1),(g_2,w_2)) &= (q(g_1,g_2), g_1\otimes w_2 + g_2\otimes w_1, w_1\wedge w_2) \\ &= (q(g_2,g_1), g_2\otimes w_1 + g_1\otimes w_2, w_2\wedge w_1) \\ &= q[W]((g_2,w_2),(g_1,w_1)). \end{aligned}$$

$Q_2:$

$$\begin{aligned} q[W]((g_1,w_1),(g_1,w_1)) &= (q(g_1,g_1), g_1\otimes w_1 + g_1\otimes w_1, w_1\wedge w_1) \\ &= (0, 1\otimes w_1, 0) \\ &= (0,0,0) = 0 \in Q[W]. \end{aligned}$$

$Q_3:$ Suppose that

$$q[W]((g_1,w_1),(g_2,w_2)) = q[W]((g_1,w_1),(g_3,w_3)).$$

Since (G,Q,q) is a quaternionic structure so Q_3 is true for q, and by Proposition 2.2.2.3, the map $\wedge$ satisfies Q_3 as well, then we need only concern ourselves with the portion of $q[W]$ in $G\otimes_{\mathbb{F}_2} W$. So we check

$$g_1\otimes w_2 + g_2\otimes w_1 = g_1\otimes w_3 + g_3\otimes w_1 \iff g_1\otimes w_2w_3 + g_2g_3\otimes w_1 = 0,$$

since $G\otimes_{\mathbb{F}_2} W$ is a vector space over $\mathbb{F}_2$, and since additive inverses are unique, then we have shown that

$$q[W]((g_1,w_1),(g_2,w_2)) = q[W]((g_1,w_1),(g_3,w_3)) \iff q[W]((g_1,w_1),(g_2g_3,w_2w_3)) = 0.$$

We have shown that $(G[W], Q[W], q[W])$ satisfies $Q_1 - Q_3$. Previously, we saw that any purely rigid structure satisfies the linkage axiom trivially, so we introduce the following lemma as it will be necessary to simplify the argument of Q_4 in general.

Lemma 2.2.2.7 *Let $g \in G$, and $w \in W^*$, be arbitrary. Then (g, w) is rigid.*

Proof: Recall that (g, w) is rigid by definition if $\mathrm{Rad}((g, w)) = \{(1, 1), (g, w)\}$. So let $(h, v) \in \mathrm{Rad}((g, w))$, then

$$(q(g, h), g \otimes v + h \otimes w, w \wedge v) = (0, 0, 0).$$

We know by the proof of Proposition 2.2.2.3 that $w \wedge v = 0$ if $v = w$ or $v = 1$, since we have assumed $w \neq 1$. If $v = 1$, then $q(g, h) = 0$, and $h \otimes w = 0$ must be the case. Since $w \neq 1$, again by assumption, then $h = 1$, because elementary tensors over $\mathbb{F}_2$ are non-degenerate. Thus, $q(g, h) = 0$, since $h = 1$ by Observation 2.0.0.4, and we have shown $(h, v) = (1, 1)$ so (g, w) is rigid. On the other hand, if $v = w$, then we require that $gh \otimes w = 0$. Since $w \neq 1$, then $gh = 1$, so $g = h$, and we get $q(g, h) = q(g, g) = 0$ by the excision axiom. Thus we have shown that $(h, v) = (g, w)$, so (g, w) is rigid, and thus we have proved the claim. ■

The preceding lemma demonstrates that in a group extension, we have $q[W](g, w)(1, w) = q[W](g, 1)(x, w) = (0, g \otimes w, 0)$, for all $x \in G$. So if these quaternions are to be linked, the linkage occurs at $(g, 1)$, (g, w) or $(1, w)$. We return to the proof of Proposition 2.2.2.6.

Q_4 Suppose that

$$q[W]((g_1, w_1), (g_2, w_2)) = q[W]((g_3, w_3), (g_4, w_4)).$$

We are left to verify the existence of $(g', w') \in G[W]$, which links these quaternions. If $w_1, w_2 \in W^*$ and $w_1 \neq w_2$, then linkage is trivial by the proof of Proposition 2.2.2.3.

We saw in Lemma 2.2.2.7 that if any $w_i \neq 1$, then then (g_i, w_i) is rigid then (g', w') is of the form (g_i, w_i), or $(1, w_i)$. If on the other hand $w_i = 1$ for each i, that is in the case of $(g', w') = (g_i, 1)$, then linkage is satisfied since q is linked by assumption, and since all other terms are the zero quaternion in each coordinate. Thus we have demonstrated that linkage holds.

Thus we may conclude that $(G[W], Q[W], q[W])$ is a quaternionic structure.

■

Lemma 2.2.2.8 *Any purely rigid quaternionic structure,* $(W, W \wedge W, \wedge)$ *is realizable as a group extension of the trivial quaternionic structure* $G = \{1\}$.

Proof: This is immediate by Definition 2.2.2.6, since $G[W] = \{1\} \oplus W \cong W$, $Q[W] = (0, 0, W \wedge W) \cong W$, and $q[W] = (0, 0, \wedge)$ is equivalent to $\wedge$, where 0 denotes the constant function. Thus we have shown that any purely rigid structure is a group extension of the trivial quaternionic structure. ■

The following lemma is a useful inductive tool, for general proofs surrounding group extensions of quaternionic structures.

Lemma 2.2.2.9 *Let* (G, Q, q) *be any finite characteristic 2 quaternionic structure. Let* W' *be any finite elementary 2 group. Assume* $W = W' \oplus \langle w_1 \rangle$. *Then*

$$(G[W'][w_1], Q[W'][w_1], q[W'][w_1]) \cong (G[W], Q[W], q[W]).$$

Proof: Certainly, $G[W] \cong G[W'] \oplus \langle w_1 \rangle$. So let $\alpha : G[W] \longrightarrow G[W'] \oplus \langle w_1 \rangle$, be the natural isomorphism, and note that α restricted to $G \oplus \{1\}$ is the identity map, so then α is the identity on $(G, Q, q) \hookrightarrow (G[W], Q[W], q[W])$. We need only show that α is a quaternionic morphism, and we will have proved the claim. Since $-1 = 1$ the first condition is trivial since α is a group homomorphism. Let $(g, w), (h, v) \in G[W]$, and suppose $q[W]((g, w), (h, v)) = 0$, then

$$q(g, h) = 0, \quad g \otimes v + h \otimes w = 0, \quad w \wedge v = 0$$

so $q(g, h) = 0$, and by Proposition 2.2.2.3, $vw = 1$, so we may assume $w = v$. By Lemma 2.2.2.7, it must then be the case that either $g = h$, or $w = 1$, because we need $g \otimes w + h \otimes w = gh \otimes w = 0$. If $w = 1$, we have that $q[W]((g, 1), (h, 1)) = 0 \implies q[W'][w_1](\alpha(g, 1), \alpha(h, 1)) = 0$, since α restricted to (G, Q, q) is the identity map. If $g = h$, then $q[W]((g, w), (g, w)) = 0 \implies q[W'][w_1](\alpha(g, w), \alpha(g, w)) = 0$, is true by the excision axiom. Thus we have shown that the natural isomorphism α is an isomorphism of quaternionic structures. ■

This next lemma serves as an example of how group extensions interact with the set of quaternions of a purely radical structure G. This inflation of the set of quaternions preserves the structure we're extending completely, that is, none of the original quaternions are replicated, in a dual sense to that of a product where the replication number of every quaternion is inflated due to the induced relative radical structure.

Lemma 2.2.2.10 *Let $(G[W], Q[W], q[W])$ be a group extension of some finite characteristic 2 quaternionic structure, and let $(1, w), (g, 1) \in G[W]^*$. Then every quaternion $(0, g \otimes w, 0) \in Q \vee (G \otimes W) \oplus (W \wedge W)$ has replication number $|\operatorname{Rad}(g)| - 1$, that is $|q[W]^{-1}(\{(0, g \otimes w, 0)\})| = |\operatorname{Rad}(g)| - 1$.*

Proof:

For any $(1, w), (g, 1) \in G[W]^*$, we have

$$q[W]((1, w), (g, 1)) = (0, g \otimes w, 0) \neq 0,$$

and by weak bilinearity, for any $x \in \mathrm{Rad}(g)$, we may take $(x, w) \in G[W]$

$$q[W]((1,w),(g,1)) = q[W]((x,w),(g,1)) \iff q[W]((x,1),(g,1)) = (q(g,x),0,0) = (0,0,0),$$

by the assumption that $x \in \mathrm{Rad}(g)$. Thus we have shown that if $(x, w) \in G[W]$, and $w \neq 1$, then $q[W]((x,w),(g,1)) = (0, g \otimes w, 0)$, so we may conclude $q[W]((1,w),(g,1))$ has replication number $|\mathrm{Rad}(g)^*| = |\mathrm{Rad}(g)| - 1$, since $x \in \mathrm{Rad}(g)$ was arbitrary, and no other pairings yield this quaternion by Proposition 2.2.2.3, we have proved the claim.

■

The significance of this lemma is if $a \in G$ is rigid, then it isn't *necessarily* the case that $q(a, b)$ is uniquely associated to a. This motivates the following lemma.

Lemma 2.2.2.11 *Let (G, Q, q) be some characteristic 2 quaternionic structure, and suppose $a \in G$ is rigid. Let $\langle a, b\rangle^* \neq \langle c, d\rangle^*$ for some $b, c, d \in G$ and furthermore suppose $q(a, b) = q(c, d)$. If $\langle a, b\rangle^* \cap \langle c, d\rangle^* \neq \emptyset$, then $\langle a, b\rangle^* \cap \langle c, d\rangle^* \neq \{a\}$.*

Proof: Let $\langle a, b\rangle^*$, and $\langle c, d\rangle^*$ be so that $\langle a, b\rangle^* \neq \langle c, d\rangle^*$, and let $\langle a, b\rangle^* \cap \langle c, d\rangle^* = \{x\}$. If $x = a$, then without loss of generality $\langle c, d\rangle^* = \langle a, d\rangle^*$. So $q(a, b) = q(a, c) \iff q(a, bc) = 0$ and since a is rigid, then it must be the case that $bc \in \{1, a\}$. If $bc = 1$, then $b = c$, and we have contradicted the assumption that $\langle a, b\rangle^* \neq \langle c, d\rangle^*$. On the other hand, if $bc = a$, then $c = ba$, and $b = ca$, which again contradicts the assumption that $\langle a, b\rangle^* \neq \langle c, d\rangle^*$. Thus $\langle a, b\rangle^* \cap \langle c, d\rangle^* \neq \{a\}$, and we have proved the claim. ■

Definition 2.2.2.12 *We say that (G, Q, q) is of* ***elementary type*** *it may be formed using the operations of product and group extension of the basic indecomposable quaternionic structures as set forth in definitions 2.0.0.5 and 2.1.0.2.*

The elementary type conjecture as posed in Marshall's works [4], [5], is an open problem concerning finitely generated Witt rings, and thus, finitely generated quaternionic structures. If the elementary type conjecture is true, then all Witt rings are realizable as the Witt ring of a field. Another significant result of the elementary type conjecture is that the classification of quadratically equivalent fields would be complete.

Chapter 3

Graphs of Quaternionic Structures

Here we introduce the notion of the graph of a quaternionic structure. To motivate this topic, a brief digression into the difficulty of trying to deal with quaternionic structures via tables. Let $(\langle x_1, y_1 \rangle, \mathbb{F}_2, q)$ be the symplectic quaternionic structure of dimension 2. A natural way to deal with these quaternionic structures is by computing a "quaternion table."

q	1	x_1	y_1	x_1y_1
1	0	0	0	0
x_1	0	0	1	1
y_1	0	1	0	1
x_1y_1	0	1	1	0

One notices that given a sufficiently complicated structure, any table of this form would quickly become cumbersome and difficult to extract information from. On the other hand, one might recognize a familiar construction, that of an adjacency matrix of a graph

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$

If we take our vertex set to be G, then it is clear that 1 as a vertex has no edges, according to this adjacency matrix. For our purposes, we make some modifications when viewing these quaternionic structures as graphs. By removing 1 from G in our construction, this array of information has a natural translation in to a complete graph, with suitable edge coloring rather than incidence relation recorded in this adjacency matrix, as motivated in the next section. We delete loops, and preserve edges labeled 0, by giving them a "color assignment" of 0, which will preserve the data set forth by the weak bilinearity condition. By removing 1 from our graphs, we can now use Q_2, or the **excision** axiom in a meaningful way. We have no "loops" from a vertex to itself, that is, we don't represent the quaternion $q(a, a)$ in our graph, and furthermore, we saw that $q(a, a) = q(a, a) \iff q(a, 1) = 0$, and so removing 1 from our graph we equivalently remove the loops from our graph. So the name excision is less mysterious in this context.

3.0.1 Graph Theoretic Conventions

We assume all standard conventions of graph theory; however, to avoid ambiguity, we will define the essential notation. Let Γ be any graph, then we write $V(\Gamma)$ to denote the set of vertices of Γ, and $E(\Gamma)$ for the set of edges.

Definition 3.0.1.1 *If Γ is any graph, and $V(H) \subseteq V(\Gamma)$ is any non-empty subset, and $E(H) \subseteq E(\Gamma)$ is any subset of edges in Γ with vertices in $V(H)$, then we call H a **subgraph of** Γ, and write $H \subset \Gamma$.*

Definition 3.0.1.2 *For any $v, w \in V(\Gamma)$ we denote the edge $vw \in E(\Gamma)$ as the **edge** connecting v to w. A **complete graph** is any graph K so that for all $v, w \in V(K)$, with $v \neq w$ we have $vw \in E(K)$. For some $\{v_1, v_2, \ldots, v_m\} \subset V(\Gamma)$ we call $(v_1, v_2, \ldots, v_m)$ a **path** in Γ if $v_i v_{i+1} \in E(\Gamma)$ for all $1 \leq i \leq m$.*

Definition 3.0.1.3 *Let C be any fixed non-empty set. Then any surjective function $c : E(\Gamma) \longrightarrow C$ is an called an edge coloring of Γ, with "colors" as elements of C. And we define Γ together with c as an edge colored graph.*

Definition 3.0.1.4 *We call a graph **connected** if for any $v, w \in V(\Gamma)$ there exists $v_1, \ldots, v_m$ so that $v_1 = v$, and $v_m = w$ so that $(v_1, \ldots, v_m)$ is a path in Γ.*

Definition 3.0.1.5 *Let p be any path from v to w, say $(v_1, \ldots, v_m)$ where $v = v_1$, and $w = v_m$. Then we define **len**$(p) = m - 1$ is the **length of the path** p.*

Definition 3.0.1.6 *Given a graph Γ, and $v \in V(\Gamma)$, define*

$$N_v = \{w \in V(\Gamma) \mid vw \in E(\Gamma)\}.$$

*We call N_v the **neighborhood subgraph** of v.*

Definition 3.0.1.7 *For any connected graph Γ, define the **distance** function*

$$\partial : V(\Gamma) \times V(\Gamma) \longrightarrow \mathbb{Z} \geq 0.$$

Let $v, w \in V(\Gamma)$, and P be the set of all paths from v to w.

$$\partial(v, w) = \begin{cases} \min_{p \in P}\{len(p)\} & \text{if } v \neq w, \\ 0 & \text{if } v = w \end{cases}$$

Definition 3.0.1.8 *We define*

$$diam(\Gamma) = \max_{v,w \in V(\Gamma)} \{\partial(v,w)\}$$

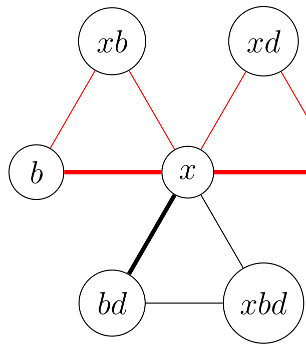
3.1 Quaternionic Graphs

Definition 3.1.0.1 *Define the **Graph of a Quaternionic Structure** is a complete graph on vertex set G^*, with Q as a set of edge colors. Denote this graph as $\mathcal{G}_q$. For clarity, to any $v, w \in G^*$ we assign the "color" $q(v,w)$ to the edge $vw \in E(\mathcal{G}_q)$.*

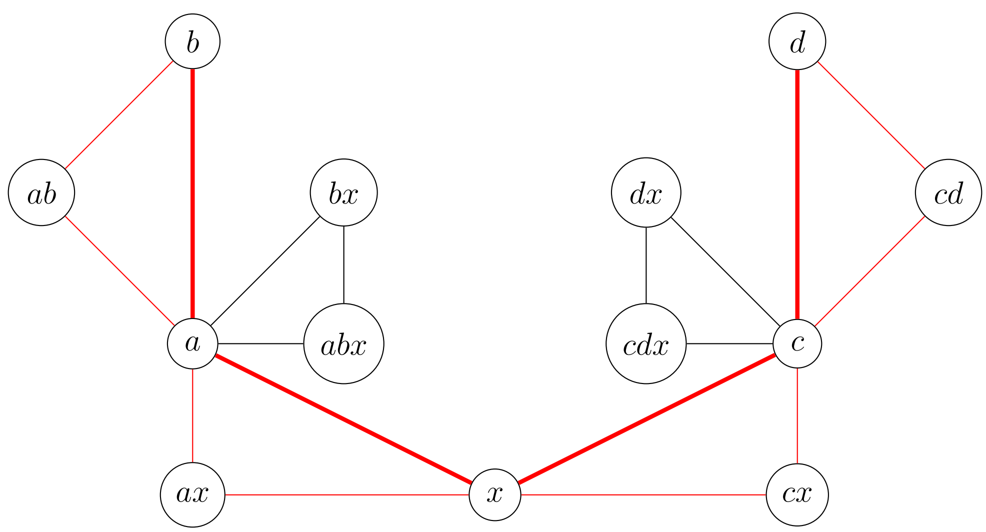
We now have a visual, as well as mathematical representation of a characteristic 2 quaternionic structure, (G, Q, q), whenever $|G| > 1$. The edge colors capture the function $q : G \times G \longrightarrow Q$, with the exception of $q(a,a)$ and $q(a,1)$ where $a \in G$. Moreover, G^* is represented in the vertex set. We will see that the graph theoretic realization of the axioms will provide subgraphs which encode the axioms. The first realization, is that our graphs being undirected is representative of the the symmetry of q. Furthermore, we will demonstrate the distinguishing axioms, weak bilinearity, and linkage as encoded by the following subgraphs.

3.1.1 Axioms via Graphs

Let (G, Q, q) be any characteristic 2 quaternionic structure. The bolded lines indicate what is required by the axioms, the additional edges and vertices are determined by the (potentially repeated) application of the axioms. We regard the color red as representative of the quaternion $p \in Q$ and black as representative of 0 in the following edge coloring. Let $b, d, x \in G$ be distinct, and suppose that $q(x,b) = q(x,d) = p \in Q^*$, then $q(x, bd) = 0$. Then weak bilinearity gives a subgraph structure as follows.



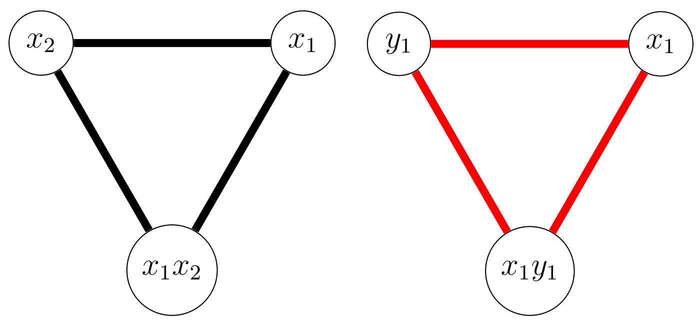
Then suppose $a, b, c, d \in G$ are all distinct, and furthermore, suppose that $q(a, b) = q(c, d) = k \in Q \setminus \{0\}$. Then the linkage axiom forces the following subgraph, where the bold edges denote what is required by linkage, and the thinner edges are applications of Observation 2.0.0.10, together with weak bilinearity.



Note that symmetry gives $q(b, a) = q(d, c)$, this gives that the subgraphs possessing a single edge coloring are diameter 2.

3.1.2 Fundamental Examples when $|G| = 4$.

For the following, consider the structures, $(\langle x_1, x_2\rangle, \{0\}, 0)$, and $(\langle x_1, y_1\rangle, \mathbb{F}_2, q)$. Regard black colored edges as 0, and red colored edges as $1 \in \mathbb{F}_2$. Then the quaternionic graphs are as follows.



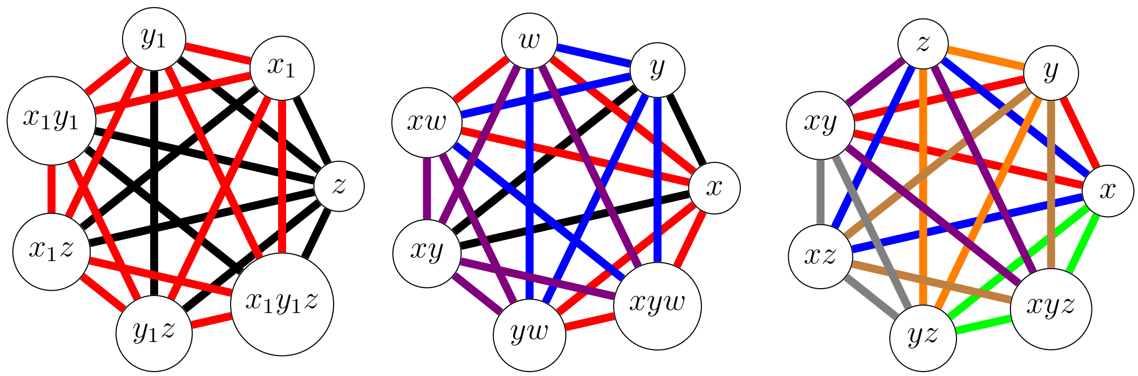
(3.1)

$(\langle x_1, x_2\rangle, \{0\}, 0)$, $\qquad$ $(\langle x_1, y_1\rangle, \mathbb{F}_2, q)$.

One may observe that in (3.1) that the graph with red edge coloring is both a quaternionic structure of symplectic type, a group extension $\langle x_1\rangle[y_1]$, and also the purely rigid structure on $\langle x_1, y_1\rangle$. A more substantive set of examples follows, and will provide a useful reference the reader to clearly distinguish these objects moving forward.

3.1.3 Three Fundamental Examples when $|G| = 8$

The following are quaternionic graphs of a product of a symplectic structure and a purely radical structure, a purely radical group extension, and a purely rigid structure.



$$(\langle x_1, y_1\rangle, \mathbb{F}_2, q) \times \langle z\rangle, \qquad \langle x, y\rangle[w], \qquad G = \langle x, y, z\rangle;\ (G, G \wedge G, \wedge). \tag{3.2}$$

Since $|G| = 2^n$ in general, when $n > 3$ the number of vertices in a given quaternionic graph become large. One also notes that the edge set grows even more rapidly, since any two vertices must share an edge due to these graphs being complete, so our examples terminate here.

3.2 Steiner Triple Systems

One observes that we may partition the edge set of our quaternionic graphs into "triangles," or "triples." This type of partition on the edge set of a complete graph is a type of Steiner Triple System, which will be important in our construction of a truly combinatorial approach to quaternionic structures. Steiner triple systems, and in general, Steiner systems have a rich history in combinatorics, and in algebra.

Definition 3.2.0.1 *A Steiner Triple System, or STS, is a set $\mathcal{S}$ together with a set, $\mathcal{B}$, which consists of all three element subsets of $\mathcal{S}$ with the property that for any distinct*

$g, h \in \mathcal{S}$, there exists precisely one block $B \in \mathcal{B}$, so that $g, h \in B$.

For any characteristic 2 quaternionic structure, say (G, Q, q), we have a natural "triple" to consider. Take any distinct $x, y \in G^*$, then $B = \{x, y, xy\}$, is a block, and B uniquely determined by x, y. Furthermore, it is clear that since G is of exponent 2, then x, xy, and y, xy determine the same block as well. Thus, any two distinct elements of G determine a block uniquely.

Proposition 3.2.0.2 *Every group of exponent 2 naturally induces a STS.*

Proof: Let G be any group of exponent 2 and let $g, h \in G^*$ be arbitrary and distinct. Put $H = \langle g, h \rangle = \{1, g, h, gh\}$. It is clear that H is the unique subgroup of order 4 generated by g, h, or more precisely,

$$H = \langle g, h \rangle = \langle g, hg \rangle = \langle h, hg \rangle. \tag{3.3}$$

For uniqueness, consider any $k, \ell \in G^*$ and let $K = \langle k, \ell \rangle$, then if $g, h \in K$, then $H = \langle g, h \rangle = K$, since K is of order 4, and contains g, h. So we may take our triples as the sets $\langle x, y \rangle^*$, where $x, y \in G^*$ are arbitrary, and distinct. Let

$$\mathcal{S} = G^*, \text{ and } \mathcal{B} = \{\langle x, y \rangle^* \mid x \neq y \in G^*\},$$

then $(G^*, \mathcal{B})$ is a STS, as desired. ■

Definition 3.2.0.3 *To any group of exponent 2, we will denote the* ***STS induced by*** $\boldsymbol{G}$ *as $(G^*, \mathcal{B})$, and whenever we write this pair it will be assumed that G is a group of exponent 2 with block set as defined in Proposition 3.2.0.2.*

3.2.1 Quaternionic Steiner Triple Systems

We saw that for any characteristic 2 (G, Q, q), we developed a natural STS associated to G. We will now develop a theory for capturing the totality of the data in (G, Q, q) as a STS equipped with a suitable block coloring function, as described in the following definition, and demonstrate an equivalence between these block colored STS and characteristic 2 quaternionic structures.

Definition 3.2.1.1 *Let (G, Q, q) be any quaternionic structure, and let $(G^*, \mathcal{B})$ be the STS associated to G. For any $B \in \mathcal{B}$, $B = \langle a, b\rangle^*$ for some $a, b \in G^*$. Define*

$$\tilde{q} : \mathcal{B} \longrightarrow Q, \text{ by } \tilde{q}(B) = q(a, b).$$

This map is well defined, since q is, and by Observation 2.0.0.10. Furthermore, it is clear that this map is symmetric, since $\langle a, b\rangle = \langle b, a\rangle$ for all $a, b \in G$.

By weak bilinearity, or Q_3, we have that $q(a, b) = q(a, c) \iff q(a, bc) = 0$. When considering blocks and the induced block coloring, we get $\tilde{q}(\langle a, b\rangle^*) = \tilde{q}(\langle a, c\rangle^*)$, together with $\tilde{q}(\langle a, bc\rangle^*) = 0$. Furthermore we note that $\langle a, b\rangle^* \cap \langle a, b\rangle^* = \{a\}$, $\langle a, b\rangle^* \cap \langle a, bc\rangle^* = \{a\}$, so

$$\langle a, b\rangle^* \cap \langle a, c\rangle^* \cap \langle a, bc\rangle^* = \{a\}.$$

For the situation where G induces some STS $(G^*, \mathcal{B})$, which is equipped with a block coloring, not necessarily originating from a quaternionic structure, we define a weak bilinear block coloring as follows.

Definition 3.2.1.2 *Any map $q : G \times G \longrightarrow Q$ is called weak bilinear if Q_1, Q_2, and Q_3 hold as described in section 2.*

Definition 3.2.1.3 *Let G be some group of exponent 2, and let $(G^*, \mathcal{B})$ be the associated STS, with some block coloring function $\tilde{q}$. If for any $B_1, B_2 \in \mathcal{B}$ where $B_1 = \langle a, b\rangle^*$, and*

$B_2 = \langle a, c\rangle^*$, *are so that* $\tilde{q}(B_1) = \tilde{q}(B_2)$ *if and only if* $B_3 = \langle a, bc\rangle^*$, *and* $\tilde{q}(B_3) = 0$. *Then we define* $\tilde{q}$ *to be* ***a weak bilinear block coloring.***

Lemma 3.2.1.4 *Let* G *be any group of exponent 2, let* $(G^*, \mathcal{B})$ *be the associated STS, and let*

$$\tilde{q} : \mathcal{B} \longrightarrow Q$$

be a block coloring function. Then $\tilde{q}$ *is a weak bilinear block coloring if and only if* $\tilde{q}$ *induces a surjective weak bilinear map*

$$q : G \times G \longrightarrow Q.$$

Proof: Let $\tilde{q}$ be a weak bilinear block coloring function. For all $\langle a, b\rangle^*$, define $\tilde{q}(\langle a, b\rangle^*) = q(a, b)$. The $\tilde{q} : \mathcal{B} \longrightarrow Q$ is onto with the possible exception of 0, so $q : G \times G \longrightarrow Q$ is as well. Extend q to all of $G \times G$ by defining $q(a, a) = 0$ for all $a \in G$, and as a consequence we have shown Q_2 holds for q. We show Q_1, by considering that for any $\langle a, b\rangle^* = B = \langle b, a\rangle^*$, so

$$q(a, b) = \tilde{q}(\langle a, b\rangle^*) = \tilde{q}(\langle b, a\rangle^*) = q(b, a),$$

and so we have shown Q_1 holds. We must now demonstrate weak bilinearity of the induced map q.

Since $\tilde{q}$ is assumed to be weak bilinear, then suppose for some $B_1, B_2 \in \mathcal{B}$ that $B_1 \neq B_2$, $\tilde{q}(B_1) = \tilde{q}(B_2)$, and let $B_1 \cap B_2 = \{x\}$. By the definition of $(G^*, \mathcal{B})$, there exists $g, h \in G$ so that $B_1 = \langle g, x\rangle^*$, and $B_2 = \langle h, x\rangle^*$. By Definition 3.2.1.4 and by our definition of q, we have

$$q(x, g) = \tilde{q}(B_1) = \tilde{q}(B_2) = q(x, h),$$

if and only if the block $B_3 = \langle x, gh\rangle^*$ is so that $\tilde{q}(B_3) = \tilde{q}(\langle x, gh\rangle^*) = 0$ but we defined q so that $\tilde{q}(\langle x, gh\rangle^*) = q(x, gh)$, so we conclude that $q(x, gh) = 0$, and since $g, h \in G^*$ were arbitrary, then we have shown that q satisfies Q_3. Thus we have shown that the map $q : G \times G \longrightarrow Q$ is surjective and satisfies $Q_1 - Q_3$.

Conversely, suppose that $q : G \times G \longrightarrow Q$ is surjective and weak bilinear, then the induced map as defined in Definition 3.2.1.1, is symmetric, so we need only show that $\tilde{q}$ is weak bilinear as in Definition 3.2.1.4. Suppose for distinct $a, b, c \in G^*$, that $q(a, b) = q(a, c)$. Since q is weak bilinear, then $q(a, bc) = 0$. Since $b = c$ by assumption, then let $B_1 = \langle a, b\rangle^*$, $B_2 = \langle a, c\rangle^*$ and $B_3 = \langle a, bc\rangle^*$. Then the induced map $\tilde{q}(B_1) = \tilde{q}(B_2)$, and we have constructed B_3 so that $\tilde{q}(B_3) = 0$, and thus we have shown that the induced map $\tilde{q}$ satisfies Definition 3.2.1.3, and thus we have shown the equivalence.

■

Definition 3.2.1.5 *We will say any weak bilinear block coloring function, $\tilde{q} : \mathcal{B} \longrightarrow Q$, is **linked** if for any $B_1, B_2 \in \mathcal{B}$ so that $\tilde{q}(B_1) = \tilde{q}(B_2)$, then there exists B_3 and B_4, with $\tilde{q}(B_3) = \tilde{q}(B_4)$, so that $B_1 \cap B_3 \neq \emptyset$, $B_2 \cap B_4 \neq \emptyset$, $B_3 \cap B_4 \neq \emptyset$, and $\tilde{q}(B_1) = \tilde{q}(B_2) = \tilde{q}(B_3) = \tilde{q}(B_4)$.*

The following theorem is of deep significance to our work. For any group of exponent 2, G, if the associated STS, $(G^*, \mathcal{B})$ possessed a weak bilinear block coloring, $\tilde{q} : \mathcal{B} \longrightarrow Q$, then we saw that $\tilde{q}$ induces a surjective weak bilinear map $q : G \times G \longrightarrow Q$. We will now demonstrate that characteristic 2 quaternionic structure (G, Q, q) is equivalent to STS of the form $(G^*, \mathcal{B})$ together with a linked block coloring function.

Theorem 3.2.1.6 *Let (G, Q, q) be any characteristic 2 quaternionic structure. Then the map q is a quaternionic mapping if and only if the block coloring function associated to $(G^*, \mathcal{B})$ is linked.*

Proof: Let (G, Q, q) be as stated in the hypotheses. Then Q_1 through Q_4 are true by assumption. In particular, for any $a, b, c, d \in G$, we have

$$q(a, b) = q(c, d) \implies \exists x \in G \text{ s.t. } q(a, b) = q(a, x) \text{ and } q(c, x) = q(c, d) \tag{3.4}$$

Let $(G^*, \mathcal{B})$ be the STS associated to G and let $\tilde{q}$ be the induced block coloring function. Put $B_1 = \langle a, b\rangle^*$ and $B_2 = \langle c, d\rangle^*$, Take $B_3 = \langle a, x\rangle^*$ and $B_4 = \langle c, x\rangle^*$. Now we know that

$$\tilde{q}(B_1) = \tilde{q}(B_2) = \tilde{q}(B_3) = \tilde{q}(B_4),$$

by (3.4). It is clear that $B_1 \cap B_3 = \{a\}$, $B_2 \cap B_4 = \{b\}$ and finally $B_3 \cap B_4 = \{x\}$, and thus we have demonstrated the a linked quaternionic mapping induces a linked block coloring.

Conversely, suppose $(G^*, \mathcal{B})$ is a STS associated to some group G of exponent 2, and let $\tilde{q}$ be any be any linked block coloring function. Without loss of generality, take Q to be the set of distinct block colors of $(G^*, \mathcal{B})$ together with the color 0. By Proposition 3.2.1.4 $\tilde{q}$ induces a map q which is surjective and weak bilinear. Since $\tilde{q}$ is linked, there exists $B_1, B_2 \in \mathcal{B}$, not necessarily distinct, so that $\tilde{q}(B_1) = \tilde{q}(B_2)$, and similarly $B_3, B_4 \in \mathcal{B}$, with $\tilde{q}(B_3) = \tilde{q}(B_4)$, so that $B_1 \cap B_3 \neq \emptyset$, $B_2 \cap B_4 \neq \emptyset$, and $B_3 \cap B_4 \neq \emptyset$. Represent $B_1 = \langle a, b\rangle^*$, and $B_2 = \langle c, d\rangle^*$. Without loss of generality, put $B_1 \cap B_3 = \{a\}$, $B_2 \cap B_4 = \{c\}$, and similarly $B_2 \cap B_3 = \{x\}$. Since blocks are uniquely determined by any two of their elements, $B_3 = \langle a, x\rangle^*$ and $B_4 = \langle c, x\rangle^*$. Therefore, we may conclude that the induced map $q : G \times G \longrightarrow Q$, as defined in Definition 3.2.1.1 satisfies

$$q(a, b) = q(c, d) \implies \exists x \in G \text{ s.t. } q(a, b) = q(a, x) \text{ and } q(c, x) = q(c, d),$$

and thus satisfies $Q_1 - Q_4$, as desired. ■

Now that the equivalence has been demonstrated, we should define this particular family of STS.

Definition 3.2.1.7 *Given any group of exponent 2, call it G, define the STS $(G^*, \mathcal{B})$ together with any fixed linked block coloring $\tilde{q}$ to be a **Quaternionic STS** or **QSTS**.*

We saw previously that from a complete graph on $2^n - 1$ vertices, we obtained a Steiner Triple System, and now we develop another graph known as a "block intersection graph," or BIG. These BIG are useful representatives of these QSTS, as the block coloring function yields a vertex coloring. Representing the blocks of the STS as vertices with colors dictated by these block coloring functions provides insight into when an arbitrary quaternionic structure may be decomposed into a product of quaternionic structures, or when the structure at hand is realizable as a group extension of some other structure.

Chapter 4

Block Intersection Graphs

Associated to any STS $(\mathcal{S}, \mathcal{B})$, there is a "block intersection graph." The vertex set is the set of blocks $\mathcal{B}$, and the edges describe when any two distinct blocks intersect, and thus the block intersection graph will be useful for our purposes, as it captures the totality of data of the STS. In particular, we are interested in the situation where this STS comes equipped with a linked block coloring function. The linkage axiom forces an interesting property for monochromatic subgraphs, namely, they are connected. These Quaternionic Block Intersection Graphs, or QBIG as we will call them, provide new tools for determining what quaternionic structures are realizable as products. We will begin with definitions, and then move on to classify the basic indecomposable quaternionic structures, moving forward to our final goal of classifying the family of QBIG of so-called *elementary type.* For more information about the Block Intersection Graphs of Steiner Triple Systems and their properties we refer the reader to [1].

4.0.1 QBIG Terminology

Definition 4.0.1.1 *Given any STS* $(G^*, \mathcal{B})$*, define* ***the Block Intersection Graph*** *or* ***BIG*** *as a graph* Γ*, with* $V(\Gamma) = \mathcal{B}$*, and for any* $B_1, B_2 \in \mathcal{B}$ *define the edge set by the* *following condition*

$$B_1B_2 \in E(\Gamma) \iff B_1 \neq B_2 \text{ and } B_1 \cap B_2 \neq \emptyset.$$

Definition 4.0.1.2 *Given any QSTS associated to (G, Q, q), we have the induced linked block coloring $\tilde{q} : \mathcal{B} \longrightarrow Q$. Let Γ be the associated BIG. Since $\mathcal{B} = V(\Gamma)$ in Definition 4.0.1.1, we will write $\tilde{q} : V(\Gamma) \longrightarrow Q$, as the induced vertex coloring on the associated BIG. Define Γ together with $\tilde{q}$ as the* ***Quaternionic BIG*** *or* ***QBIG*** *associated to (G, Q, q).*

Observation 4.0.1.3 *If $(G^*, \mathcal{B})$ and $(H^*, \mathcal{B}')$ are two STS, and $|G| \neq |H|$, then the resulting STS are not equivalent, and furthermore the resulting BIG are not equivalent.*

Definition 4.0.1.4 *Let Γ be any QBIG, and $M \subset \Gamma$ be a subgraph with $E(M) \subset E(\Gamma)$ maximal. Then we say that M is a* ***monochromatic subgraph*** *or if the color needs specification, a $p-$**monochromatic subgraph**, of Γ if $V(M) \subseteq \tilde{q}^{-1}(\{p\})$. That is, $M \subset \Gamma$ is any subgraph so that $\tilde{q}(v) = p$ for all $v \in V(M)$.*

Our goal here is to understand this induced combinatorial data in the context of its relationship with the quaternionic structures which they represent. The following result is an immediate result of the linkage axiom.

Proposition 4.0.1.5 *Let Γ be the QBIG of some characteristic 2 quaternionic structure (G, Q, q), and $p \in Q$. If $M \subset \Gamma$ is so that $V(M) = \tilde{q}^{-1}\{p\}$, then M is connected.*

Proof: Since Γ is a QBIG, and for any two vertices $v, w \in V(M)$, we have $\tilde{q}(v) = p = \tilde{q}(w)$ by assumption. If $vw \in E(\Gamma)$, then (v, w) is a path connecting v and w. On the other hand, if $vw \notin E(\Gamma)$, then by the fact that $\tilde{q}$ is a linked block coloring, there exists blocks $v', w' \in V(M)$, so that $vv' \in E(\Gamma)$, $v'w' \in E(\Gamma)$, and $w'w \in E(\Gamma)$, so (v, v', w', w)

is a path in Γ connecting v and w. Since v, w were arbitrary, then it follows that M is connected. ■

Lemma 4.0.1.6 *Let Γ be QBIG of (G, Q, q) an arbitrary quaternionic structure with $|G| > 4$, then diam$(\Gamma) = 2$.*

Proof: Consider Γ as stated, and let $x_1, x_2, x_3, x_4 \in G$ be pairwise distinct, let $\langle x_1, x_2\rangle^* = v_1$, and let $\langle x_3, x_4\rangle^* = v_2$, be arbitrary in $V(\Gamma)$. If $v_1 \cap v_2 \neq \emptyset$, then $\partial(v_1, v_2) = 0$ if $v_1 = v_2$, else $\partial(v_1, v_2) = 1$. Now suppose $v_1 \cap v_2 = \emptyset$, then we claim that there exists $w \in V(\Gamma)$ so that (v_1, w, v_2) is a path in Γ. Put $w = \langle x_1, x_3\rangle^*$. Then it is clear that $v_1 w$ and $v_2 w \in E(\Gamma)$. So (v_1, w, v_2) is a path of length 2, and thus we have shown that any vertices in Γ are connected by a path of length 2, and thus diam$(\Gamma) = 2$, since this is the minimal path length greater than 1, and thus we have proved the claim. ■

Let $(\mathcal{S}, \mathcal{B})$ be any STS, and suppose $B_1, B_2 \in \mathcal{B}$ so that $B_1 \cap B_2 \neq \emptyset$, and let $v \in B_1 \cap B_2$. The intersection is uniquely determined by the definition of STS, so $\{v\} = B_1 \cap B_2$. If Γ is the associated BIG to $(\mathcal{S}, \mathcal{B})$, then we may label $B_1 B_2 \in E(\Gamma)$ by v, and since the intersection is unique, then the labeling is unique.

Definition 4.0.1.7 *Given any BIG of some STS $(\mathcal{S}, \mathcal{B})$, call it Γ, and define the edge labeling described above as the* ***natural edge labeling*** *of $E(\Gamma)$. For any $v \in \mathcal{S}$, denote $\Gamma(v)$, as the subgraph of Γ defined on the vertex set $V(\Gamma(v)) = \{A, B \in \mathcal{B} \mid A \cap B = \{v\}\}$. We will call $\Gamma(v)$ the $v-$**incident subgraph** of Γ.*

Lemma 4.0.1.8 *Let Γ be the BIG associated to some STS $(\mathcal{S}, \mathcal{B})$. Then for any $v \in \mathcal{S}$, the subgraph $\Gamma(v)$ is a complete subgraph of Γ.*

Proof: Fix $v \in \mathcal{S}$. For any distinct $B_1, B_2 \in \Gamma(v)$, by assumption we have $B_1 \cap B_2 = \{v\}$, so $B_1 B_2 \in E(\Gamma)$. Thus we have shown that $\Gamma(v)$ is complete, since B_1 and B_2 were arbitrary vertices, and $B_1 B_2 \in E(\Gamma(v))$. ■

Lemma 4.0.1.9 *Suppose* $(G^*, \mathcal{B})$ *is some STS associated to a group of exponent 2. Let* Γ *be the associated BIG. For distinct* $g, h, g', h' \in G^*$, *if* $A = \langle g, h\rangle^*$, *and* $B = \langle g', h'\rangle^*$ *are distinct vertices in* $V(\Gamma)$, *and* $AB \notin E(\Gamma)$, *then* $\{g, h, g', h'\}$ *is linearly independent.*

Proof: Suppose $A \neq B$, and $AB \notin E(\Gamma)$. Let $g, h \in G^*$, and $g', h' \in G^*$, be distinct and so that $A = \langle g, h\rangle^*$, and $B = \langle g', h'\rangle^*$. Since $A \neq B$, and since $AB \notin E(\Gamma)$, then $A \cap B = \emptyset$. So without loss of generality, $g' \notin A$, so the set $\{g, h, g'\}$, is linearly independent. If $h' \in \langle g, h, g'\rangle^*$, then $h' = gg'$ or $h' = hg'$, since $h' \notin \langle g, h\rangle^*$, but since G is of exponent 2, then $g'h' = h$, or $g'h' = g$, contradicting our assumption that $A \cap B = \emptyset$, so conclude that $h' \notin \langle g, h, g'\rangle^*$, so the set $\{g, h, g', h'\}$ is linearly independent, as desired.

■

Definition 4.0.1.10 *Let* $(G^*, \mathcal{B})$ *be some STS corresponding to a finite group of exponent 2. Let* $\{v_1, \ldots, v_n\} \subset \mathcal{B}$, *so that* $v_i \cap \bigcup_{i \neq j} v_j = \emptyset$. *Then we call* $\{v_1, \ldots, v_n\}$ *a set of* ***independent blocks****.*

Definition 4.0.1.11 *Let* $(G^*, \mathcal{B})$ *be some STS corresponding to a finite group of exponent 2. Suppose* $\{v_1, \ldots, v_n\}$ *is any set of blocks in* $\mathcal{B}$, *with* $v_i = \langle g_i, h_i\rangle^*$. *Then we call* $\{v_1, \ldots, v_n\}$ *a set of* ***spanning blocks****, if* $\langle g_1, h_1, \ldots, g_n, h_n\rangle = G$.

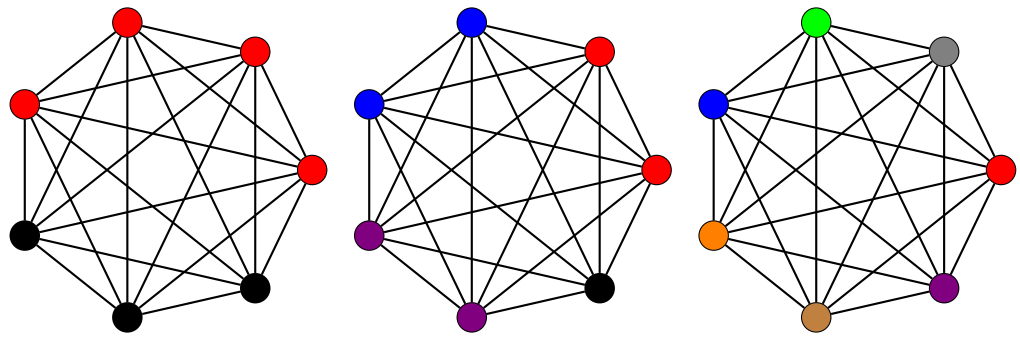
Lemma 4.0.1.12 *Let* $(G^*, \mathcal{B})$ *be some STS, and suppose* $\{v_1, \ldots, v_n\}$ *is a set of blocks which is both independent and spanning, then* $\dim(G)$ *is even.*

Proof: It is clear from definitions 4.0.1.10 and 4.0.1.11, that $\bigcup_{i=1}^{n}\{g_i, h_i\}$ is a basis for G, and $\dim(G) = 2n$ is even, as desired. ■

We require this the preceding lemma to classify all QBIG corresponding to symplectic quaternionic structures. This notion of linearly independent spanning blocks gives a generating set of blocks, which is valuable in determining the dimension of an underlying group.

4.0.2 Examples of QBIG

Here are the QBIG associated to the graphs given in section 3.1.3 (3.2).



$((\langle x_1, y_1\rangle, \mathbb{F}_2, q) \times \langle z\rangle,$ $\qquad \langle x, y\rangle[z],$ $\qquad (G, G \wedge G, \wedge).$ (4.1)

Let G be any purely radical quaternionic structure, and let $|G| = 2^n$. We may readily compute the number of vertices of the associated QBIG, call it Γ, by computing the number of subspaces of dimension 2. There are $2^n - 1$ non-zero vectors, and any two of them will span a 2 dimensional subspace. So by observation 2.0.0.10 we have the total number of subspaces of dimension 2 in G is $\frac{2^n-1(2^n-2)}{6}$. Since each vertex in a BIG represents a subspace of dimension 2, it is clear that the number of vertices in any given BIG will typically be rather large. Due to this, our examples terminate here.

Lemma 4.0.2.1 *If* $\alpha : (G, Q, q) \longrightarrow (G', Q', q')$, *is any injective morphism of quaternionic structures,* Γ *is the QBIG of* (G, Q, q), *and if* Γ' *is the QBIG of* (G', Q', q'), *then we may identify* $\Gamma \subset \Gamma'$.

Proof: Let $g, h \in G^*$, be distinct. Then $\langle g, h\rangle^* \in V(\Gamma)$, and $\langle \alpha(g), \alpha(h)\rangle^* \in V(\Gamma')$, since α is injective. If $B_1, B_2 \in V(\Gamma)$, are distinct, if $B_1B_2 \in E(\Gamma)$, then $B_1 \cap B_2 = \{k\}$ for some $k \in G^*$, then and $\alpha(B_1) \cap \alpha(B_2) = \{\alpha(k)\}$, again since α is injective, so $\alpha(B_1)\alpha(B_2) \in E(\Gamma')$. Thus we may conclude that $\Gamma \subset \Gamma'$, is a subgraph, as desired. ■

4.1 The Basic Indecomposable QBIG for $|Q| \leq 2$.

In this section describe the QBIG of the so-called "basic indecomposable" type. For any basic structure no elements are rigid, and yet the quaternionic structure is not realizable as a product, as in Definition 2.2.1.5. Recall that any (G, Q, q) is called purely radical if $Q = \{0\}$. In such a case, we motivate the following lemma as a result of taking any QBIG with precisely one vertex color.

Theorem 4.1.0.1 *Let Γ be any QBIG, with vertex coloring so that Γ is monochromatic, then Γ is the QBIG of some purely radical quaternionic structure.*

Proof: Since $\tilde{q} : V(\Gamma) \longrightarrow Q = \{0\}$, so $\tilde{q}$ is constant, and thus Γ is the QBIG of some purely radical structure. ■

Here we determine the necessary and sufficient conditions for a QBIG to correspond to the symplectic type quaternionic structures according to Definition 2.1.0.2.

Theorem 4.1.0.2 *The quaternionic structure $(G, \mathbb{F}_2, q)$ is a symplectic quaternionic structure of dimension $2n$, if and only if the corresponding QBIG, Γ, has precisely two vertex colors, possesses a set of independent spanning blocks $\{v_1, v_2, \ldots, v_n\} \subset V(\Gamma)$, where*

$$\tilde{q}(v_1) = \tilde{q}(v_2) = \cdots = \tilde{q}(v_n) = 1, \quad v_i v_j \notin E(\Gamma),$$

and

$$V(N_{v_i} \cap N_{v_j}) \subset \tilde{q}^{-1}\{0\}, \quad \forall i \neq j. \tag{4.2}$$

Proof: First, suppose $(G, \mathbb{F}_2, q)$ is finite and of symplectic type of dimension $2n$, and let Γ be the associated QBIG. It is clear then that Γ has precisely two vertex colors. Give G the symplectic basis, and for every i, define $v_i = \langle x_i, y_i \rangle^*$. Thus $\tilde{q}(v_i) = q(x_i, y_i) = 1$, for every i, and we have demonstrated the first assertion. Now, it is clear that $v_i v_j \notin E(\Gamma)$, since $\langle x_i, y_i\rangle^* \cap \langle x_j, y_j\rangle^* = \emptyset$ as basis vectors are linearly independent. So it is clear that $\{v_1, \ldots, v_n\}$ is a set of independent spanning blocks. Let $w \in V(N_{v_i} \cap N_{v_j})$ be arbitrary, and let $w = \langle x, y\rangle^*$. By assumption $w \in V(N_{v_i})$ and $w \in V(N_{v_j})$, so $wv_i \in E(\Gamma)$ implies that $\langle x, y\rangle^* \cap \langle x_i, y_i\rangle^* \neq \emptyset$, so without loss of generality, let $x \in \langle x_i, y_i\rangle^*$. Similarly, $\langle x, y\rangle^* \cap \langle x_j, y_j\rangle^* \neq \emptyset$, so let $y \in \langle x_j, y_j\rangle^*$. Then it is clear by the definition of $(G, \mathbb{F}_2, q)$ that $q(x, y) = 0$, so $\tilde{q}(w) = 0$, and we have shown that $V(N_{v_i} \cap N_{v_j}) \subset \tilde{q}^{-1}\{0\}$ since w was arbitrary, and thus we have established the desired properties of Γ.

Conversely, suppose Γ is as described above. We wish to show that the induced quaternionic structure is of symplectic type. Since Γ has precisely two vertex colors, and by Theorem 3.2.1.6, we have an induced quaternionic map $q : G \times G \longrightarrow \mathbb{F}_2$. Without loss of generality, let $v_i = \langle x_i, y_i\rangle^*$, for each i in the independent spanning blocks $\{v_1, \ldots, v_n\}$. By Lemma 4.0.1.12, we may take $G = \langle x_1, y_1, \ldots, x_n, y_n\rangle$. We wish to show that the induced map map is so that $q(x_i, y_j) = \delta_{ij}$, and all other basis vector pairings are trivial. To do so, consider by (4.2) that if $w \in V(N_{v_i} \cap N_{v_j})$, then $v_i w$, and $v_j w$ are both edges in $E(\Gamma)$. By definition of BIG, $v_i w, v_j w \in E(\Gamma)$, means for some $x \in \langle x_i, y_i\rangle^*$ we have $\langle x, y\rangle^* \cap \langle x_i, y_i\rangle^* = \{x\}$, and similarly for some y we have $\langle x, y\rangle^* \cap \langle x_j, y_j\rangle^* = \{y\}$. So let $w = \langle x, y\rangle^*$. By the assumption in (4.2) $\tilde{q}(w) = q(x, y) = 0$, and since x, y were arbitrary, and our assumption about the intersections of our blocks being represented by x_i, x_j and y_i, y_j were arbitrary, it is now clear that $q(x_i, y_j) = \delta_{ij}$, and all other basis pairings are trivial. Since G is of even dimension, and since q is alternating (remember this is trivial since $1 = -1$) and non-degenerate on the basis $\{x_1, y_1, \ldots, x_n, y_n\}$ then we may conclude that q is non-degenerate, and therefore the symplectic pairing. Thus we have shown that $(G, \mathbb{F}_2, q)$ is a symplectic quaternionic structure according to Definition 2.1.0.2, and thus we have proved that Γ is a QBIG of a symplectic quaternionic structure.

■

Definition 4.1.0.3 *Every QBIG as described in Theorem 4.1.0.2, will be called a **QBIG of symplectic type**.*

We have now classified the finitely generated basic indecomposable QBIG where $|Q| \leq 2$, namely, the purely radical 1 dimensional structure and the symplectic structures of dimension $2n$. The elementary type conjecture states "*every finitely generated quaternionic structure can be constructed, up to isomorphism, by means of the operations of product and group extensions of the basic indecomposable quaternionic structures.*" In the next section we will classify products and group extensions of the basic indecomposable quaternionic structures where $|Q| \leq 2$. If the elementary type conjecture is false, then there must exist some basic indecomposable quaternionic structure with $|Q| > 2$.

Chapter 5

The QBIG of Elementary Type

We begin with products of quaternionic structures, and categorize the relationship between products and their corresponding QBIG. Marshall's approach to classifying finitely generated torsion-free Witt rings involved spaces of orderings to identify projections. We will dualize his technique in a manor speaking, looking for inclusions rather than projections to deal with special cases of finitely generated characteristic 2 Witt rings. In our situation, the Witt ring is of characteristic 2, and the associated quaternionic structure induces a QBIG which possesses subgraphs isomorphic to the QBIG of the product factors. We begin this section lemma about the nature of products and what we will come to know as relative radicals, realized in the QBIG as subgraphs we will define as "shadows." These shadows are essential in that they are intimately connected to the notion of elements being radical relative to some fixed subgroup.

We begin with a preliminary theorem which we will use to reduce our arguments, that is, that $\mathrm{Rad}(q)$ is a purely radical quaternionic substructure. We will use this theorem to avoid any ambiguity surrounding $\mathrm{Rad}(q)$ being non-trivial.

Theorem 5.0.0.1 *Let (G, Q, q) be any characteristic 2 quaternionic structure. Then $(G, Q, q) \cong (G', Q, q') \times \mathrm{Rad}(q)$.*

Proof: Since $\operatorname{Rad}(q) = \{a, b \in G \mid q(a, b) = 0, \}$, then $\operatorname{Rad}(q)$ is a purely radical quaternionic structure. Let $G' = G/\operatorname{Rad}(q)$, and take $\pi_1 : G \longrightarrow G/\operatorname{Rad}(q)$ to be the canonical homomorphism. Since $1 = -1$, then π_1 satisfies the first axiom of a morphism of quaternionic structures trivially. Consider non-trivial cosets $x\operatorname{Rad}(q), y\operatorname{Rad}(q) \in G/\operatorname{Rad}(q)$, then for any $\tau \in \operatorname{Rad}(q)$, we may take $\pi_1(x) = x\tau$, and $\pi_1(y) = y\tau$, as representatives of these cosets and $q(x\tau, y\tau) = q(x, y)$, by Observation 2.0.0.10, we have $q(x, y) = 0$, implies $q'(\pi_1(x)), q'(\pi_1(y)) = 0$, and thus $\pi_1 : G \longrightarrow G/\operatorname{Rad}(q)$, is a quaternionic morphism. Similarly, it is clear that we may take $G \cong G' \times \operatorname{Rad}(q)$, so let $\pi_2 : G \longrightarrow G/G'$, be the canonical homomorphism, and since $G/G' \cong \operatorname{Rad}(q)$, and furthermore since all quaternion pairings are zero in the image, then it is clear that π_2 satisfies the criterion for $(G', Q, q') \times \operatorname{Rad}(q)$, being a product, and thus we have proved the claim. ■

By the previous theorem, we may assume that $\operatorname{Rad}(q) = \{1\}$ for the remainder of these classification theorems.

5.1 Products as QBIG

Lemma 5.1.0.1 *Let (G_1, Q_1, q_1) and (G_2, Q_2, q_2) be non-trivial finite characteristic 2 quaternionic structures, and let $(G, Q, q) = (G_1, Q_1, q_1) \times (G_2, Q_2, q_2)$. If for some $g, g' \in G^*$, if $q_1(g, g') = k \in Q_1 \backslash \{0\}$, then the replication number of $(k, 0) \in Q$, $|q^{-1}(k, 0)| \geq |G_2|$.*

Proof: Let $g, g' \in G_1$ be as described above. Then $q((g, 1), (g', 1)) = (q_1(g, g'), 0) = (k, 0) \in Q$. By Lemma 2.2.1.3, for all $h \in G_2$, we have $(1, h) \in \operatorname{Rad}((g, 1))$, and $(1, h) \in \operatorname{Rad}((g', 1))$, so $(1, h) \in \operatorname{Rad}((g, 1)) \cap \operatorname{Rad}((g', 1))$. Since $(1, h)$ is arbitrary, then $\{1\} \times G_2 \subset \operatorname{Rad}((g, 1)) \cap \operatorname{Rad}((g', 1))$, so by Lemma 2.0.0.11 we have that $|q^{-1}(k, 0)| \geq |G_2|$, and thus we have proved the claim. ■

Lemma 5.1.0.2 *If (G, Q, q) is a product of any two, non-trivial characteristic 2 quaternionic structures, then (G, Q, q) is basic. That is, no element of (G, Q, q) is rigid.*

Proof: Suppose (G, Q, q) is a product of two non-trivial characteristic 2 quaternionic structures, $(G, Q, q) \cong (G_1, Q_1, q_1) \times (G_2, Q_2, q_2)$. Let $(x, z) \in G$. To show that this element is basic, we need only construct $(y, w) \in G^*$, where $(y, w) \neq (x, z)$. So consider $q((x, z), (y, w)) = (q_1(x, y), q_2(z, w))$. So we choose $y = x$, and $w = 1$. Then $q((x, z), (x, 1)) = (q_1(x, x), q_2(z, 1))$, is true by Q_2 and by Observation 2.0.0.4. Since $(x, 1) \notin \{(1, 1), (x, z)\}$ we have shown that (x, z) is not rigid, and since $(x, z) \in G$ was arbitrary, we have shown that no element of (G, Q, q) is rigid, and therefore (G, Q, q) is basic. ■

The following lemma supplies the graph theoretic equivalent of the relative radical property established in Lemma 2.2.1.3. We will observe that products induce $0-$chromatic subgraphs of a very specific nature.

Lemma 5.1.0.3 *Let Γ be the QBIG of $(G, Q, q) = (G_1, Q_1, q_1) \times (G_2, Q_2, q_2)$, a product of non-trivial characteristic 2 quaternionic structures. Then Γ possesses a $0-$chromatic complete subgraph for each $(1, h) \in \{1\} \times G_2^*$, denote this graph as $\mathcal{K}_h \subset \Gamma$ whose vertices are blocks of the form $\mathcal{B}_1 = \{\langle (g, 1), (1, h) \rangle^* \mid g \in G_1^*\}$.*

Proof: Fix $(1, h) \in G_1 \times G_2^*$, and let $(g, 1) \in G_1^* \times G_2$ be arbitrary. Then by Lemma 2.2.1.3, every block of the form $\langle (g, 1), (1, h) \rangle^* \in V(\Gamma)$ will be colored 0 by $\tilde{q}$. So $\Gamma((1, h))$ is a complete subgraph of Γ by Lemma 4.0.1.8, and $V(\Gamma((1, h))) = \{A, B \in V(\Gamma) \mid A \cap B = \{(1, h)\}\}$. Let $\mathcal{B}_1 = \{\langle (g, 1), (1, h) \rangle^* \mid g \in G_1^*\}$. It is clear that $\mathcal{B}_1 \subset V(\Gamma(1, h))$, and since $\Gamma((1, h))$, is complete, then we may take $\mathcal{K}_h$ to be the complete subgraph of $\Gamma((1, h))$ with vertex set $\mathcal{B}_1$, and thus we have proved the claim. ■

Theorem 5.1.0.4 *Let (G, Q, q) be any characteristic 2 quaternionic structure, and let Γ be the corresponding QBIG. Then $x \in \mathrm{Rad}(q)$ if and only if $V(\Gamma(x)) \subset \tilde{q}^{-1}(\{0\})$, that is $\Gamma(v)$ is $0-$chromatic.*

Proof: If $x \in \mathrm{Rad}(q)$, then for all $g \in G \setminus \{1, x\}$, we have $\tilde{q}(\langle x, g\rangle^*) = 0$, by the definition of the induced vertex coloring on Γ. Since $V(\Gamma(x)) = \{\langle x, g\rangle^* \mid g \in G \setminus \{1, x\}\}$, then we have $V(\Gamma(x)) \subset \tilde{q}^{-1}\{0\}$, as desired.

Conversely, suppose that $V(\Gamma(x)) \subset \tilde{q}^{-1}\{0\}$, for some $x \in G^*$. Since $\tilde{q}$ is the linked block coloring induced by the QBIG Γ, then the induced quaternionic map q has the property that $q(x, g) = 0$ for all $g \in G$, and thus $x \in \mathrm{Rad}(q)$, and we have established the equivalence. ■

Definition 5.1.0.5 *We call the complete subgraph $\mathcal{K}_h$ in Lemma 5.1.0.3, a **factor shadow of** $(1, h)$.*

We saw in Lemma 5.1.1.2 that for a product of characteristic 2 quaternionic structures, we are able to identify sub QBIGs which correspond to the factors of our product. Upon identifying these, Lemma 2.2.1.3 forces the existence of factor shadows as in Definition 5.1.0.5, and once identified the remaining structure is determined by observing that the structure is basic.

5.1.1 Products of Symplectic Type and Purely Radical Type

Given any product of the form $\prod_{i=1}^{N} G_i$, of purely radical structures, it is clear that the set of quaternions remains the zero quaternion, $\prod_{i=1}^{N}\{0\} = \{(0, 0, \ldots, 0)\}$, and is therefore classified by Theorem 4.1.0.1. A more substantive family of QBIG are the products of symplectic type.

Definition 5.1.1.1 *Let* $(G,Q,q) = \prod_{i=1}^{N}(G_i,Q_i,q_i)$, *be any product of quaternionic structures. Define*

$$(H_j,Q_j,q_j) = \prod_{i=1,\ i\neq j}^{N}(G_i,Q_i,q_i).$$

Before we proceed, we need a lemma about subgraphs of products.

Lemma 5.1.1.2 *Let* $(G,Q,q) = \prod_{i=1}^{N}(G_i,Q_i,q_i)$ *be the product of some finite characteristic 2 quaternionic structures. Let* Γ *be the QBIG associated to* (G,Q,q)*. Then* Γ *possesses a subgraph* $\Gamma_i \subset \Gamma$ *for each* i *which is the QBIG of* (G_i,Q_i,q_i)*.*

Proof: Let $\eta_i : G_i \longrightarrow G$, be canonical inclusion maps given by Lemma 2.2.1.2. For each i, denote the QBIG associated to (G_i,Q_i,q_i) as Γ_i. If $g,h \in G_i^*$, are distinct, then $\langle \eta_i(g), \eta_i(h)\rangle^*$ is a block in G, and therefore a vertex in $V(\Gamma_i)$, and similarly if $B_1, B_2 \subset G_i$ are any distinct blocks so that $B_1 \cap B_2 = \{g'\}$, for some $g' \in G^*$, then $\eta_i(B_1) \cap \eta_i(B_2) = \{\eta_i(g')\}$, so if $B_1B_2 \in E(\Gamma_i)$, then $\eta_i(B_1)\eta_i(B_2) \in E(\Gamma)$. Denote the QBIG of the image of η_i as Γ_i, then $\Gamma_i \subset \Gamma$ is a subgraph, and thus we have proved the claim. ∎

Theorem 5.1.1.3 *Let* $\{(G_i,\mathbb{F}_2,q_i)\}_{i=1}^{N}$ *be a family of finite symplectic quaternionic structures each of dimension* $2n_i$*. Let*

$$(G,Q,q) = \prod_{i=1}^{N}(G_i,\mathbb{F}_2,q_i).$$

Then Γ *is the QBIG of (G,Q,q) if and only if* Γ *has subgraphs* $\Gamma_i \subset \Gamma$*, each of which being symplectic type of dimension* $2n_i$*, and* Γ *has precisely* 2^N *vertex colors, and for all* $g \in G^*$*, and for every element* $g \in G_j$*, there exists a factor shadow of* $(g_j, 1)$ *with vertex set* $\mathcal{B}_j = \{\langle (g_j,1),(1,h_j)\rangle^* \mid h_j \in H_j^*\}$*.*

Proof: Since (G, Q, q), is a product of symplectic structures, and since there exist injections η_i for every i by Lemma 2.2.1.2, then there exist $\Gamma_i \subset \Gamma$ for each i, by Lemma 4.0.2.1. Furthermore, since Q is the set of colors for Γ, and $Q = {\mathbb{F}_2}^N$, then it is clear that Γ has 2^N colors. For every element $g_j \in G_j$, it is clear that there exists a factor shadow of $(g_j, 1)$ with vertex set $\mathcal{B}_j = \{\langle (g_j, 1), (1, h_j)\rangle^* \mid h_j \in H_j^*\}$, by Lemma 5.1.0.3.

On the other hand, suppose Γ is as stated in the hypotheses, and let (G, Q, q) be the induced quaternionic structure. To each Γ_i, by Theorem 4.1.0.2, we may associate a symplectic quaternionic structure of dimension $2n_i$, call it $(G_i, \mathbb{F}_2, q_i)$. It must be the case that $G \cong \prod_{i=1}^{N} G_i$, for Γ to even potentially be the QBIG of the product. Let $\pi_i : G \longrightarrow G_i$ be the canonical projections. Since $H_i = \ker(\pi_i)$ for every i, then it is clear that $\pi_i : (G, Q, q) \longrightarrow (G_i, \mathbb{F}_2, q_i)$, is a quaternionic morphism, since for every element $g_i \in G_i$, there exists a factor shadow of $(g_i, 1)$ with vertex set $\mathcal{B}_i = \{\langle (g_i, 1), (1, h_i)\rangle^* \mid h_i \in H_i^*\}$, identify $G_i \hookrightarrow G$, and $H_i \hookrightarrow G$, then $H_i \subset \mathrm{Rad}(g_i)$ for all i, so π_i restricted to $(G_i, \mathbb{F}_2, q_i) \hookrightarrow (G, Q, q)$ is an isomorphism, since π_i restricted to $G_i \hookrightarrow G$ is, and thus π_i is certainly a quaternionic morphism. So we may conclude that $(G, Q, q) \cong \prod_{i=1}^{N} (G_i, \mathbb{F}_2, q_i)$, and we have proved the desired equivalence. ■

In 1980, Marshall classified all finitely generated torsion-free abstract Witt rings, and demonstrated that any such Witt ring is realizable as the Witt ring of some pythagorean field with finitely generated square class group. Marshall developed the theory of spaces of orderings associated to these Witt rings, and found product decompositions for the torsion-free, finitely generated Witt rings. To continue the work toward a resolution of Elementary Type Conjecture, we will now use our combinatorial techniques to describe when a characteristic 2 quaternionic structure (G, Q, q) admits a product decomposition by classifying all QBIG which are products of Elementary Type.

5.1.2 Products in General

Theorem 5.1.2.1 *Suppose* $(G, Q, q) = \prod_{i=1}^{N}(G_i, Q_i, q_i)$. *Then* Γ *is the QBIG of* (G, Q, q) *if and only if there exist* $\Gamma_i \subset \Gamma$, *each the QBIG of* (G_i, Q_i, q_i), *so that for all* $g_i \in G_i^*$, *it is the case that* $\Gamma(\eta_i(g_i))$ *possesses a* H_i *factor shadow, and furthermore,* $|Q| = \prod_{i=1}^{N} |Q_i|$.

Proof: By Lemma 2.2.1.2 $\eta_i : (G_i, Q_i, q_i) \hookrightarrow (G, Q, q)$, and so by Lemma 4.0.2.1 there is an inclusion of QBIGs $\Gamma_i \subset \Gamma$. Furthermore, Lemma 5.1.0.3 demonstrates that for each $g_i \in G_i^*$, there exists a factor shadow subgraph of $\Gamma(\eta_i(g_i))$. Since Γ is the QBIG of (G, Q, q) and since $Q = \prod_{i=1}^{N} Q_i$, then it is clear that $|Q| = \prod_{i=1}^{N} |Q_i|$, by the definition of QBIG.

Conversely, suppose Γ has $\Gamma_i \subset \Gamma$ each the QBIG of (G_i, Q_i, q_i). Let (G, Q, q) be the quaternionic structure induced by the QSTS which defines Γ, induced according to Theorem 3.2.1.1. Then it is clear that $G \cong \prod_{i=1}^{N} G_i$, and q agrees with $q_1 \times q_2 \times \cdots \times q_N$ on all zero quaternion pairings. Since $|Q| = \prod_{i=1}^{N} |Q_i|$, then the maps q and $q_1 \times q_2 \times \cdots \times q_N$ are equivalent up to some bijection, since they have the same domain, and their codomains are in bijection. So we may conclude that $(G, Q, q) \cong \prod_{i=1}^{N}(G_i, Q_i, q_i)$, and thus we have the desired equivalence. ∎

5.2 QBIG of Group Extensions

The notion of a group extension is dual to that of a product in a sense, as we observed that products of quaternionic structures induce factor shadows, which were defined as complete $0-$chromatic subgraphs. Similarly, we will observe that rigid elements of a given quaternionic structure will induce a dual structure; namely incident subgraphs which whose vertices all possess unique colors. We motivate this observation with the following theorem, and define these subgraphs which correspond to rigid elements.

Theorem 5.2.0.1 *Let (G, Q, q) be any characteristic 2 quaternionic structure, and Γ the associated QBIG. Then $a \in G$ is rigid, if and only if $\Gamma(a)$ is so that no distinct vertices share the same color.*

Proof: If $a \in G$ is rigid, then for any $b, c \in G$, $q(a, b) = q(a, c) \iff bc \in \{1, a\}$, by definition of rigid. If $bc = 1$, then $b = c$, otherwise if $bc = a$, then $c = ba$, and Observation 2.0.0.10 gives $q(a, c) = q(a, ba) = q(a, b)$, and so $q(a, b)$ is unique for any $b \in G \setminus \{1, a\}$, so every block $\langle a, b \rangle^* \in V(\Gamma)$ possesses a unique color.

On the other hand, $\Gamma(a)$ has the property that for all distinct $A, B \in V(\Gamma(a))$, $A \cap B = \{a\}$, and $\tilde{q}(A) \neq \tilde{q}(B)$, then for some $b, c \in G^*$ we have $A = \langle a, b \rangle^*$, and $B = \langle a, c \rangle^*$, so the quaternionic map induced by Theorem 3.2.1.1 has the property that $q'(a, b) \neq q'(a, c)$, and by weak bilinearity $q'(a, bc) \neq 0$, for all $b, c \in G^*$, with the exception of $bc = 1$ or $bc = a$, so conclude that $a \in G$ is rigid. ■

If a is rigid, then $\Gamma(a)$ appears to be a "rainbow" of colors, or thinking of light through a prism, it appears to be a refraction, and this observation motivates the following definition.

Definition 5.2.0.2 *Let (G, Q, q) be any characteristic 2 quaternionic structure, and suppose that Γ is the QBIG. If $a \in G$ is rigid, define $\Gamma(a)$ as the $a-$**refraction subgraph** or more generally a **refraction subgraph.***

Corollary 5.2.0.3 *Suppose (G, Q, q) is any characteristic 2 quaternionic structure, and let $R \subset G$ be the set of rigid elements, then for all $a \in R$, $\Gamma(a)$ is a refraction subgraph.*

By Lemma 2.2.2.8, every purely rigid structure is realizable as a group extension of the trivial quaternionic structure. So our classification of purely rigid QBIG is a consequence of classifying the QBIG of general group extensions. However, the notion of a purely rigid substructure will be useful so we begin by classifying purely rigid QBIG as a special case.

5.2.1 Purely Rigid QBIG

Dual to the purely radical structures are the purely rigid structures. Rather than having the property of being $0-$chromatic, these structures are adorned with a maximal vertex coloring where no two distinct possess the same color. Similarly to that of the purely radical structure, linkage is trivially satisfied,

Theorem 5.2.1.1 *Any quaternionic structure* (G, Q, q) *is purely rigid if and only if the induced QBIG possesses a totally non-zero vertex coloring, with every vertex possessing a unique color.*

Proof: Suppose (G, Q, q) is purely rigid, then $Q = G \wedge G$, and $q = \wedge$, and by Proposition 2.2.2.3, for any distinct $v, w \in G^*$, $v \wedge w$, is unique. So let Γ be the induced QBIG. For $A, B \in V(\Gamma)$ if $A = \langle v, w \rangle^*$, and $v', w' \in W^*$, with $B = \langle v', w' \rangle^*$ are so that $A \neq B$, then $\tilde{q}(A) \neq \tilde{q}(B)$, since $v \wedge w \neq v' \wedge w'$, so every vertex has a unique color, and is non-zero, since $v \wedge w = 0$ if $v = 1$ or $v = w$, but blocks are dimension 2 subspaces, and therefore this is absurd, since these would not constitute blocks.

Conversely, suppose Γ is some QBIG with any distinct $A, B \in V(\Gamma)$ $\tilde{q}(A) \neq \tilde{q}(B)$, and no vertex is colored zero. Then by Theorem 3.2.1.1 there is an induced (G, Q, q), so that for any distinct $v, w \in G^*$ $q(v, w) \neq 0$, and is unique. So by weak bilinearity of q, for any distinct $w, w' \in G$, with $w' \neq vw$, so that $\langle v, w \rangle^* \neq \langle v, w' \rangle^*$. Then we have $q(v, w) \neq q(v, w') \iff q(v, ww') \neq 0$, so $\text{Rad}(v) = \{1, v\}$, so $v \in G^*$ is rigid, and since v was arbitrary we have proved the claim. ■

We conclude this section with the following theorem for characteristic 2 quaternionic structures At first glance, the theorem may appear to be insignificant; however, it will allow us to decompose arbitrary group extensions in the case of characteristic 2 quaternionic structures.

Theorem 5.2.1.2 *Let $B \subset G$ be the set of basic elements in some characteristic 2 quaternionic structure (G, Q, q). Then B is a subgroup of G.*

Proof: Let $a, b \in B$, be distinct. Since G is of exponent 2, then we need only show $ab \in B$. Since a is basic, then there exist some $x \in G \setminus \{1, a\}$ so that $q(a, x) = 0$, and similarly, there exists $y \in G \setminus \{1, b\}$, so that $q(b, y) = 0$. By the linkage axiom, there exists $g \in G$, so that

$$q(a, x) = q(a, g) = q(b, g) = q(b, y) = 0,$$

and by weak bilinearity $q(ab, g) = 0$. If $g = ab$, then since $q(a, x) = 0$ by assumption, then $q(a, b) = 0$. Then by Observation 2.0.0.10, we have $q(a, ab) = 0$, and since $a \notin \{1, ab\}$, then ab is basic. On the other hand, if $ab = 1$, then $a = b$, and this contradicts our assumption, so this is absurd. We may conclude $B \subset G$ is a subgroup, and thus we have proved the claim. ■

A natural question to ask: "*Does the set of all rigid elements constitute a subgroup*?" To see that the set of rigid elements does not form a subgroup in general, let G be any purely radical structure of so that $|G| > 2$, and let $\langle w \rangle = W$, and consider $(G[W], [W], q[W])$. If $g \in G^*$, and $w \in W^*$, then $(g, w) \in G[W]$ is rigid by Lemma 2.2.2.7, and $(1, w) \in G[W]$, is rigid by Proposition 2.2.2.3; however $(g, w)(1, w) = (g, 1)$ is basic, since $\mathrm{Rad}((g, 1)) = \{(h, 1) \mid h \in G\}$, and since $|G| > 2$, then $\mathrm{Rad}((g, 1)) \neq \{(1, 1), (g, 1)\}$. Now, we provide the following theorem which will allow us to construct a maximal subgroup of rigid elements.

Theorem 5.2.1.3 *Let (G, Q, q) be any finite characteristic 2 quaternionic structure and let B be the subgroup of basic elements of G. Then there exists a maximal subgroup P consisting of only rigid elements together with 1, so that $G = B \oplus P$.*

Proof: Take G to be a vector space over $\mathbb{F}_2$. Since B is a subgroup of G by Theorem 5.2.1.2 then it is a subspace, so let $\alpha : G \longrightarrow \tilde{P}$ be any fixed linear transformation so that $\ker(\alpha) = B$. Let $\{b_1, \ldots, b_n\}$ be a basis for B. Extend $\{b_1, \ldots, b_n\}$ to a basis for all of G, let us call it $\{b_1, \ldots, b_n, p_1, \ldots, p_m\}$. Let $P = \langle p_1, \ldots, p_m \rangle$, then it is clear that $G = B \oplus P$, and since $P \cap B = \{1\}$, it is clear that no element of P is basic, so P consists of only rigid elements, and thus we have proved the claim. ■

Definition 5.2.1.4 *Let (G, Q, q) be any finite characteristic 2 quaternionic structure, and let $P \subset G$ be the subgroup as defined in Theorem 5.2.1.3. We will refer to any $p \in P^*$ as **purely rigid**.*

Lemma 5.2.1.5 *Let (G, Q, q) be any finite characteristic 2 quaternionic structure. Let $a \in G$ be rigid, and let $B \in G$ denote the subgroup of basic elements of G. Then for all $b \in B$, ba is rigid.*

Proof: Let $B \subset G$ be the collection of basic elements of G. This follows immediately from Theorem 5.2.1.3, by the direct sum decomposition of $G = B \oplus P$, where P consists of purely rigid elements. Suppose $a \in G$ is rigid. Then there exists $b' \in B$ and $p \in P$ so that $(b', p) = a$ since G is a direct sum. Let $b \in B$ be arbitrary, then $(b, 1)(b', p) = ba$. If $ba \in B$, then $p = 1$, since $B \cap P = \{1\}$, but this is absurd, since we chose $a \notin B$, so $p \neq 1$. Thus we may conclude $ba \notin B$ for all $b \in B$, and thus ba is rigid. ■

Lemma 5.2.1.6 *Let (G, Q, q) be any finite characteristic 2 quaternionic structure. If $B \subset G$ is the subgroup of basic elements of G then the canonical homomorphism $\alpha : G \longrightarrow G/B$, induces a canonical surjective morphism*

$$\alpha : (G, Q, q) \longrightarrow (G/B, G/B \wedge G/B, \wedge).$$

Proof: Let $a, b \in G$ be so that $q(a,b) = 0$. If $a \in B$, then $\alpha(a) \wedge \alpha(b) = 0$, since $B \wedge bB = 0$ by the definition of the exterior power. Suppose $a \in G \setminus B$, then by Lemma 5.2.1.5, then a is rigid. Then if $q(a,b) = 0$, then $b = 1$ or $b = a$, so $\alpha(a) \wedge \alpha(b) = \alpha(a) \wedge \alpha(a)$, or $\alpha(a) \wedge \alpha(b) = \alpha(a) \wedge \alpha(1)$, both of which are $0 \in G/B \wedge G/B$, and thus α is a quaternionic morphism. Furthermore it is clear that α is surjective since $\alpha(a) \wedge \alpha(b) = 0$, implies $a = b$, or without loss of generality $b = 1$, so the quaternions are preserved, along with their parings, and thus ■

We may perform a similar construction with $\alpha : G \longrightarrow G/P$.

Lemma 5.2.1.7 *Let (G, Q, q) be any finite characteristic 2 quaternionic structure. If $P \subset G$ is the subgroup of purely rigid elements of G then the canonical homomorphism $\alpha : G \longrightarrow G/P \cong B$, induces a canonical surjective morphism*

$$\alpha : (G, Q, q) \longrightarrow (G/P, Q', q'),$$

where Q' be the range of the map

$$q' = q\Big|_{B \times B} : B \times B \longrightarrow Q' \subseteq Q.$$

Proof: Let $\alpha : G \longrightarrow G/P$ be the canonical surjection, and suppose $q(a,b) = 0$, for some $a, b \in G$. If a is rigid, then $q(a,b) = 0$, is to say $b \in \{1, a\}$, since q' is the restriction of q, then $q'(\alpha(a), \alpha(b)) = q'(\alpha(a), \alpha(a)) = 0$ or $q'(\alpha(a), \alpha(1)) = 0$. On the other hand if $a \in G$ is basic, and $q(a,b) = 0$, then since q' is the restriction of q, to $B \times B$, then it is clear that $q'(\alpha(a), \alpha(b)) = q(\alpha(a), \alpha(b)) = q(a,b) = 0$, and thus α is a surjective quaternionic morphism, since q is surjective, and Q' is the image of q'.

■

Theorem 5.2.1.8 *Let (G, Q, q) be any finite quaternionic structure. Then there exists* *a group of exponent 2, call it W, and finite basic characteristic 2 quaternionic structure (G', Q', q') so that $(G, Q, q) \cong (G'[W], Q'[W], q'[W])$.*

Proof: Let (G, Q, q) be any finite quaternionic structure, and denote $B \subset G$ the subgroup of basic elements of G, and take $W \cong G/B$, and it is clear that W is the purely rigid subgroup of G. Define (G', Q', q') as follows, $G' = B$, $q' = q\Big|_{B \times B}$, and $Q' = \text{range}(q')$. Since $G = B \oplus W$, we take $\phi : G \longrightarrow B \oplus W$, as the group isomorphism determined by the direct sum decomposition.

Since for any $a, b \in G$ so that $q(a, b) = 0$, then let $\phi(a) = (b_1, a_1) \in B \oplus W$, and $\phi(b) = (b_2, a_2) \in B \oplus W$ be the unique representatives in the direct sum. We see that

$$q'[W]((b_1, a_1), (b_2, a_2)) = (q'(b_1, b_2), b_1 \otimes a_2 + b_2 \otimes a_1, a_1 \wedge a_2) = 0$$

by the following cases. By Proposition 2.2.2.3 if $a_1 \neq 1$, then $a_1 = a_2$ or without loss of generality $a_2 = 1$. If $a_1 \neq 1$, then by Lemma 2.2.2.7, then $b_1 = b_2$, since (a_1, b_1) is rigid by Lemma 5.2.1.5, so the pairing is trivially zero. If on the other hand $a_1 = a_2 = 1$, then $a = b_1$, and $b = b_2$, are both basic, and so $q(a, b) = q(b_1, b_2) = 0$, and since q' is the restriction of q to $B \times B$, then $q'(b_1, b_2) = 0$, and since $a_1 = a_2 = 1$, then the other coordinates are zero. Thus we have shown, for any $a, b \in G$, that if $q(a, b) = 0$, then

$$q'[W](b_1 a_1, b_2 a_2) = (q'(b_1, b_2), b_1 \otimes a_2 + b_2 \otimes a_1, a_1 \wedge a_2) = 0,$$

and thus $\alpha : (G, Q, q) \longrightarrow (B[W], Q'[W], q'[W])$, is a morphism of quaternionic structures. Since α is an isomorphism, and since every $(g, p) \in B \oplus W$, is rigid for any $p \neq 1$, by Lemma 5.2.1.5, it is clear that α is an isomorphism of quaternionic structures. ■

5.2.2 The QBIG of Group Extensions

Theorem 5.2.2.1 *Let (G, Q, q) be any group extension. Then Γ is the QBIG of (G, Q, q) if and only if Γ possesses subgraphs $\Gamma_B \subset \Gamma$, the QBIG of a basic quaternionic structure, and $\Gamma_W \subset \Gamma$, the QBIG of a purely rigid structure, so that $G \cong B \oplus W$, and for all $(x, w) \in B \oplus W^*$, $\Gamma((x, w))$ is a refraction graph, and for all $(b, 1) \in B^* \oplus W$, and $\Gamma((b, 1))$ is not a refraction graph.*

Proof: By Theorem 5.2.1.8, every quaternionic structure (G, Q, q) with basic part B and purely rigid part W gives the existence of an isomorphism $(G, Q, q) \cong (B[W], Q'[W], q'[W])$. So it is clear that if Γ is the QBIG of (G, Q, q), then there exists $\Gamma_B \subset \Gamma$ and $\Gamma_W \subset \Gamma$ the QBIG of (B, Q', q') and $(W, W \wedge W, \wedge)$ respectively. By Lemma 2.2.2.7 every $(x, w) \in G \oplus W^*$ is rigid, and by Theorem 5.2.0.1, $\Gamma((x, w))$ is a refraction graph if and only if (x, w) is rigid. Since $(b, 1)$ is basic, then by Theorem 5.2.0.1 $V(\Gamma((b, 1)))$ cannot be a refraction graph, and thus we have proved the claim.

On the other hand, let (G, Q, q) be some finite characteristic 2 quaternionic structure and suppose Γ has the properties as stated above. Then we may take $G = B \oplus W$. By Theorem 5.2.0.1 every element of the form (x, w) is rigid in $B \oplus W^*$, so we may conclude that $B \subset G$ is the collection of basic elements, since $\Gamma((b, 1))$ is not a refraction graph for all $b \in B^*$, and thus $(1, w) \in B \oplus W^*$, is purely rigid. By Theorem 5.2.1.8, it follows that $G \cong B \oplus W$ gives $(G, Q, q) \cong (B[W], Q'[W], q'[W])$, so (G, Q, q) is a group extension, and we have proved the claim. ■

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