

University of California  
Santa Barbara

# **Robust stability theory for stochastic dynamical systems**

A dissertation submitted in partial satisfaction  
of the requirements for the degree

Doctor of Philosophy  
in  
Electrical and Computer Engineering

by

Anantharaman Subbaraman

Committee in charge:

Professor Andrew R. Teel, Chair  
Professor João P. Hespanha  
Professor Francesco Bullo  
Professor Katie Byl

December 2015

The Dissertation of Anantharaman Subbaraman is approved.

---

Professor João P. Hespanha

---

Professor Francesco Bullo

---

Professor Katie Byl

---

Professor Andrew R. Teel, Committee Chair

December 2015

Robust stability theory for stochastic dynamical systems

Copyright © 2015

by

Anantharaman Subbaraman

## Acknowledgements

I am grateful to my advisor Dr. Andrew Teel for motivating me to simultaneously strive for elegant and rigorous methods for solving problems. I thank Dr. João Hespanha for giving me an opportunity to work with his research group. I thank Dr. Francesco Bullo and Dr. Katie Byl for being in my dissertation committee.

I would like to thank Sergio Grammatico and Antonino Sferlazza for working with me on the area of stochastic systems. I have thoroughly enjoyed the many discussions, technical and otherwise with CCDC members Matthew Hartman, Nicholas Cox, Jorge Poveda, John Simpson, Jason Isaacs and Rush Patel. Finally, I would like to thank my family for their support.

The research presented in this dissertation was supported in part by the Air Force Office of Scientific Research grant FA9550-12-1-0127 and National Science Foundation grant ECCS-1232035.

# Curriculum Vitæ

## Anantharaman Subbaraman

### Education

- 2011 M.S. Electrical and Computer Engineering, University of California, Santa Barbara.
- 2010 B. Tech., Instrumentation and Control Engineering, National Institute of Technology-Trichy, India.

### Experience

- 2011-2015 Graduate Student Researcher, University of California, Santa Barbara.
- 2014 Research Intern, Mitsubishi Electric Research Laboratories, Cambridge.

### Publications

- Subbaraman, A. and Teel, A. R. *On the equivalence between global recurrence and the existence of a smooth Lyapunov function for hybrid systems*. Systems and Control Letters (Accepted), 2015.
- Subbaraman, A. and Teel, A. R. *Recurrence principles and their application to stability theory for a class of stochastic hybrid systems*. IEEE Transactions on Automatic Control (Conditionally accepted), 2015.
- Subbaraman, A. and Teel, A. R. *Robustness of recurrence for a class of stochastic hybrid systems*. Proceedings of Analysis and Design of Hybrid Systems, 304–309, 2015.
- Teel, A. R., Subbaraman, A. and Sferlazza, A. *Stability analysis for stochastic hybrid systems: a survey*. Automatica, 50(10), 2435–2456, 2014.
- Teel, A. R., Hespanha, J. P. and Subbaraman, A. *Equivalent characterizations of input-to-state stability for stochastic discrete-time systems*. IEEE Transactions on Automatic Control, 59(2), 516–522, 2014.
- Subbaraman, A. and Teel, A.R. *A Krasovskii-LaSalle function based recurrence principle for a class of stochastic hybrid systems*. Proceedings of the 53rd IEEE Conference on Decision and Control, 2014.
- Teel, A. R., Hespanha, J. P. and Subbaraman, A. *A converse Lyapunov theorem and robustness of asymptotic stability in probability*. IEEE Transactions on Automatic Control, 59(9), 2426–2441, 2014.

- Subbaraman, A. and Teel, A. R. *A converse Lyapunov theorem for strong global recurrence*. *Automatica*, 49(10), 2963–2974, 2013.
- Subbaraman, A. and Teel, A. R. *A Matrosov theorem for strong global recurrence*. *Automatica*, 49(11), 3390–3395, 2013.
- Grammatico, S., Subbaraman, A. and Teel, A. R. *Discrete-time stochastic control systems: a continuous Lyapunov function implies robustness to strictly causal perturbations*. *Automatica*, 49(10), 2939–2952, 2013.
- Hartman, M., Subbaraman, A. and Teel, A. R. *Robust global almost sure synchronization on a circle via stochastic hybrid control*. *Control of Cyber-Physical Systems*. Springer International Publishing, 2013.
- Subbaraman, A., Hartman, M. and Teel, A. R. *A stochastic hybrid algorithm for robust global almost sure synchronization on a circle*. *Proceedings of the 52nd IEEE Conference on Decision and Control*, 2013.
- Grammatico, S., Subbaraman, A. and Teel, A. R. *Discrete-time stochastic control systems: examples of robustness to strictly causal perturbations*. *Proceedings of the 52nd IEEE Conference on Decision and Control*, 2013.
- Teel, A. R., Hespanha, J. and Subbaraman, A. *Stochastic difference inclusions: results on recurrence and asymptotic stability in probability*. *Proceedings of the 51st IEEE Conference on Decision and Control*, 2012.

## Abstract

Robust stability theory for stochastic dynamical systems

by

Anantharaman Subbaraman

In this work, we focus on developing analysis tools related to stability theory for certain classes of stochastic dynamical systems that permit non-unique solutions. The non-unique nature of solutions arise primarily due to the system dynamics that are modeled by set-valued mappings. There are two main motivations for studying such classes of systems. Firstly, understanding such systems is crucial to developing a robust stability theory. Secondly, such system models allow flexibility in control design problems.

We begin by developing analysis tools for a simple class of discrete-time stochastic system modeled by set-valued maps and then extend the results to a larger class of stochastic hybrid systems. Stochastic hybrid systems are a class of dynamical systems that combine continuous-time dynamics, discrete-time dynamics and randomness. The analysis tools are established for properties like global asymptotic stability in probability and global recurrence. We focus on establishing results related to sufficient conditions for stability, weak sufficient conditions for stability, robust stability conditions and converse Lyapunov theorems. In this work a primary assumption is that the stochastic system satisfies some mild regularity properties with respect to the state variable and random input. The regularity properties are needed to establish the existence of random solutions and results on sequential compactness for the solution set of the stochastic system.

We now explain briefly the four main types of analysis tools studied in this work. Sufficient conditions for stability establish conditions involving Lyapunov-like functions satisfying strict decrease properties along solutions that are needed to verify stability

properties. Weak sufficient conditions relax the strict decrease nature of the Lyapunov-like function along solutions and rely on either knowledge about the behavior of the solutions on certain level sets of the Lyapunov-like function or use multiple nested non-strict Lyapunov-like functions to conclude stability properties. The invariance principle and Matrosov function theory fall in to this category. Robust stability conditions determine when stability properties are robust to sufficiently small perturbations of the nominal system data. Robustness of stability is an important concept in the presence of measurement errors, disturbances and parametric uncertainty for the nominal system. We study two approaches to verify robustness. The first approach to establish robustness relies on the regularity properties of the system data and the second approach is through the use of Lyapunov functions. Robustness analysis is an area where the notion of set-valued dynamical systems arise naturally and it emphasizes the reason for our study of such systems. Finally, we focus on developing converse Lyapunov theorems for stochastic systems. Converse Lyapunov theorems are used to illustrate the equivalence between asymptotic properties of a system and the existence of a function that satisfies a decrease condition along the solutions. Strong forms of the converse theorem imply the existence of smooth Lyapunov functions. A fundamental way in which our results differ from the results in the literature on converse theorems for stochastic systems is that we exploit robustness of the stability property to establish the existence of a smooth Lyapunov function.



# Contents

<b>Curriculum Vitae</b>	<b>v</b>
<b>Abstract</b>	<b>vii</b>
<b>List of Figures</b>	<b>xiv</b>
<b>1 Introduction</b>	<b>1</b>
1.1 Outline of the results . . . . .	4
<b>2 Hybrid systems</b>	<b>6</b>
2.1 Introduction . . . . .	6
2.2 Preliminaries on hybrid systems . . . . .	6
2.3 Recurrence and Uniform recurrence . . . . .	11
2.4 Recurrence and other properties . . . . .	14
2.5 Invariance principle for recurrence . . . . .	17
2.6 Robust global recurrence and a converse Lyapunov theorem . . . . .	18
2.7 Hitting time to open sets - an equivalent characterization . . . . .	28
2.8 Proof of Theorem 2.5 . . . . .	30
2.9 Summary of results for global asymptotic stability . . . . .	35
<b>3 Stochastic difference inclusions</b>	<b>37</b>
3.1 Introduction . . . . .	37
3.2 Preliminaries on difference inclusions with random inputs . . . . .	38
3.3 Recurrence and asymptotic stability in probability . . . . .	41
3.4 Stability in terms of probability functions . . . . .	45
3.5 Global recurrence . . . . .	49
3.6 Global asymptotic stability in probability . . . . .	65
<b>4 Robust stochastic stability under discontinuous stabilization</b>	<b>69</b>
4.1 Introduction . . . . .	69
4.2 Constrained stochastic systems with control inputs . . . . .	71
4.3 Continuous Lyapunov function implies robustness . . . . .	73

4.4	Strictly causal generalized random solutions . . . . .	76
<b>5</b>	<b>Stochastic hybrid systems</b>	<b>82</b>
5.1	Introduction . . . . .	82
5.2	Preliminaries on stochastic hybrid systems . . . . .	83
5.3	Weak total recurrence . . . . .	86
5.4	The recurrence principles . . . . .	89
5.5	Corollaries of the recurrence principle . . . . .	92
5.6	Comparison to invariance properties . . . . .	95
5.7	Application to stability theory . . . . .	100
<b>6</b>	<b>Robust global recurrence in stochastic hybrid systems</b>	<b>111</b>
6.1	Introduction . . . . .	111
6.2	Recurrence and Uniform recurrence . . . . .	112
6.3	Viability and reachability probabilities . . . . .	114
6.4	Preliminary bounds on viability and reachability probabilities . . . . .	116
6.5	Robustness of recurrence . . . . .	118
6.6	Necessary and sufficient condition for global recurrence . . . . .	124
<b>7</b>	<b>Conclusions</b>	<b>141</b>
7.1	Summary . . . . .	141
7.2	Future directions . . . . .	142
<b>A</b>	<b>Mathematical review</b>	<b>145</b>
<b>B</b>	<b>Sequential compactness: Hybrid systems and stochastic hybrid systems</b>	<b>148</b>
B.1	Hybrid systems . . . . .	148
B.2	Stochastic hybrid systems . . . . .	150
<b>C</b>	<b>Stochastic stability properties</b>	<b>153</b>
<b>D</b>	<b>Proofs</b>	<b>156</b>
D.1	Proof of Proposition 2.4 . . . . .	156
D.2	Proof of Theorem 3.1 . . . . .	156
D.3	Proof of Proposition 3.5 . . . . .	160
D.4	Proof of Theorem 3.2 . . . . .	163
D.5	Proof of Theorem 3.3 . . . . .	167
D.6	Proof of Theorem 3.4 . . . . .	170
D.7	Proof of Theorem 3.7 . . . . .	172
D.8	Proof of Proposition 4.1 . . . . .	176
D.9	Proof of Theorem 4.1 . . . . .	178
D.10	Proof of Proposition 4.2 . . . . .	182
D.11	Proof of Lemma 4.3 . . . . .	183

D.12 Proof of Theorem 5.1 . . . . .	184
D.13 Proof of Theorem 5.2 . . . . .	189
D.14 Proof of Lemma 5.1 . . . . .	206
D.15 Proof of Theorem 5.3 . . . . .	207
D.16 Proof of Proposition 6.3 . . . . .	208
D.17 Proof of Proposition 6.4 . . . . .	210
D.18 Proof of Proposition 6.5 . . . . .	212
D.19 Proof of Proposition 6.6 . . . . .	213
D.20 Proof of Proposition 6.2 . . . . .	214
D.21 Proof of Proposition 6.8 . . . . .	215
D.22 Proof of Theorem 6.1 . . . . .	216
D.23 Proof of Theorem 6.2 . . . . .	217
D.24 Proof of Proposition 6.9 . . . . .	219
D.25 Proof of Theorem 6.3 . . . . .	220
<b>Bibliography</b>	<b>221</b>

## Notation

- $\mathbb{R}_{\geq 0}$  denotes the non-negative real numbers.
- $\mathbb{Q}$  denotes the set of rational numbers.
- $\mathbb{Z}_{\geq 0}$  denotes the non-negative integers.
- For  $S \subset \mathbb{R}^n$ , the symbol  $\mathbb{I}_S$  denotes the indicator function of  $S$  i.e.,  $\mathbb{I}_S(x) = 1$  for  $x \in S$  and  $\mathbb{I}_S(x) = 0$  otherwise.
- For vectors  $f_1, f_2 \in \mathbb{R}^n$ ,  $\langle f_1, f_2 \rangle$  denotes the inner product.
- For  $c \geq 0$  and a function  $V : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ ,  $L_V(c) := \{x \in \mathbb{R}^n : V(x) = c\}$  denotes the level set of the function  $V$ .
- $\mathbb{B}, \mathbb{B}^\circ$  denote the closed and open unit ball in  $\mathbb{R}^n$ .
- $\partial S, \bar{S}$  and  $\overline{\text{co}}S$  represents the boundary of the set  $S$ , closure of the set  $S$  and the closed convex hull of the set  $S$  respectively.
- Given a closed set  $S \subset \mathbb{R}^n$  and  $\epsilon > 0$ ,  $S + \epsilon\mathbb{B}$  represents the set  $\{x \in \mathbb{R}^n : |x|_S \leq \epsilon\}$  and  $S + \epsilon\mathbb{B}^\circ$  represents the set  $\{x \in \mathbb{R}^n : |x|_S < \epsilon\}$ .
- $\mathbf{B}(\mathbb{R}^m)$  denotes the Borel  $\sigma$ -field, the subsets of  $\mathbb{R}^m$  generated from all open subsets of  $\mathbb{R}^m$  through complements and finite and countable unions.
- For a compact set  $\mathcal{A} \subset \mathbb{R}^n$ , a function  $V : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$  belongs to the class  $\mathcal{PD}(\mathcal{A})$  if  $V(x) = 0$  for  $x \in \mathcal{A}$  and positive elsewhere.
- For  $\tau \geq 0$ , we define the sets  $\Gamma_{\leq \tau} := \{(s, t) \in \mathbb{R}^2 : s + t \leq \tau\}$  and  $\Gamma_{\geq \tau} := \{(s, t) \in \mathbb{R}^2 : s + t \geq \tau\}$

- The functions  $\pi_i : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \times \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$  are such that  $\pi_i(t_1, t_2, z) = t_i$  for each  $i \in \{1, 2\}$ .
- For sets  $S_1, S_2 \subset \mathbb{R}^n$ ,  $\mathbb{I}_{\subset S_1}(S_2) = 1 - \sup_{x \in S_2} \mathbb{I}_{\mathbb{R}^n \setminus S_1}(x)$  and  $\mathbb{I}_{\cap S_1}(S_2) = \sup_{x \in S_2} \mathbb{I}_{S_1}(x)$  with the convention that the maximum's are zero when  $S_2 = \emptyset$ .

# List of Figures

2.1	A solution to a hybrid system - hybrid arc $\phi$ . . . . .	8
2.2	Perturbation of nominal system models . . . . .	10
2.3	Illustration of recurrence . . . . .	12
2.4	Converse Lyapunov theorem for recurrence . . . . .	34
2.5	Summary of recurrence results for (2.1) . . . . .	35
3.1	Summary of recurrence results for (3.1) . . . . .	65
3.2	Summary of stability results for (3.1) . . . . .	68
6.1	Summary of results for recurrence in (6.1) . . . . .	140

# Chapter 1

## Introduction

Stability theory for dynamical systems is one of the most well studied topics in control theory and is pioneered by the work in [1]. For continuous-time systems, results for certifying asymptotic stability of the origin are found in the seminal work by Lyapunov in [1]. In particular, through the use of a Lyapunov function that is decreasing strictly along solutions, asymptotic stability of the origin can be concluded for the system without explicit knowledge of the actual solution to the ordinary differential equation.

The sufficient conditions proposed in [1] is one among the many different analysis tools studied in the literature. A natural relaxation of the sufficient condition leads to the use of Lyapunov-like functions satisfying non-strict decrease conditions to establish asymptotic stability of the origin. The invariance principle in [2], [3] characterize the behavior of complete, bounded solutions using the notion of  $\Omega$ -limit sets which are invariant. Under the additional assumption of a non-increasing Lyapunov-like function it is also established that complete, bounded solutions converge to the largest invariant sets inside level sets of the Lyapunov-like function. This key result is then used to establish weak sufficient conditions for asymptotic stability that do not insist on the existence of a Lyapunov function satisfying strict decrease properties.

In [4] and [5] instead of using one non-strict Lyapunov function, multiple non-strict but nested functions (referred to as Matrosov functions) are utilized to conclude asymptotic stability. In essence, the invariance principle and Matrosov function theory provide weak sufficient conditions for the asymptotic stability property. However, the invariance principle requires knowledge about solution behavior on certain level sets of the Lyapunov function to conclude asymptotic stability whereas the Matrosov function based approach does not.

Another important problem studied in the literature relates to robustness of stability properties. In particular, conditions for which the stability property of the nominal system is preserved under the action of sufficiently small perturbations are studied. The perturbations affecting the nominal system can be measurement errors related to the state or modeling uncertainties in the system description. Robustness properties can be studied from the viewpoint of the regularity properties of the nominal system ([6], [7]), or through a Lyapunov function approach which usually involves an assumption regarding the existence of a Lyapunov function satisfying good regularity properties for the nominal system ([8], [7]) or by explicitly considering the disturbance/ noise inputs to the system as in the various works on input to state stability ([9], [10]).

Converse Lyapunov theorems are used to establish the equivalence between asymptotic stability properties and the existence of Lyapunov-like functions that satisfy certain decrease conditions along solutions. Converse Lyapunov theorems for a locally Lipschitz differential inclusion appear in [11], with its discrete-time counterpart in [9]. Strong stability of the origin for a differential inclusion under mild regularity assumptions is proved to be equivalent to the existence of a smooth Lyapunov function in [12]. For difference inclusions under similar regularity assumptions a converse Lyapunov theorem is established in [7], where sufficient conditions for existence of a smooth Lyapunov function for difference equations with discontinuous right hand sides is also established. Results on



the existence of smooth Lyapunov functions under the assumption of  $\mathcal{KL}$ -stability with respect to two measures for differential and difference inclusions is in [6],[13] respectively.

Next, we briefly discuss the development of analysis tools for a larger class of hybrid systems. Hybrid systems are a class of dynamical systems that combines continuous-time dynamics and discrete-time dynamics. The developments in area of stability analysis for hybrid systems are a bit recent. In particular, [14] establishes Lyapunov function based sufficient conditions for global asymptotic stability, converse Lyapunov theorems, robust stability and the invariance principle under mild assumptions on the system data. The work by [15] and [16] establish a Matrosov theorem and input to state stability results respectively.

The literature on analysis tools for stability in stochastic systems has also taken a similar route but is more diverse. The diversity arises primarily due to many variants of the stability properties that can be studied for stochastic systems (See [17]). This is a direct consequence of the different notions of convergence that exist for sequence of random variables (See [18, Chapter 6]). In particular, for stochastic systems stability properties can be studied based on convergence in mean, convergence in probability, almost sure convergence and convergence in distribution. Some of the stability properties studied frequently in the literature are mean square asymptotic stability (asymptotic, exponential), almost sure asymptotic stability, asymptotic stability in probability, stability in distribution, positive recurrence and null recurrence. We refer the reader to [19], [20], [21], [22] and [23] for results on Lyapunov function based conditions for certifying stability, invariance principle and converse Lyapunov theorems.

Stochastic hybrid systems are a class of dynamical systems that combine continuous-time dynamics, discrete-time dynamics and randomness. In stochastic hybrid systems randomness can affect the system dynamics in a number of different ways and consequently the modeling frameworks studied in the literature vary in complexity and scope.

(See [24], [25], [23] and [26]). We also refer the reader to [27] for details on the many different classes of stochastic hybrid systems and related developments in stability theory.

Stability analysis tools for stochastic systems modeled by set-valued mappings are seldom studied in the literature. In this dissertation, we emphasize the key role set-valued stochastic systems play in the development of a robust stability theory for stochastic systems. We will focus on discrete-time stochastic systems and stochastic hybrid systems modeled by set-valued mappings and establish a range of analysis tools related to stability theory. We restrict our study and the development of analysis tools to global recurrence of open, bounded sets and global asymptotic stability in probability of compact sets.

## 1.1 Outline of the results

In the following chapters we aim to establish a variety of analysis tools related to stability theory for stochastic systems. In particular, we study properties like recurrence and asymptotic stability in probability for a class of discrete-time stochastic systems and stochastic hybrid systems.

In Chapter 2, we study the recurrence property for non-stochastic hybrid systems. Under mild regularity properties for the hybrid system we establish that recurrence of bounded sets is equivalent to the well studied property of ultimate boundedness. We also establish that the recurrence property is robust to sufficiently small state dependent perturbations and develop a converse Lyapunov theorem. Chapter 2 serves as an introduction to the recurrence property and to the type of analysis tools that will be studied for a more general class of stochastic systems in the subsequent chapters.

In Chapter 3, we introduce a class of discrete-time stochastic systems modeled by set-valued mappings (stochastic difference inclusions). We characterize the notion of a random solution to the stochastic difference inclusion and establish sufficient conditions

for stability, an invariance principle, conditions for robust stability and a converse Lyapunov theorem under good regularity properties for the stochastic difference inclusion.

In Chapter 4, we analyze robustness for a class of discrete-time stochastic systems stabilized by discontinuous feedback laws. The results from Chapter 3 on robustness are generally not applicable in the case of discontinuities in the control law. Hence, we develop a Lyapunov function based approach to verify robustness as opposed to relying on the regularity properties of the closed loop system.

In Chapter 5, we study a class of stochastic hybrid systems modeled by set-valued mappings. In particular, we focus on systems where the randomness is restricted to the discrete-time dynamics. For this class of systems, we present a result related to the invariance principle for characterizing the behavior of bounded random solutions. Application of this result to establishing weak sufficient conditions for recurrence and asymptotic stability in probability is also presented.

In Chapter 6, we provide a Lyapunov function based characterization of the recurrence property for the class of stochastic hybrid systems studied in Chapter 5. In particular, we establish robustness results and a converse Lyapunov theorem for global recurrence of open, bounded sets.

In Chapter 7, we summarize the contributions of this dissertation and point out future research directions.

# Chapter 2

## Hybrid systems

### 2.1 Introduction

Hybrid systems are a class of dynamical systems that combine continuous-time dynamics and discrete-time dynamics. Several frameworks have been proposed in the literature for the modeling and analysis of hybrid systems. We refer the reader to [28], [29] and [30] for details. The aim of this chapter is to review a mathematical framework for hybrid system models proposed in [28], give the reader an introduction to the study of a property called recurrence, and, establish a Lyapunov function based characterization for the recurrence property. The main results presented in this chapter are from [31]. The subsequent chapters will build upon the fundamental results in this chapter and extend the results to a larger class of systems affected by randomness.

### 2.2 Preliminaries on hybrid systems

We follow the mathematical framework in [28] for modeling hybrid systems. As explained in [14, Chapter 1] other models for describing hybrid systems can be encom-

passed within the framework of [28]. So, we consider a class of hybrid systems with a state  $x \in \mathbb{R}^n$  written formally as

$$\dot{x} \in F(x), \quad x \in C \quad (2.1a)$$

$$x^+ \in G(x), \quad x \in D \quad (2.1b)$$

where  $C, D \subset \mathbb{R}^n$  represent the flow and jump sets (where continuous and discrete evolution of the state is permitted) respectively and  $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n, G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$  represent the set-valued flow and jump maps respectively. In essence, the continuous-time dynamics is modeled by a differential inclusion and the discrete-time dynamics is modeled by a difference inclusion.

### 2.2.1 Solution concept

We define solutions to the hybrid system on a generalized time domain that uses two variables  $t, j$  to keep track of the continuous evolution of the state and the number of jumps elapsed respectively. To define solutions to (2.1) we require the notion of a *hybrid time domain*: a subset  $E$  of  $(\mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0})$ , which is the union of infinitely many intervals of the form  $[t_j, t_{j+1}] \times \{j\}$ , where  $0 = t_0 \leq t_1 \leq t_2 \leq \dots$ , or finitely many of such intervals, with the last one possibly of the form  $[t_j, t_{j+1}] \times \{j\}$ ,  $[t_j, t_{j+1}) \times \{j\}$ , or  $[t_j, \infty) \times \{j\}$ . A function  $\phi : E \rightarrow \mathbb{R}^n$  that maps a hybrid time domain to the Euclidean space and for which  $t \mapsto \phi(t, j)$  is locally absolutely continuous for fixed  $j$  is called a *hybrid arc*.

A hybrid arc is a *solution* to (2.1) if  $\phi(0, 0) \in C \cup D$  and:

- 1) for all  $j \in \mathbb{Z}_{\geq 0}$  and almost all  $t$  such that  $(t, j) \in \text{dom } \phi$ :  $\phi(t, j) \in C$ ,  $\dot{\phi}(t, j) \in F(\phi(t, j))$
- 2) for all  $(t, j) \in \text{dom } \phi$  such that  $(t, j+1) \in \text{dom } \phi$ :  $\phi(t, j) \in D$ ,  $\phi(t, j+1) \in G(\phi(t, j))$ .

A solution to the hybrid system is called *maximal* if it cannot be extended, and

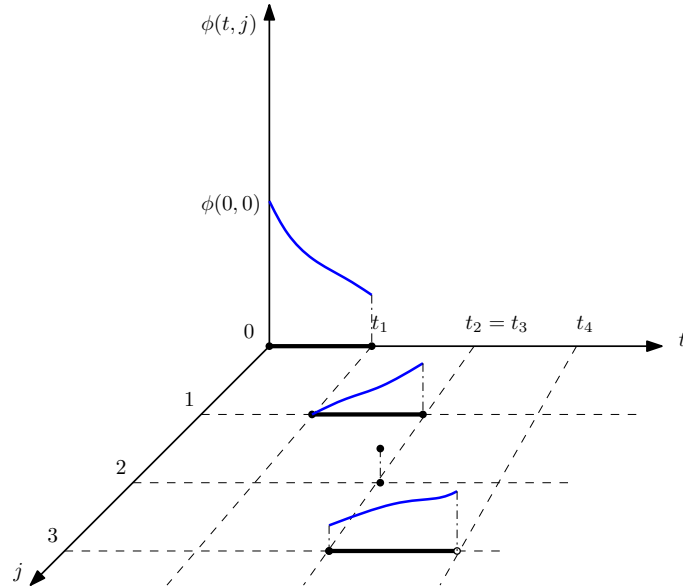


Figure 2.1: A solution to a hybrid system - hybrid arc  $\phi$

*complete* if its domain is unbounded. See Figure 2.1 for a representation of a hybrid arc.

We will represent the hybrid system (2.1) through its data as

$$\mathcal{H} := (C, F, D, G). \quad (2.2)$$

We denote by  $\mathcal{S}_{\mathcal{H}}(K)$  the set of all maximal solutions starting from the set  $K \subset \mathbb{R}^n$  for the hybrid system  $\mathcal{H}$ . We assume throughout this chapter that  $\mathcal{H}$  satisfies certain regularity properties listed below.

**Standing Assumption 2.1** *The data  $\mathcal{H}$  of the hybrid system (2.1) satisfies the following conditions:*

1. *The sets  $C, D \subset \mathbb{R}^n$  are closed.*
2. *The mapping  $F$  is outer semicontinuous, locally bounded, convex valued and non-empty on  $C$ .*
3. *The mapping  $G$  is outer semicontinuous, locally bounded and non-empty on  $D$ .*

If  $F, G$  are single-valued mappings, then Standing Assumption 2.1 reduces to the mappings  $f, g$  being continuous on  $C$  and  $D$  respectively. The system (2.1a) is said to have no finite escape times if there are no solutions of (2.1a) that escape to infinity at a finite time.

### 2.2.2 Systems modeled by set-valued mappings

In this section we explain the motivations for studying systems modeled by set-valued mappings and the system regularity properties imposed in Standing Assumption 2.1.

The main reasons for studying systems modeled by set-valued mappings are listed below.

- Firstly, set-valued mappings arise in the context of analysis of systems in the presence of disturbances. This notion is illustrated for the simple case of a discrete-time system  $x^+ = g(x)$  in Figure 2.2. Analysis of the nominal system  $x^+ = g(x)$  in the presence of measurement errors in the state  $x$  and modeling uncertainties in the mapping  $g$  leads to the study of the difference inclusion  $x^+ \in g(x + \delta\mathbb{B}) + \delta\mathbb{B}$ , where  $\delta > 0$  is the size of the perturbation. Hence, the study of set-valued mappings is crucial for the development of a robust stability theory for dynamical systems. This aspect will be explored further in this chapter and in the subsequent chapters for a larger class of stochastic systems. We refer the reader to [32, Chapter 1] for more details.
- Secondly, allowing set-valued mappings can provide a degree of flexibility in the control design process and also be a useful technical tool in solving control synthesis problems. For example, [33], [34, Chapter 7] and [32, Chapter 1] present scenarios in the context of control system analysis related to optimal control synthesis, local controllability analysis, study of constrained control systems where set-valued

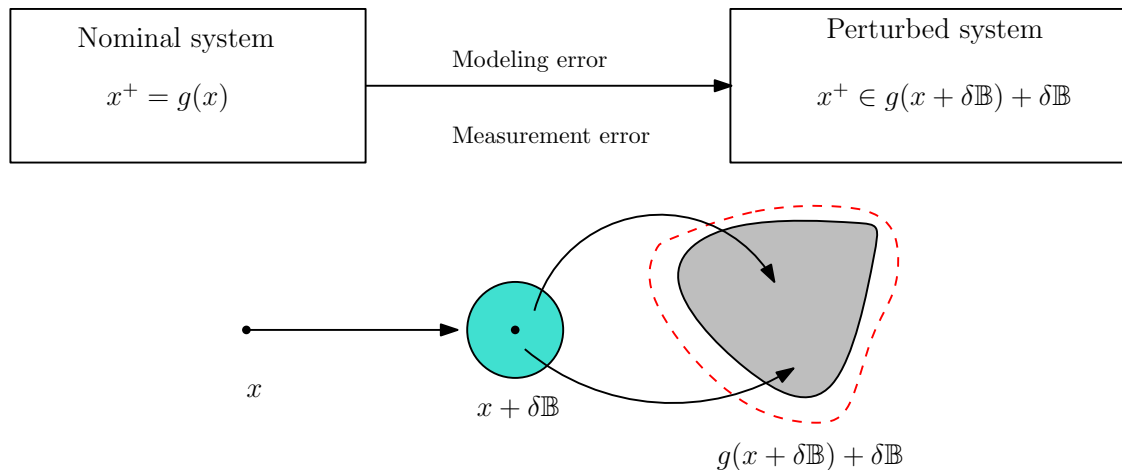


Figure 2.2: Perturbation of nominal system models

mappings arise. We refer the reader to [35] for more examples that illustrate the importance of set-valued analysis in control systems. Set-valued mappings arising in control design oriented problems are in [36], [37], [38], [39] and [40].

- Finally, set-valued systems also arise frequently in the study of discontinuous systems of the form  $\dot{x} = f(x)$  or  $x^+ = g(x)$  through the Krasovskii/ Filippov regularization and in defining notions of generalized solutions for discontinuous systems. For example, the Krasovskii regularization for discontinuous flow and jump maps  $f, g$  are given by  $\cap_{\delta>0} \overline{\text{con}} f(x + \delta\mathbb{B})$  and  $\cap_{\delta>0} \overline{\text{con}} g(x + \delta\mathbb{B})$  where  $\overline{\text{con}}$  refers to the closed convex hull. The Krasovskii regularization can also be used to infer robustness of stability properties for the original discontinuous system. We refer the reader to [14, Lemma 5.16] for more details.

The primary motivation for imposing the regularity properties in Standing Assumption 2.1 are now stated. Standing Assumption 2.1 is crucial to establishing the notion of nominal well-posed and well-posed hybrid systems (See [14, Chapter 6] and the Appendix for more details). These notions are then used to prove the equivalence between uniform and non-uniform versions of stability properties, establish robustness of stability



properties and consequently aid in the development of converse Lyapunov theorems.

## 2.3 Recurrence and Uniform recurrence

In this section we define the notion of recurrence for sets. Recurrence is a weak property that is frequently studied in the literature for stochastic systems. It is a weaker property compared to asymptotic stability but nevertheless useful in many applications where stronger properties are difficult to establish. Recurrence proves to be a useful alternative particularly in the study of systems affected by persistent disturbances. In this chapter, we study the recurrence property not for stochastic systems, but for a class of non-stochastic hybrid systems. Subsequent chapters will explore the recurrence property in detail for discrete-time stochastic systems and stochastic hybrid systems.

**Definition 2.1** *A set  $\mathcal{O} \subset \mathbb{R}^n$  is said to be globally recurrent for the hybrid system  $\mathcal{H}$  in (2.2) if there are no finite escape times for (2.1a) and for each complete solution  $\phi \in \mathcal{S}_{\mathcal{H}}(C \cup D)$ , there exists  $(t, j) \in \text{dom } \phi$  such that  $\phi(t, j) \in \mathcal{O}$ .*

Loosely speaking, the definition means that from every initial condition, solutions either stop or hit the set  $\mathcal{O}$  and solutions do not exhibit finite escape times. An illustration of the recurrence property is in Figure 2.3. The recurrence definition does not impose any invariance-like property for the set  $\mathcal{O}$ . Hence, solutions that start from the set  $\mathcal{O}$  can leave the set. Recurrence of the set  $\mathcal{O}$  also does not impose any stability-like conditions since solutions that start close to the set  $\mathcal{O}$  need not stay close. In this respect, recurrence is different from the frequently studied asymptotic stability property. Nevertheless, there are some connections between recurrence and properties like ultimate boundedness and asymptotic stability which will be explored in the subsequent sections. Another consequence of the recurrence property is that for complete solutions, recurrence

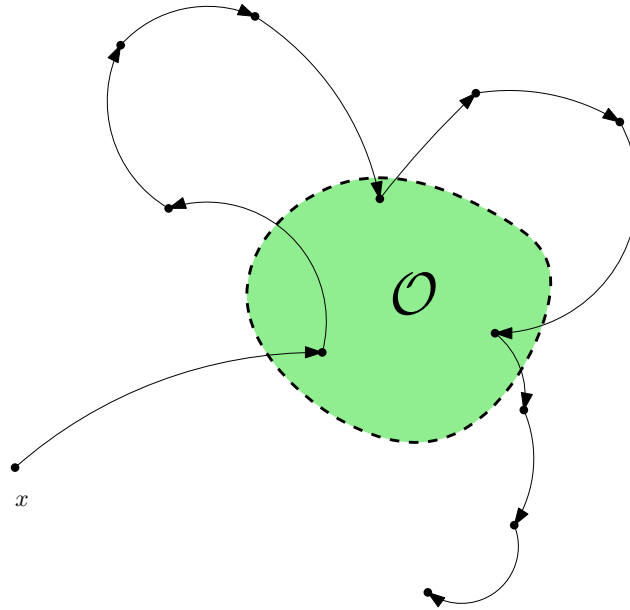


Figure 2.3: Illustration of recurrence

for the set  $\mathcal{O}$  implies that solutions have to visit the set  $\mathcal{O}$  infinitely often.

**Definition 2.2** *A set  $\mathcal{O} \subset \mathbb{R}^n$  is said to be uniformly globally recurrent for  $\mathcal{H}$  in (2.2) if there are no finite escape times for (2.1a) and for each compact set  $K$ , there exists  $T > 0$  such that for each solution  $\phi \in \mathcal{S}_{\mathcal{H}}(K)$ , either  $t + j < T$  for all  $(t, j) \in \text{dom } \phi$  or there exists  $(t, j) \in \text{dom } \phi$  such that  $t + j \leq T$  and  $\phi(t, j) \in \mathcal{O}$ .*

Recurrence is a property that is studied with respect to open sets for a variety of reasons. Firstly, we consider open, bounded sets to establish robustness of the recurrence property. Secondly, equivalence between uniform and non-uniform versions of recurrence hold only for open sets. These aspects will be illustrated through examples in the sections that follow.

The following result establishes equivalence between uniform and non-uniform recurrence when  $\mathcal{O}$  is open and bounded under mild regularity properties for  $\mathcal{H}$  stated in Standing Assumption 2.1.

**Proposition 2.1** *An open, bounded set  $\mathcal{O}$  is globally recurrent for  $\mathcal{H}$  in (2.2) if and only if it is uniformly globally recurrent for  $\mathcal{H}$ .*

*Proof:*  $\Leftarrow$  Follows immediately from the definitions.

$\Rightarrow$  Suppose  $\mathcal{O}$  is not uniformly globally recurrent. Then, there exists a compact set  $K$  such that for every  $i \in \mathbb{Z}_{>0}$ , there exists a solution  $\phi_i \in \mathcal{S}_{\mathcal{H}}(K)$  such that there exists  $(t, j) \in \text{dom } \phi_i$  satisfying  $t + j > i$  and for all  $(t, j) \in \text{dom}(\phi_i)$  satisfying  $t + j \leq i$ ,  $\phi_i(t, j) \notin \mathcal{O}$ . Due to compactness of  $K$ , and absence of finite escape times, it follows from [14, Prop 6.13] that the sequence of solutions  $\phi_i$  is locally eventually bounded<sup>1</sup>. Then, [41, Thm 4.4] states that the sequence  $\phi_i$  admits a converging subsequence  $\psi_i$  that converges to a complete solution  $\psi \in \mathcal{S}_{\mathcal{H}}(K)$ . From recurrence of  $\mathcal{O}$ , there exists  $(t, j)$  such that  $\psi(t, j) \in \mathcal{O}$ . From the definition of convergence of hybrid arcs, there exists a sequence  $\{t_i, j_i, \psi_i(t_i, j_i)\}$  such that  $t_i \rightarrow t, j_i \rightarrow j$  and  $\psi_i(t_i, j_i) \rightarrow \psi(t, j)$ . Since  $\mathcal{O}$  is open, for  $i$  large enough  $\psi_i(t_i, j_i) \in \mathcal{O}$ . This contradicts the initial assumption and establishes uniform global recurrence of  $\mathcal{O}$ .  $\blacksquare$

Without Standing Assumption 2.1, Proposition 2.1 is not necessarily true and the following example illustrates it.

**Example 2.1** Consider  $\mathcal{H} = (\emptyset, \emptyset, \mathbb{R}, g)$  where  $g(x) = (\max\{0, x\})^2$  if  $x < 1$  and  $g(x) = 0$  otherwise. Consider an open neighborhood of the origin of the form  $\mathcal{O} := (-\varepsilon, \varepsilon)$  for any  $\varepsilon \in (0, 1)$ . Then for every initial condition  $x \in \mathbb{R}^n$ , all solutions reach the set  $\mathcal{O}$  and hence  $\mathcal{O}$  is globally recurrent. Now for every compact set  $K \subset \mathbb{R}$  such that  $\{1\} \in \text{int}(K)$ , the solutions that start arbitrarily close to the left of  $x = 1$  takes arbitrarily long times to reach the set  $\mathcal{O}$  and hence the set  $\mathcal{O}$  is not uniformly globally recurrent. The mapping  $g$  is discontinuous at the point  $x = 1$  and hence does not satisfy Standing Assumption

<sup>1</sup>A sequence of solutions  $\phi_i$  is called locally eventually bounded if for every  $\tau \geq 0$ , there exists  $i^*$  and  $M > 0$  such that for all  $i \geq i^*$  and all  $(t, j) \in \text{dom}(\phi_i)$  with  $t + j \leq \tau$ ,  $\phi_i(t, j) \in M\mathbb{B}$ . We refer the reader to [28] and [14, Definition 5.24] for more details.

2.1.

The next example illustrates how Proposition 2.1 can fail if the set  $\mathcal{O}$  is not open. The equivalence between recurrence and uniform recurrence is crucial to the development of the converse Lyapunov theorem established later in this chapter.

**Example 2.2** Consider a continuous-time system with  $C = \mathbb{R}^2$  and the following dynamics

$$\dot{x}_1 = x_2 - x_1(x_1^2 + x_2^2 - 1), \quad \dot{x}_2 = -x_1 - x_2(x_1^2 + x_2^2 - 1).$$

Let  $\epsilon \in (0, 1)$ . It can be observed that the closed set  $\mathcal{O} := \{0\} \cup \{x : |x| \in [1 - \epsilon, 1 + \epsilon]\}$  is globally recurrent. Moreover, for solutions starting closer to the origin, it takes arbitrarily long time to reach the set  $\mathcal{O}$ . Hence, uniform recurrence fails even though the system satisfies the conditions of Standing Assumption 2.1.

## 2.4 Recurrence and other properties

In this section we make connections between recurrence and other well studied properties like ultimate boundedness and asymptotic stability.

### 2.4.1 Connection to Ultimate boundedness

**Definition 2.3** *The solutions of  $\mathcal{H}$  in (2.2) are uniformly ultimately bounded if there are no finite escape times for (2.1a) and there exists  $M > 0$  such that for each  $\Delta > 0$  there exists  $T > 0$  such that for every  $\phi \in \mathcal{S}_{\mathcal{H}}(\Delta\mathbb{B})$  either  $t + j < T$  for all  $(t, j) \in \text{dom } \phi$  or  $\phi(t, j) \in M\mathbb{B}$  for all  $(t, j) \in \text{dom } \phi$  satisfying  $t + j \geq T$ .*

**Proposition 2.2** *The solutions of  $\mathcal{H}$  in (2.2) are uniformly ultimately bounded if and only if there exists an open, bounded set  $\mathcal{O}$  that is globally recurrent for  $\mathcal{H}$ .*

*Proof:* It follows from the definition that ultimate boundedness of solutions of  $\mathcal{H}$  implies that the set  $\mathcal{O} = (M + 1)\mathbb{B}^o$  is globally recurrent for  $\mathcal{H}$ .

Next, we establish that recurrence of an open, bounded set  $\mathcal{O}$  for  $\mathcal{H}$  implies uniform ultimate boundedness of solutions of  $\mathcal{H}$ . We first claim that the reachable<sup>2</sup> set (in infinite hybrid time) from the compact set  $\overline{\mathcal{O}}$  is bounded. It follows from [14, Prop 6.13] that there exists a compact set  $K_1$  such that  $\mathcal{R}_{\leq 2}(\overline{\mathcal{O}}) \subset K_1$ . Let  $T > 0$  be such that the condition of uniform recurrence holds from the set  $K_1$ . Then, using [14, Prop 6.13], it follows that there exists a set  $K_2$  such that  $\mathcal{R}_{\leq T}(K_1) \subset K_2$ . Then, we claim that  $\cup_{\tau \geq 0} \mathcal{R}_{\tau}(\overline{\mathcal{O}}) \subset K_2$ . Let  $\phi \in \mathcal{S}_{\mathcal{H}}(\overline{\mathcal{O}})$ . If  $t + j \leq T + 2$  for all  $(t, j) \in \text{dom } \phi$  then  $\phi(t, j) \in K_2$  for all  $(t, j) \in \text{dom } \phi$ . If not, there exists  $(t_1, j_1)$  such that  $t_1 + j_1 \in [1, 2]$  and  $\phi(s, i) \in K_1 \subset K_2$  for  $s + i \leq t_1 + j_1$  and  $\phi(t_1, j_1) \in \mathcal{O}$  or  $\phi(t_1, j_1) \in K_1 \setminus \mathcal{O}$ . For the second case, there exists  $(t_2, j_2)$  such that  $0 \leq t_2 + j_2 - (t_1 + j_1) \leq T$  such that  $\phi(t_2, j_2) \in \mathcal{O}$  and  $\phi(s, i) \in K_2$  for  $s + i \leq t_2 + j_2$ . We now iterate the same argument to prove that the reachable set from  $\overline{\mathcal{O}}$  is bounded. Next, we choose  $M > 0$  sufficiently large so that  $K_2 \subset M\mathbb{B}$ . Now let  $\Delta > 0$ . From the definition of uniform global recurrence there exists  $T > 0$  such that for each solution  $\phi \in \mathcal{S}_{\mathcal{H}}(\Delta\mathbb{B})$ , either  $t + j < T$  for all  $(t, j) \in \text{dom } \phi$  or there exists  $(t, j) \in \text{dom } \phi$  such that  $t + j \leq T$  and  $\phi(t, j) \in \mathcal{O}$ . Then, from  $\mathcal{O}$  being recurrent, and the reachable set from  $\mathcal{O}$  being bounded, it follows that there are no finite escape times and for every  $\phi \in \mathcal{S}_{\mathcal{H}}(\Delta\mathbb{B})$  either  $t + j < T$  for all  $(t, j) \in \text{dom } \phi$  or  $\phi(t, j) \in M\mathbb{B}$  for all  $(t, j) \in \text{dom } \phi$  satisfying  $t + j \geq T$ . This establishes uniform ultimate boundedness. ■

<sup>2</sup>The reachable set from a set  $S$  within hybrid time  $\tau$  is defined as  $\mathcal{R}_{\leq \tau}(S) := \{\phi(t, j) : \phi(0, 0) \in S \text{ and } t + j \leq \tau\}$ . The reachable set from a set  $S$  in infinite hybrid time is  $\cup_{\tau \geq 0} \mathcal{R}_{\tau}(S)$ .

## 2.4.2 Connection to asymptotic stability

Asymptotic stability is a widely studied property for dynamical systems. In this section, we adopt the definition of asymptotic stability of closed sets for hybrid systems from [28].

**Definition 2.4** *A closed set  $\mathcal{A}$  is uniformly globally stable (UGS) for  $\mathcal{H}$ , if there exists a class- $\mathcal{K}_\infty$  function  $\alpha$  such that for every solution  $\phi$  to  $\mathcal{H}$ ,  $|\phi(t, j)|_{\mathcal{A}} \leq \alpha(|\phi(0, 0)|_{\mathcal{A}})$  for every  $(t, j) \in \text{dom}(\phi)$ .*

**Definition 2.5** *A closed set  $\mathcal{A}$  is uniformly globally attractive for  $\mathcal{H}$ , if there are no finite escape times for (2.1a) and for every  $\varepsilon > 0, r > 0$  there exists a  $T > 0$  such that for every solution  $\phi$  to  $\mathcal{H}$  with  $|\phi(0, 0)|_{\mathcal{A}} \leq r$ ,  $(t, j) \in \text{dom}(\phi)$  and  $t + j \geq T$  imply  $|\phi(t, j)|_{\mathcal{A}} \leq \varepsilon$ .*

A closed set  $\mathcal{A}$  is *uniformly globally asymptotically stable* (UGAS) for  $\mathcal{H}$  if it is uniformly globally stable and uniformly globally attractive for  $\mathcal{H}$ . In [25], it is established that UGAS of a closed set can be expressed in terms of UGS and uniform global recurrence of open neighborhoods of the closed set. The following result is proved in [25, Prop 2.2] for a wide class of stochastic hybrid systems, so we only state the result here.

**Proposition 2.3** *If the closed set  $\mathcal{A} \subset \mathbb{R}^n$  is UGS for  $\mathcal{H}$  and, for every  $\varepsilon > 0$ , the open set  $\mathcal{A} + \varepsilon\mathbb{B}^o$  is uniformly globally recurrent for  $\mathcal{H}$ , then the set  $\mathcal{A}$  is UGAS for  $\mathcal{H}$ .*

In fact, the existence of an open, bounded recurrent set  $\mathcal{O}$  for  $\mathcal{H}$  implies that there exists a compact set  $\mathcal{A}$ , that is UGAS for  $\mathcal{H}$ . This is a consequence of the reachable set from  $\mathcal{O}$  being bounded and [14, Corollary 7.7]. The proof of the following result is presented in the Appendix.

**Proposition 2.4** *Let the open, bounded set  $\mathcal{O} \subset \mathbb{R}^n$  be globally recurrent for  $\mathcal{H}$  in (2.2). Then, there exists a compact set  $\mathcal{A}$  that is UGAS for  $\mathcal{H}$ .*

## 2.5 Invariance principle for recurrence

In this section we state weak sufficient conditions for recurrence in terms of non-strict Lyapunov-like functions. The invariance principle utilizes the existence of weak-Lyapunov function that is non-increasing along solutions outside the set  $\mathcal{O}$  and knowledge about behavior of solutions in level sets of this weak-Lyapunov function to conclude global recurrence. The invariance principle is proved in a subsequent chapter for a larger class of stochastic hybrid systems and hence we only state the result here. Let  $\mathcal{S}_{C \setminus \mathcal{O}}^F(x)$  refer to solutions of the constrained system  $\dot{x} \in F(x), x \in C \setminus \mathcal{O}$  from initial condition  $x$ .

**Definition 2.6** *A continuous function  $\widehat{V} : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$  is a weak-Lyapunov function relative to an open, bounded set  $\mathcal{O} \subset \mathbb{R}^n$  for  $\mathcal{H}$  in (2.2) if  $\widehat{V}$  is radially unbounded and satisfies*

$$\widehat{V}(\phi(t)) \leq \widehat{V}(x_0), \forall x_0 \in C \setminus \mathcal{O}, t \in \text{dom}(\phi), \phi \in \mathcal{S}_{C \setminus \mathcal{O}}^F(x_0) \quad (2.3)$$

$$\max_{g \in G(x_0) \cap (\mathbb{R}^n \setminus \mathcal{O})} \widehat{V}(g) \leq \widehat{V}(x_0), \forall x_0 \in D \setminus \mathcal{O}. \quad (2.4)$$

The conditions (2.3) and (2.4) state that the function  $\widehat{V}$  is non-increasing along flows outside the set  $\mathcal{O}$  and is non-increasing along jumps when the solutions are restricted to points outside the set  $\mathcal{O}$ . The next result states that by ruling out solutions that stay in level sets of the function  $\widehat{V}$  outside the set  $\mathcal{O}$ , global recurrence can be established without the existence of Lyapunov functions satisfying strict decrease properties.

**Theorem 2.1** *Let  $\widehat{V}$  be a weak-Lyapunov function relative to an open, bounded set  $\mathcal{O} \subset \mathbb{R}^n$  for the system  $\mathcal{H}$ . Then,  $\mathcal{O}$  is globally recurrent if and only if for every  $c \geq 0$  for which  $L_{\widehat{V}}(c) \cap (\mathbb{R}^n \setminus \mathcal{O})$  is non-empty there does not exist a complete solution  $\phi$  that remains in the set  $L_{\widehat{V}}(c) \cap (\mathbb{R}^n \setminus \mathcal{O})$ .*

The following sections will focus on establishing necessary and sufficient conditions for global recurrence using Lyapunov functions satisfying strict decrease conditions. Robustness of recurrence is crucial to establishing the equivalence between recurrence of open, bounded sets and the existence of smooth Lyapunov functions satisfying strict decrease conditions.

## 2.6 Robust global recurrence and a converse Lyapunov theorem

Robustness can be loosely defined as the stability property being preserved for the nominal system under sufficiently small perturbations. In this section we present results on robustness of recurrence to sufficiently small state dependent perturbations and also describe a Lyapunov function based characterization for recurrence that relies on strict decrease conditions along solutions.

We establish three types of robustness results. Firstly, we establish that recurrence of an open bounded set implies recurrence of a smaller open set within the original set. This type of result can be viewed as robustness to perturbations in the set. Secondly, we prove recurrence is preserved when the data of the hybrid system is modified to slow down recurrence. Slowing down the recurrence property loosely means that we make quantities related to the worst case first hitting time to the recurrent set for solutions from every initial condition increase with the distance of the initial condition to the recurrent set. Finally, we show that by perturbing the system data in a sufficiently small manner we preserve recurrence. This property establishes robustness of recurrence to measurement noise, additive disturbances and parameter uncertainty in system data.

The robustness results developed in this section will play an important role in es-



establishing necessary conditions for recurrence in terms of Lyapunov functions. We also illustrate using examples the importance of Standing Assumption 2.1 in issues relating to robustness and the existence of smooth Lyapunov functions.

### 2.6.1 Robustness of recurrence to state dependent inflations

We now establish a series of robustness results that will eventually be applied to the development of converse Lyapunov theorems. Firstly, since we do not insist on solutions to  $\mathcal{H}$  being complete we analyze an inflated system of  $\mathcal{H}$  for which maximal solutions are complete and for which recurrence properties are preserved. This inflated system is later used in the construction of a Lyapunov function to certify recurrence of an open, bounded set for  $\mathcal{H}$ . If the open, bounded set  $\mathcal{O}$  is globally recurrent for  $\mathcal{H}$ , consider the inflated system

$$\widehat{\mathcal{H}} := (C, F, \mathbb{R}^n, \widehat{G}), \quad (2.5)$$

where  $\widehat{G}(x) = G_1(x) \cup G_2(x)$  with  $G_1(x) = G(x)$  for  $x \in D$  and  $G_1(x) = \emptyset$  for  $x \notin D$ , and  $G_2(x) = x^*$  for some  $x^* \in \mathcal{O}$  and for all  $x \in \mathbb{R}^n$ . From the data of the hybrid system  $\widehat{\mathcal{H}}$  and recurrence of the set  $\mathcal{O}$  for  $\mathcal{H}$ , it follows that the maximal solutions of  $\widehat{\mathcal{H}}$  are complete.

**Lemma 2.1** *The data of the hybrid system  $\widehat{\mathcal{H}}$  in (2.5) satisfies Standing Assumption 2.1.*

*Proof:* Since  $\mathcal{H}$  satisfies Standing Assumption 2.1, only the outer semicontinuity and local boundedness of  $\widehat{G}$  needs to be verified. Since  $G_1$  and  $G_2$  are locally bounded, this implies the local boundedness of  $\widehat{G}$ . The outer semicontinuity of  $G_1$  follows from outer semicontinuity of  $G$  and the set  $D$  being closed. The mapping  $G_2$  is continuous.

The outer semicontinuity of  $\widehat{G}$  follows from [42, Proposition 2] since it is the union of two outer semicontinuous mappings. ■

**Lemma 2.2** *If the open, bounded set  $\mathcal{O}$  is globally recurrent for  $\mathcal{H}$  in (2.2), then  $\mathcal{O}$  is globally recurrent for  $\widehat{\mathcal{H}}$  in (2.5).*

*Proof:* Since the flow map for the hybrid system  $\widehat{\mathcal{H}}$  is the same as  $\mathcal{H}$ , the solutions generated by  $\widehat{\mathcal{H}}$  do not exhibit finite escape times. Let  $\psi$  be any solution to  $\widehat{\mathcal{H}}$ . If  $\psi$  is a solution of  $\mathcal{H}$ , then there exists  $(t, j)$  such that  $\psi(t, j) \in \mathcal{O}$ . If  $\psi$  is not a solution of  $\mathcal{H}$ , then there exists  $(t, j)$  such that  $\psi(t, j) = x^* \in \mathcal{O}$ . Hence global recurrence of  $\mathcal{O}$  for  $\widehat{\mathcal{H}}$  follows. ■

The next theorem states that recurrence of an open bounded set  $\mathcal{O}$  implies the existence of a smaller recurrent set inside  $\mathcal{O}$ . This result is primarily used to obtain a smooth Lyapunov function that certifies recurrence of  $\mathcal{O}$ .

**Theorem 2.2** *If the open, bounded set  $\mathcal{O} \subset \mathbb{R}^n$  is globally recurrent for  $\widehat{\mathcal{H}}$  in (2.5), then there exists  $\varepsilon > 0$  and an open set  $\widehat{\mathcal{O}}$  satisfying  $\widehat{\mathcal{O}} + \varepsilon\mathbb{B}^o \subset \mathcal{O}$  such that  $\widehat{\mathcal{O}}$  is globally recurrent for  $\widehat{\mathcal{H}}$ .*

*Proof:* We prove the theorem by contradiction. Suppose there does not exist a smaller globally recurrent set inside  $\mathcal{O}$ . Then, for every  $i \in \mathbb{Z}_{\geq 1}$ , there exists a complete solution  $\phi_i$  such that  $\phi_i(t, j) \in \mathbb{R}^n \setminus \mathcal{O} + 1/i\mathbb{B}$  for all  $(t, j) \in \text{dom}(\phi_i)$  and  $\phi_i(\hat{t}_i, \hat{j}_i) \in \mathcal{O}$  for some  $(\hat{t}_i, \hat{j}_i) \in \text{dom}(\phi_i)$ . We now define

$$t_i^* := \inf\{t : \phi_i(t, j) \in \overline{\mathcal{O}}\}, \quad j_i^* := \inf\{j : \phi_i(t, j) \in \overline{\mathcal{O}}\}.$$

Define new solutions  $\psi_i$  such that  $\psi_i(t, j) = \phi_i(t + t_i^*, j + j_i^*)$ . Hence  $\psi_i(0, 0) \in \overline{\mathcal{O}}$  for all  $i \in \mathbb{Z}_{\geq 1}$  and  $\psi_i(t, j) \in (\mathbb{R}^n \setminus \mathcal{O}) + (1/i)\mathbb{B}$  for all  $(t, j) \in \text{dom}(\psi_i)$ . Since  $\psi_i(0, 0) \in \overline{\mathcal{O}}$ ,

and there are no finite escape times due to recurrence of the set  $\mathcal{O}$ , it follows from [14, Prop 6.13] that the sequence of solutions  $\psi_i$  is locally eventually bounded. Then, from [41, Thm 4.4] there exists a subsequence (which we do not relabel) that converges to a solution  $\psi$  which is complete since  $\psi_i$  are complete. The solutions  $\psi_i$  stay in the closed set  $S_i := (\mathbb{R}^n \setminus \mathcal{O}) + 1/i\mathbb{B}$  for all time. Let  $(t, j) \in \text{dom}(\psi)$ . Then, there exists  $(t_i, j_i, \psi_i(t_i, j_i)) \rightarrow (t, j, \psi(t, j))$  with  $\psi_i(t_i, j_i) \in S_i$ . Then, the limit  $\psi(t, j) \in \mathbb{R}^n \setminus \mathcal{O}$ . Since  $(t, j)$  are arbitrary, this implies that the solution  $\psi$  stays in the set  $\mathbb{R}^n \setminus \mathcal{O}$  for all time. This contradicts the global recurrence of the set  $\mathcal{O}$ . ■

The next inflation of the data of  $\widehat{\mathcal{H}}$  results in preserving recurrence while slowing down the worst case first hitting time for solutions. This inflation will help in constructing a Lyapunov function that will be radially unbounded. If the open, bounded set  $\mathcal{O}$  is globally recurrent for  $\widehat{\mathcal{H}}$ , let  $x^* \in \mathcal{O}$  and define the continuous set-valued mapping  $M_\nu(x) := \{x^*\} + \nu(|x - x^*|)\mathbb{B}$  where  $\nu \in \mathcal{K}_\infty$ . Consider the inflated mapping

$$\widehat{\mathcal{H}}_\nu := (C, F, \mathbb{R}^n, \widehat{G}_\nu), \quad (2.6)$$

where  $\widehat{G}(x) = G_1(x) \cup M_\nu(x)$ . The proof of the next lemma is very similar to Lemma 2.1.

**Lemma 2.3** *For every  $\nu \in \mathcal{K}_\infty$ , the data of the hybrid system  $\widehat{\mathcal{H}}_\nu$  in (2.6) satisfies Standing Assumption 2.1.*

The next theorem claims the existence of a  $\nu \in \mathcal{K}_\infty$  small enough to preserve recurrence of the set  $\mathcal{O}$  for the inflated system  $\widehat{\mathcal{H}}_\nu$  if  $\mathcal{O}$  is globally recurrent for  $\widehat{\mathcal{H}}$ .

**Theorem 2.3** *If the open, bounded set  $\mathcal{O} \subset \mathbb{R}^n$  is globally recurrent for  $\widehat{\mathcal{H}}$  in (2.5), then there exists  $\nu \in \mathcal{K}_\infty$  such that  $\mathcal{O}$  is globally recurrent for  $\widehat{\mathcal{H}}_\nu$  in (2.6).*

*Proof:* Let  $S_i \subset \mathbb{R}^n$  be a sequence of compact sets such that  $S_i \subset S_{i+1}$ ,  $\cup_{i \in \mathbb{Z}_{\geq 0}} S_i = \mathbb{R}^n$  and  $S_0$  is a small neighborhood of  $x^*$  that is contained in the set  $\mathcal{O}$ . It follows from Proposition 2.2 that global recurrence of  $\mathcal{O}$  for  $\widehat{\mathcal{H}}$  implies that the reachable set (in infinite hybrid time) from  $\mathcal{O}$  is bounded. Uniform global recurrence of  $\mathcal{O}$  for  $\widehat{\mathcal{H}}$  implies that for every compact set  $S_i$ , there exists a time  $J_i$  such that solutions from  $S_i$  for the system  $\widehat{\mathcal{H}}$  reach the set  $\mathcal{O}$  within time  $J_i$ . Then, the reachable set (in infinite hybrid time) from  $S_i$  is  $\mathcal{R}(S_i) := \mathcal{R}_{\leq J_i}(S_i) \cup \Gamma$  where  $\mathcal{R}_{\leq J_i}(S_i)$  is the reachable set from  $S_i$  within time  $J_i$  and  $\Gamma$  is the reachable set from  $\mathcal{O}$  for the system  $\widehat{\mathcal{H}}$ . Since both  $\Gamma$  and  $\mathcal{R}_{\leq J_i}(S_i)$  are bounded,  $\mathcal{R}(S_i)$  is also bounded. Define  $\gamma_i := \sup_{x \in \mathcal{R}(S_i)} |x - x^*|$  and  $r_i = \inf_{x \in \partial S_{i-1}} |x - x^*|$  for  $i \in \mathbb{Z}_{\geq 1}$ . Let  $\nu \in \mathcal{K}_\infty$  be such that for every  $i \in \mathbb{Z}_{\geq 1}$ ,  $\nu(\gamma_i) < r_i/2$ .

We now claim that for each  $i \in \mathbb{Z}_{\geq 1}$ , every solution  $\phi \in \mathcal{S}_{\widehat{\mathcal{H}}_\nu}(S_i)$  there exists  $(t, j) \in \text{dom}(\phi)$  such that  $\phi(t, j) \in S_{i-1} \cup \mathcal{O}$ . Let a solution  $\phi \in \mathcal{S}_{\widehat{\mathcal{H}}_\nu}(S_i)$  be given. If the solution  $\phi$  can also be generated by  $\widehat{\mathcal{H}}$  from the set  $S_i$ , then global recurrence of  $\mathcal{O}$  for  $\widehat{\mathcal{H}}$  implies that there exists  $(t, j) \in \text{dom}(\phi)$  such that  $\phi(t, j) \in \mathcal{O}$ . If  $\phi$  is not a solution generated by  $\widehat{\mathcal{H}}$ , then there exists  $(t, j) \in \text{dom}(\phi)$  such that  $(t, j)$  is the first jump time satisfying  $\phi(t, j+1) \in \{x^*\} + \nu(|\phi(t, j)| - x^*)$ . Then, necessarily  $\phi(t, j) \in \mathcal{R}(S_i)$ . From the construction of the mapping  $\nu$ , it follows that  $\phi(t, j+1) \in S_{i-1}$ . This establishes the claim.

We now establish that for every solution  $\phi \in \mathcal{S}_{\widehat{\mathcal{H}}_\nu}(\mathbb{R}^n)$ , there exists  $(t, j) \in \text{dom}(\phi)$  such that  $\phi(t, j) \in \mathcal{O}$ . Let the  $\phi \in \mathcal{S}_{\widehat{\mathcal{H}}_\nu}(\mathbb{R}^n)$  be given. Then, there exists  $i \in \mathbb{Z}_{\geq 1}$  such that  $\phi(0, 0) \in S_i$ . If  $\phi$  is a solution of  $\widehat{\mathcal{H}}$ , the result follows from global recurrence of  $\mathcal{O}$  for  $\mathcal{H}$ . If  $\phi$  is not a solution of  $\widehat{\mathcal{H}}$ , it follows from the above claim that there exists  $(t_i, j_i) \in \text{dom}(\phi)$  such that  $\phi(t_i, j_i) \in S_{i-1}$ . We now apply the result of the above claim in an iterative manner from the set  $S_{i-1}$  till the solution  $\phi$  reaches the set  $S_0$ . Hence, there exists positive constants  $\{t_k\}_{k=0}^{i-1}$  and  $\{j_k\}_{k=0}^{i-1}$  such that  $\phi(\sum_{k=j}^i t_k, \sum_{k=j}^i j_k) \in S_{j-1}$  for every  $j \in \{1, \dots, i\}$ . Since  $S_0 \subset \mathcal{O}$ , it follows that  $\phi(\sum_{k=1}^i t_k, \sum_{k=1}^i j_k) \in S_0 \subset \mathcal{O}$ . This

establishes global recurrence of the set  $\mathcal{O}$  for the hybrid system  $\widehat{\mathcal{H}}_\nu$ . ■

Finally, we introduce state dependent perturbations typically used in robustness analysis. For a continuous, positive valued function  $\delta : \mathbb{R}^n \rightarrow \mathbb{R}_{>0}$ , we denote the perturbed version of  $\widehat{\mathcal{H}}$  by

$$\widehat{\mathcal{H}}_\delta := (C_\delta, F_\delta, \mathbb{R}^n, \widehat{G}_\delta), \quad (2.7)$$

where

$$\begin{aligned} C_\delta &:= \{x \in \mathbb{R}^n : (x + \delta(x)\mathbb{B}) \cap C \neq \emptyset\} \\ F_\delta &:= \overline{\text{con}}F((x + \delta(x)\mathbb{B}) \cap C) + \delta(x)\mathbb{B} \\ \widehat{G}_\delta &:= \{v \in \mathbb{R}^n : v \in g + \delta(g)\mathbb{B}, g \in \widehat{G}(x + \delta(x)\mathbb{B})\} \end{aligned}$$

and  $\overline{\text{con}}$  refers to the closed convex hull. The next result follows from [14, Proposition 6.28].

**Lemma 2.4** *For every continuous  $\delta : \mathbb{R}^n \rightarrow \mathbb{R}_{>0}$ , the data of the hybrid system  $\widehat{\mathcal{H}}_\delta$  in (2.7) satisfies Standing Assumption 2.1.*

We now establish that the recurrence property is robust to sufficiently small state dependent perturbation. So, under the regularity conditions in Standing Assumption 2.1 if there exists an open, bounded set that is globally recurrent for the nominal system, then there exists a sufficiently small perturbation such that the recurrence property is preserved for the inflated system  $\widehat{\mathcal{H}}_\delta$  in (2.7). The following lemma will be used to construct the state dependent perturbation in the robustness result.

**Lemma 2.5** *Let the open, bounded set  $\mathcal{O}$  be globally recurrent for  $\widehat{\mathcal{H}}$  in (2.5). Then, for each compact set  $K$ , there exists  $\delta > 0$  such that every solution from the set  $K$  reaches*

the set  $\mathcal{O}$  for  $\widehat{\mathcal{H}}_\delta$  in (2.7).

*Proof:* We establish the result by contradiction. Suppose the statement of the lemma is not true, then there exists a compact set  $K$  such that for every  $i \in \mathbb{Z}_{\geq 1}$ , there exists a solution  $\phi_i \in \mathcal{S}_{\widehat{\mathcal{H}}_{1/i}}(K)$  such that  $\phi_i(t, j) \notin \mathcal{O}$  for every  $(t, j) \in \text{dom}(\phi_i)$ . Since the hybrid system is well posed [14, Def 6.29, Thm 6.30], it follows from [14, Thm 6.1, Prop 6.33] that the sequence of solutions  $\phi_i$  has a subsequence which we do not relabel that converges to the solution  $\phi \in \mathcal{S}_{\widehat{\mathcal{H}}}(K)$  for the nominal system. Since the set  $\mathcal{O}$  is globally recurrent for the nominal system, there exists  $(t, j)$  such that  $\phi(t, j) \in \mathcal{O}$ . Then from convergence of hybrid arcs, there exists a sequence  $(t_i, j_i, \phi_i(t_i, j_i))$  that converges to  $(t, j, \phi(t, j))$ . Then, since the set  $\mathcal{O}$  is open, for large enough  $i$ ,  $\phi_i(t_i, j_i) \in \mathcal{O}$ , which contradicts the initial assumption.  $\blacksquare$

**Theorem 2.4** *If the open, bounded set  $\mathcal{O} \subset \mathbb{R}^n$  is globally recurrent for  $\widehat{\mathcal{H}}$  in (2.5), then there exists a continuous function  $\delta : \mathbb{R}^n \rightarrow \mathbb{R}_{>0}$  such that  $\mathcal{O}$  is globally recurrent for  $\widehat{\mathcal{H}}_\delta$  in (2.7).*

*Proof:* It follows from Proposition 2.4 that global recurrence of  $\mathcal{O}$  for  $\widehat{\mathcal{H}}$  implies that there exists a compact set  $\mathcal{A}$  that is uniformly globally asymptotically stable for  $\widehat{\mathcal{H}}$ . Then, from [14, Thm 7.21], it follows that there exists a continuous function  $\rho_1 : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$  that is zero on the set  $\mathcal{A}$  and positive elsewhere such that  $\mathcal{A}$  is uniformly globally asymptotically stable for  $\widehat{\mathcal{H}}_{\rho_1}$ . Let  $\varepsilon \in (0, 1)$ . It follows from [14, Lemma 7.20] that there exists a constant  $\rho_2 > 0$  such that solutions of  $\widehat{\mathcal{H}}_{\rho_2}$  from the compact set  $\mathcal{A} + \mathbb{B}$  converge to  $\mathcal{A} + \varepsilon\mathbb{B}$ . Now choose a continuous function  $\rho : \mathbb{R}^n \rightarrow \mathbb{R}_{>0}$  such that  $\rho(x) \leq \rho_2$  for  $x \in \mathcal{A} + \mathbb{B}$  and  $\rho(x) \leq \min\{\rho_1(x), \rho_2\}$  otherwise. Then, it follows that solutions of the system  $\widehat{\mathcal{H}}_\rho$  reach the open set  $\mathcal{A} + \mathbb{B}^\circ$ .

Let  $\delta_1 > 0$  satisfy the condition of Lemma 2.5 for the compact set  $K = \mathcal{A} + \mathbb{B}$ . Let  $\delta : \mathbb{R}^n \rightarrow \mathbb{R}_{>0}$  be a continuous function which satisfies  $\delta(x) \leq \min\{\delta_1, \rho(x)\}$ . Let  $\phi \in$

$\mathcal{S}_{\widehat{\mathcal{H}}_\delta}(\mathbb{R}^n)$ . Since  $\delta(x) \leq \rho(x)$  for all  $x \in \mathbb{R}^n$ , it follows that there exists  $(t_1, j_1) \in \text{dom}(\phi)$  such that  $\phi(t_1, j_1) \in \mathcal{A} + \mathbb{B}^\rho$ . Define a solution  $\psi(t, j) := \phi(t_1 + t, j_1 + j)$  for all  $(t, j)$  such that  $(t_1 + t, j_1 + j) \in \text{dom}(\phi)$ . From Lemma 2.5 there exists  $(t_2, j_2) \in \text{dom}(\psi)$  such that  $\psi(t_2, j_2) \in \mathcal{O}$ . Hence, we have  $\phi(t_1 + t_2, j_1 + j_2) \in \mathcal{O}$ . This establishes global recurrence of the set  $\mathcal{O}$  for  $\widehat{\mathcal{H}}_\delta$ .  $\blacksquare$

The following example illustrates that recurrence is not necessarily robust to arbitrarily small perturbations if the conditions of Standing Assumption 2.1 are not satisfied.

**Example 2.3** Consider the system in Example 2.1. The Krasovskii regularization of the mapping  $g$  (the smallest closed set that contains all the limit points of  $g$ ) defined as  $G(x) := \bigcap_{\rho > 0} \overline{g(x + \rho\mathbb{B})}$  is set-valued at the point of discontinuity  $x = 1$ . So for  $x = 1$  we have  $G(1) = \{0, 1\}$ . Hence, when the set  $\mathcal{O}$  is a small neighborhood of the origin, the point  $x = 1$  becomes a fixed point and hence even for arbitrarily small perturbations, the recurrence property fails and consequently the set  $\mathcal{O}$  is not robustly recurrent for  $\mathcal{H}$ .

The next example shows that the recurrence property is not necessarily robust if the set  $\mathcal{O}$  is not open.

**Example 2.4** Consider the simple discrete-time system  $x^+ = 0$  for  $x \in \mathbb{R}$ . Let the set  $\mathcal{O} = \{0\}$ . Global recurrence of the set  $\mathcal{O}$  follows from the system dynamics. Let  $\delta : \mathbb{R} \rightarrow \mathbb{R}_{>0}$  be any continuous function. Then, the perturbed system is represented as  $x^+ \in \delta(x)\mathbb{B}$ . Since  $\delta(x) > 0$ , it is clear that not all solutions of the perturbed system reach the set  $\mathcal{O}$  and hence the set  $\mathcal{O}$  is not robustly recurrent.

## 2.6.2 Necessary and sufficient conditions for recurrence

Converse Lyapunov theorems are used to establish the equivalence between asymptotic stability properties and the existence of Lyapunov-like functions that satisfy certain

decrease conditions along solutions. Applications of converse theorems in stabilization and robust stability analysis can be found in [43], [44] and [11]. For continuous-time systems converse theorems related to asymptotic stability are established in [11], [12] and [6]. See [9], [7] and [13] for similar results in the discrete-time case. Converse theorems for asymptotic stability of compact sets for a class of hybrid systems is established in [45] and [46].

Converse theorems for recurrence in discrete-time deterministic systems is developed in [22, Thm 11.2.1] although the Lyapunov function generated is merely upper semicontinuous. Using the robustness of recurrence to various state dependent perturbations we construct a smooth Lyapunov function for the converse theorem. We exploit the robustness to go from a preliminary non-smooth Lyapunov function to a smooth Lyapunov function for recurrence by utilizing the construction in [45].

We now present a Lyapunov function based characterization of recurrence for open, bounded sets.

**Definition 2.7** *A smooth function  $V : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$  is a Lyapunov function with respect to an open, bounded set  $\mathcal{O}$  for  $\mathcal{H}$  if it is radially unbounded and there exists  $\mu > 0$  such that*

$$\langle \nabla V(x), f \rangle \leq -1 + \mu \mathbb{I}_{\mathcal{O}}(x), \quad \forall x \in C, f \in F(x) \quad (2.8)$$

$$V(g) - V(x) \leq -1 + \mu \mathbb{I}_{\mathcal{O}}(x), \quad \forall x \in D, g \in G(x). \quad (2.9)$$

In essence, the Lyapunov function in (2.8) and (2.9) satisfies a strict decrease condition along solutions outside the set  $\mathcal{O}$  as opposed to the non-strict decrease conditions in (2.3) and (2.4). The robustness results are utilized to establish a converse Lyapunov theorem for recurrence of an open, bounded set for the hybrid system  $\mathcal{H}$ .

The outline of the construction of the Lyapunov function used in the converse theorem



is as follows. Under the assumption that an open, bounded set  $\mathcal{O}$  is globally recurrent for  $\mathcal{H}$  in (2.2), we apply the robustness results to establish that  $\widehat{\mathcal{O}} \subset \mathcal{O} + \varepsilon\mathbb{B}^o$  is globally recurrent for the system  $\widehat{\mathcal{H}}_{\nu,\delta}$  for some  $\nu \in \mathcal{K}_\infty$ ,  $\varepsilon > 0$  and a continuous, positive function  $\delta$ . The system  $\widehat{\mathcal{H}}_{\nu,\delta}$  is a perturbed version of the system  $\widehat{\mathcal{H}}_\nu$  in (2.6). We then construct a preliminary (possibly non-smooth) Lyapunov function  $V_0$  to certify recurrence of the set  $\widehat{\mathcal{O}}$  for  $\widehat{\mathcal{H}}_{\nu,\delta}$ . The construction of  $V_0$  is related to worst-case first hitting for solutions to the set  $\mathcal{O}$ . A similar construction is used for discrete-time systems in [22]. The function  $V_0$  is then smoothed to arrive at a smooth Lyapunov function  $V$  to certify recurrence of  $\mathcal{O}$  for the system  $\mathcal{H}$  in (2.2).

We now state a necessary and sufficient condition for global recurrence of an open, bounded set for  $\mathcal{H}$ . The proof of the following theorem is in Section 2.8.

**Theorem 2.5** *The open, bounded set  $\mathcal{O} \subset \mathbb{R}^n$  is globally recurrent for  $\mathcal{H}$  in (2.2) if and only if there exists a smooth Lyapunov function relative to  $\mathcal{O}$  for  $\mathcal{H}$ .*

The following result is a simple corollary of Theorem 2.5 and Proposition 2.2 that establishes the equivalence between the solutions of  $\mathcal{H}$  being uniformly ultimately bounded and the existence of a smooth Lyapunov function that satisfies in (2.8) and (2.9) with respect to an open, bounded set  $\mathcal{O}$ .

**Corollary 2.1** *The solutions of  $\mathcal{H}$  in (2.2) are uniformly ultimately bounded if and only if there exists an open, bounded set  $\mathcal{O}$  and a smooth Lyapunov function that satisfies (2.8) and (2.9) with respect to  $\mathcal{O}$  for  $\mathcal{H}$ .*

The following example illustrates that the existence of even a continuous Lyapunov functions is not necessarily guaranteed without the conditions of Standing Assumption 2.1.

**Example 2.5** Consider the system in Example 2.1. We show that there does not exist a continuous Lyapunov function with respect to the set  $\mathcal{O} = (-\varepsilon, \varepsilon)$  for any  $\varepsilon \in (0, 1)$ . We establish the claim by contradiction. Suppose  $V$  is a continuous Lyapunov function. Then,  $V(g(x)) \leq V(x) - 1$  for  $x \in \mathbb{R}^n \setminus \mathcal{O}$ . Let  $x_i \in [\varepsilon, 1)$  be such that  $\lim_{i \rightarrow \infty} x_i = 1$ . Then  $\lim_{i \rightarrow \infty} g(x_i) = 1$  even though  $g(1) = 0$ . Then,  $\lim_{i \rightarrow \infty} V(g(x_i)) - V(x_i) = 0$ , which contradicts the strict decrease condition of  $V$ . Hence, there does not exist a continuous Lyapunov function for the system. This is due to the non-robust nature of recurrence.

## 2.7 Hitting time to open sets - an equivalent characterization

We now establish an equivalent characterization for recurrence in terms of functions that will be used to construct the Lyapunov function in the converse theorem. Let  $\mathcal{O} \subset \mathbb{R}^n$  be an open set. Let  $\tilde{\mathcal{H}}$  be a hybrid system whose data satisfies the conditions of Standing Assumption 2.1, and for which the maximal solutions are complete. We define for every hybrid arc  $\phi \in \mathcal{S}_{\tilde{\mathcal{H}}}(\mathbb{R}^n)$  the function

$$W^{\mathcal{O}}(\phi) := \inf_{(t,j) \in \text{dom}(\phi), \phi(t,j) \in \mathcal{O}} (t + j). \quad (2.10)$$

For every  $\phi$ ,  $W^{\mathcal{O}}(\phi)$  is related to the first time the solution  $\phi$  hits the set  $\mathcal{O}$ . If the solution  $\phi$  never hits the set  $\mathcal{O}$ , then  $W^{\mathcal{O}}(\phi) = \infty$ . The worst first hitting time to the set  $\mathcal{O}$  from the initial condition  $x$  for the system  $\tilde{\mathcal{H}}$  is then related to the quantity  $\sup_{\phi \in \mathcal{S}_{\tilde{\mathcal{H}}}(x)} W^{\mathcal{O}}(\phi)$ . In [47] a robust boundedness problem is studied for continuous-time systems that uses the notion of first hitting times to certain forward invariant compact sets.

The following result then establishes the connection between global recurrence and the first hitting times and the proof follows from the definition of recurrence and Proposition

2.1.

**Proposition 2.5** *Let  $\mathcal{O} \subset \mathbb{R}^n$  be open and bounded. Then, the following statements are equivalent.*

1.  $\mathcal{O}$  is globally recurrent for  $\tilde{\mathcal{H}}$ .
2.  $\mathcal{O}$  is uniformly globally recurrent  $\tilde{\mathcal{H}}$ .
3. For every compact set  $K \subset \mathbb{R}^n$ , there exists  $T_K > 0$  such that  $\sup_{\phi \in \mathcal{S}_{\tilde{\mathcal{H}}}(K)} W^{\mathcal{O}}(\phi) \leq T_K$ .
4. For every  $x \in \mathbb{R}^n$  and  $\phi \in \mathcal{S}_{\tilde{\mathcal{H}}}(x)$ ,  $W^{\mathcal{O}}(\phi) < \infty$ .

*Proof:* 1)  $\Rightarrow$  2) The implication follows from Proposition 2.1. 2)  $\Rightarrow$  3) Since the set  $\mathcal{O}$  is uniformly globally recurrent, this implies that for every compact  $K$ , there exists  $T_K > 0$  such that for every  $\phi \in \mathcal{S}_{\tilde{\mathcal{H}}}(K)$  there exists  $(t, j)$  such that  $\phi(t, j) \in \mathcal{O}$  and  $t + j \leq T_K$ . This implies that for every  $\phi \in \mathcal{S}_{\tilde{\mathcal{H}}}(K)$ ,  $W^{\mathcal{O}}(\phi) \leq T_K$ . 3)  $\Rightarrow$  4) This implication is trivial. 4)  $\Rightarrow$  1) For any  $\phi$  such that  $W^{\mathcal{O}}(\phi) < \infty$  and every  $\varepsilon > 0$  there exists  $(t, j)$  such that  $t + j \leq W^{\mathcal{O}}(\phi) + \varepsilon$  and  $\phi(t, j) \in \mathcal{O}$ . This establishes global recurrence of the set  $\mathcal{O}$ . ■

From now on we will use (2.10) only to characterize recurrence in hybrid systems for which the maximal solutions are complete. Hence, in the later sections the results of Proposition 2.5 will be applied to  $\hat{\mathcal{H}}$ ,  $\hat{\mathcal{H}}_\nu$  or  $\hat{\mathcal{H}}_\delta$ . The next result establishes that the worst case hitting time to open sets is an upper semicontinuous function.

**Proposition 2.6** *If the open, bounded set  $\mathcal{O} \subset \mathbb{R}^n$  is globally recurrent for  $\tilde{\mathcal{H}}$ , then the mapping  $x \mapsto \sup_{\phi \in \mathcal{S}_{\tilde{\mathcal{H}}}(x)} W^{\mathcal{O}}(\phi)$  is well defined and upper semicontinuous.*

*Proof:* Local boundedness of the mapping follows from Propositions 2.1 and 2.5 and hence, the mapping  $x \mapsto \sup_{\phi \in \mathcal{S}_{\tilde{\mathcal{H}}}(x)} W^{\mathcal{O}}(\phi)$  is well defined. Let  $\phi_i$  be a sequence

of hybrid arcs that converges graphically to the hybrid arc  $\phi$ . We will establish that  $\limsup_{i \rightarrow \infty} W^{\mathcal{O}}(\phi_i) \leq W^{\mathcal{O}}(\phi)$ . Let  $\varepsilon > 0$  be arbitrary. Let  $(t, j)$  be such that  $t + j \leq W^{\mathcal{O}}(\phi) + \varepsilon/2$  and  $\phi(t, j) \in \mathcal{O}$ . Since  $\phi_i$  converge to  $\phi$  graphically, there exist sequences  $(t_i, j_i) \rightarrow (t, j)$  and  $\phi_i(t_i, j_i) \rightarrow \phi(t, j)$ . Since  $\mathcal{O}$  is open, for  $i$  large enough  $\phi_i(t_i, j_i) \in \mathcal{O}$ . Hence, without loss of generality for  $i$  large enough we have  $W^{\mathcal{O}}(\phi_i) \leq t_i + j_i$  and  $t_i + j_i \leq t + j + \varepsilon/2$ . This implies that

$$\limsup_{i \rightarrow \infty} W^{\mathcal{O}}(\phi_i) \leq t_i + j_i \leq t + j + \varepsilon/2 \leq W^{\mathcal{O}}(\phi) + \varepsilon.$$

Since  $\varepsilon > 0$  is arbitrary, it follows that  $\limsup_{i \rightarrow \infty} W^{\mathcal{O}}(\phi_i) \leq W^{\mathcal{O}}(\phi)$ . The upper semi-continuity of the mapping  $x \mapsto \sup_{\phi \in \mathcal{S}_{\mathcal{H}}(x)} W^{\mathcal{O}}(\phi)$  follows directly from the proof of [25, Lemma 8.3] using [41, Thm 4.4]. ■

## 2.8 Proof of Theorem 2.5

### 2.8.1 Necessity

#### Preliminary Lyapunov function

In this section, we will construct a Lyapunov function for an inflated system which while not necessarily smooth satisfies good regularity properties and decrease conditions. The robustness results from the previous sections will then be used to establish the smoothness of the Lyapunov function.

Let the open, bounded set  $\mathcal{O} \subset \mathbb{R}^n$  be globally recurrent for the system  $\mathcal{H}$ . Then from Lemma 2.1,  $\mathcal{O}$  is globally recurrent for the system  $\widehat{\mathcal{H}}$ . From Theorem 2.2, there exists a smaller open set  $\widehat{\mathcal{O}}_2$  such that  $\widehat{\mathcal{O}}_2 + \varepsilon_2 \mathbb{B}^o \subset \mathcal{O}$  for some  $\varepsilon_2 > 0$ . From Theorem 2.3, there exists a  $\mathcal{K}_\infty$  function  $\nu$  such that  $\widehat{\mathcal{O}}_2$  is globally recurrent for  $\widehat{\mathcal{H}}_\nu$ . Finally, from Theorem

2.4, there exists a continuous state dependent perturbation  $\delta : \mathbb{R}^n \rightarrow \mathbb{R}_{>0}$  such that  $\widehat{\mathcal{O}}_2$  is globally recurrent for  $\widehat{\mathcal{H}}_{\nu,\delta}$ . We will construct a preliminary Lyapunov function using the solutions of the system  $\widehat{\mathcal{H}}_{\nu,\delta}$ . Define  $V_0 : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$  as

$$V_0(x) := \sup_{\phi \in \mathcal{S}_{\widehat{\mathcal{H}}_{\nu,\delta}}(x)} W^{\widehat{\mathcal{O}}_2}(\phi). \quad (2.11)$$

**Proposition 2.7** *The function  $V_0$  is radially unbounded and upper semicontinuous.*

*Proof:* The upper semicontinuity of  $V_0$  follows from Proposition 2.6. We just need to establish the radial unboundedness of the function  $V_0$ . Let  $Q_i = \overline{\widehat{\mathcal{O}}_2 + (i+1)\mathbb{B}^\circ} \setminus (\widehat{\mathcal{O}}_2 + i\mathbb{B}^\circ)$  be a sequence of compact sets for  $i \in \mathbb{Z}_{\geq 0}$ . We now consider the solutions of the system  $x^+ \in M_\nu(x) = x^* + \nu(|x - x^*|)\mathbb{B}$  for  $x \in \mathbb{R}^n$ . It can be observed that solutions generated by  $M_\nu$  are a subset of the solutions generated by  $\widehat{\mathcal{H}}_{\nu,\delta}$ . Let  $\alpha_i \in \mathbb{R}_{>0}$  for each  $i \in \mathbb{Z}_{\geq 0}$  be such that  $\alpha_i = \inf_{x \in Q_i} \sup_{\phi \in \mathcal{S}_{M_\nu}(x)} W^{\widehat{\mathcal{O}}_2}(\phi)$ . Since  $\nu \in \mathcal{K}_\infty$ , it follows from the structure of  $Q_i$  and  $M_\nu$  that the mapping  $i \mapsto \alpha_i$  is increasing and unbounded. Then the worst first hitting times generated by solutions hybrid system  $\widehat{\mathcal{H}}_{\nu,\delta}$  necessarily satisfies  $\alpha_i \leq \sup_{\phi \in \mathcal{S}_{\widehat{\mathcal{H}}_{\nu,\delta}}(x)} W^{\widehat{\mathcal{O}}_2}(\phi)$  for  $x \in Q_i$ . As  $i \rightarrow \infty$ , then it follows that  $V_0$  is radially unbounded. ■

**Proposition 2.8** *Let  $\phi \in \mathcal{S}_{\widehat{\mathcal{H}}_{\nu,\delta}}(x)$  be such that  $\text{graph}(\phi) \cap ([0, t] \times \{0, \dots, \max\{0, j-1\}\} \times \mathbb{R}^n) \subset \mathbb{R}^2 \times \mathbb{R}^n \setminus \widehat{\mathcal{O}}_2$  for some  $(t, j) \in \text{dom}(\phi)$  then*

$$V_0(\phi(t, j)) \leq V_0(x) - (t + j). \quad (2.12)$$

*Proof:*

Let  $\hat{\phi}^* \in \mathcal{S}_{\widehat{\mathcal{H}}_{\nu,\delta}}(\phi(t, j))$  be a solution such that  $W^{\widehat{\mathcal{O}}_2}(\hat{\phi}^*) = \sup_{\hat{\phi} \in \mathcal{S}_{\widehat{\mathcal{H}}_{\nu,\delta}}(\phi(t, j))} W^{\widehat{\mathcal{O}}_2}(\hat{\phi})$ . Such a solution  $\hat{\phi}^*$  exists due to regularity properties of hitting time function  $W^{\widehat{\mathcal{O}}_2}$  and

similar reasoning used in the proof of [25, Lemma 8.3]. Now let  $\phi^* \in \mathcal{S}_{\widehat{\mathcal{H}}_{\nu,\delta}}(x)$  be a solution satisfying  $\phi^*(\tilde{t}, \tilde{j}) = \phi(\tilde{t}, \tilde{j})$  for  $\tilde{t} \leq t$ ,  $\tilde{j} \leq j$  and  $\phi^*(t + \tilde{t}, j + \tilde{j}) = \hat{\phi}^*(\tilde{t}, \tilde{j})$  otherwise. We consider two cases depending on  $\phi(t, j)$ . First, let  $\phi(t, j) \in \mathbb{R}^n \setminus \widehat{\mathcal{O}}_2$ . Then, it can be observed that  $W^{\widehat{\mathcal{O}}_2}(\phi^*) = W^{\widehat{\mathcal{O}}_2}(\hat{\phi}^*) + t + j$ . Next, we note that if  $\phi(t, j) \in \widehat{\mathcal{O}}_2$ , then from the assumptions on the solution  $\phi$  we have  $W^{\widehat{\mathcal{O}}_2}(\hat{\phi}^*) = 0$ ,  $W^{\widehat{\mathcal{O}}_2}(\phi^*) = t + j$  and  $W^{\widehat{\mathcal{O}}_2}(\phi^*) = W^{\widehat{\mathcal{O}}_2}(\hat{\phi}^*) + t + j$ . Hence, it follows that

$$\begin{aligned} V_0(\phi(t, j)) &= \sup_{\hat{\phi} \in \mathcal{S}_{\widehat{\mathcal{H}}_{\nu,\delta}}(\phi(t, j))} W^{\widehat{\mathcal{O}}_2}(\hat{\phi}) \\ &= W^{\widehat{\mathcal{O}}_2}(\hat{\phi}^*) = W^{\widehat{\mathcal{O}}_2}(\phi^*) - t - j \\ &\leq \sup_{\phi \in \mathcal{S}_{\widehat{\mathcal{H}}_{\nu,\delta}}(x)} W^{\widehat{\mathcal{O}}_2}(\phi) - (t + j) = V_0(x) - (t + j). \end{aligned}$$

■

### Smoothing of Lyapunov function

Now choose the open set  $\widehat{\mathcal{O}}_1$  such that  $\widehat{\mathcal{O}}_2 + (\varepsilon_2/3)\mathbb{B}^o \subset \widehat{\mathcal{O}}_1$  and  $\widehat{\mathcal{O}}_1 + (\varepsilon_2/3)\mathbb{B}^o \subset \mathcal{O}$ . It follows that  $\widehat{\mathcal{O}}_1$  is also globally recurrent for  $\widehat{\mathcal{H}}_{\nu,\delta}$ . Define  $\rho(x) := \min\{\delta(x), \varepsilon_2/3\}$ . Let  $\varrho$  come from [14, Lemma 7.37] using  $\rho$ . Then, the function  $\varrho$  is continuous and positive on bounded sets. We can also conclude that if  $x \in \mathbb{R}^n \setminus \widehat{\mathcal{O}}_1$ , then  $x + \varrho(x)\mathbb{B} \subset \mathbb{R}^n \setminus \widehat{\mathcal{O}}_2$ . Let  $\Psi : \mathbb{R}^n \rightarrow [0, 1]$  be any infinitely differentiable function such that  $\Psi(x) = 0$  for  $x \notin \mathbb{B}$  and  $\int \Psi(x) dx = 1$ .

Now define  $V(x) := \int_{\mathbb{R}^n} V_0(x + \varrho(x)\eta)\Psi(\eta)d\eta$ . The local boundedness and radial unboundedness follows from the properties of  $V_0$ . The smoothness of  $V$  on  $\mathbb{R}^n$ , follows from the results in [14, Section 7.36].

From [14, Lemma 7.37], for every  $\phi \in \mathcal{S}_{\widehat{\mathcal{H}}_{\nu,\rho}}(\mathbb{R}^n)$  and  $\eta \in \mathbb{B}$ , there exists  $\psi_\eta \in \mathcal{S}_{\widehat{\mathcal{H}}_{\nu,\rho}}(\mathbb{R}^n)$  such that  $\text{dom}(\phi) = \text{dom}(\psi)$ ,  $\psi_\eta(0, 0) = \phi(0, 0) + \varrho(\phi(0, 0))\eta$  and  $\psi_\eta(t, j) =$

$\phi(t, j) + \varrho(\phi(t, j))\eta$ . Let  $\phi$  be solution of  $\widehat{\mathcal{H}}_{\nu, \varrho}$  from  $x$  such that  $\text{graph}(\phi) \cap ([0, t] \times \{0, \dots, \max\{0, j-1\}\} \times \mathbb{R}^n) \subset \mathbb{R}^2 \times \mathbb{R}^n \setminus \widehat{\mathcal{O}}_1$  for some  $(t, j) \in \text{dom}(\phi)$ . It follows from the properties of  $\varrho$ , [14, Lemma 7.37] and Proposition 2.8 that

$$\begin{aligned} V(\phi(t, j)) &= \int_{\mathbb{R}^n} V_0(\phi(t, j) + \varrho(\phi(t, j))\eta)\Psi(\eta)d\eta = \int_{\mathbb{R}^n} V_0(\psi_\eta(t, j))\Psi(\eta)d\eta \\ &\leq \int_{\mathbb{R}^n} V_0(\psi_\eta(0, 0))\Psi(\eta)d\eta - (t + j) \\ &= \int_{\mathbb{R}^n} V_0(x + \varrho(x)\eta)\Psi(\eta)d\eta - (t + j) = V(x) - (t + j). \end{aligned} \quad (2.13)$$

Then, from [45, Claim 6.3] and (2.13) it follows that for every  $x \in C \cap (\mathbb{R}^n \setminus \mathcal{O})$  and  $f \in F(x)$  and small  $t \geq 0$ ,  $V(x + tf) \leq V(x) - t$ . The smoothness of  $V$  implies that for every  $x \in C \cap (\mathbb{R}^n \setminus \mathcal{O})$  and  $f \in F(x)$   $\langle \nabla V(x), f \rangle \leq -1$ . Similarly for  $x \in D \cap (\mathbb{R}^n \setminus \mathcal{O})$ ,  $\phi(0, 1) = g$  we have  $V(g) \leq V(x) - 1$  for  $g \in G(x)$ .

Finally, we establish the existence of  $\mu > 0$  such that (2.8) and (2.9) hold. Since  $V$  is smooth,  $\mathcal{O}$  is bounded, and  $F$  is locally bounded, there exists  $\mu_1 > 0$  such that  $\langle \nabla V(x), f \rangle \leq \mu_1$  for all  $x \in C \cap \mathcal{O}$  and  $f \in F(x)$ . Similarly local boundedness of  $G$  implies the existence of  $\mu_2 > 0$  such that  $\max_{g \in G(x)} V(g) - V(x) \leq \mu_2$  for all  $x \in D \cap \mathcal{O}$ . Then conditions (2.8) and (2.9) hold with  $\mu = \max\{\mu_1, \mu_2\}$ . An illustration of the development of the converse theorem from the robustness results is presented in Figure 2.4.

### 2.8.2 Sufficiency

The proof of sufficiency follows from observing that the existence of a  $V$  satisfying (2.8) and (2.9) satisfies the conditions of Theorem 2.1. However, we will present a proof without appealing to the invariance principle in Theorem 2.1. Since  $V$  is radially unbounded and  $\mathcal{O}$  is bounded, there are no finite escape times for the solutions of  $\mathcal{H}$ .

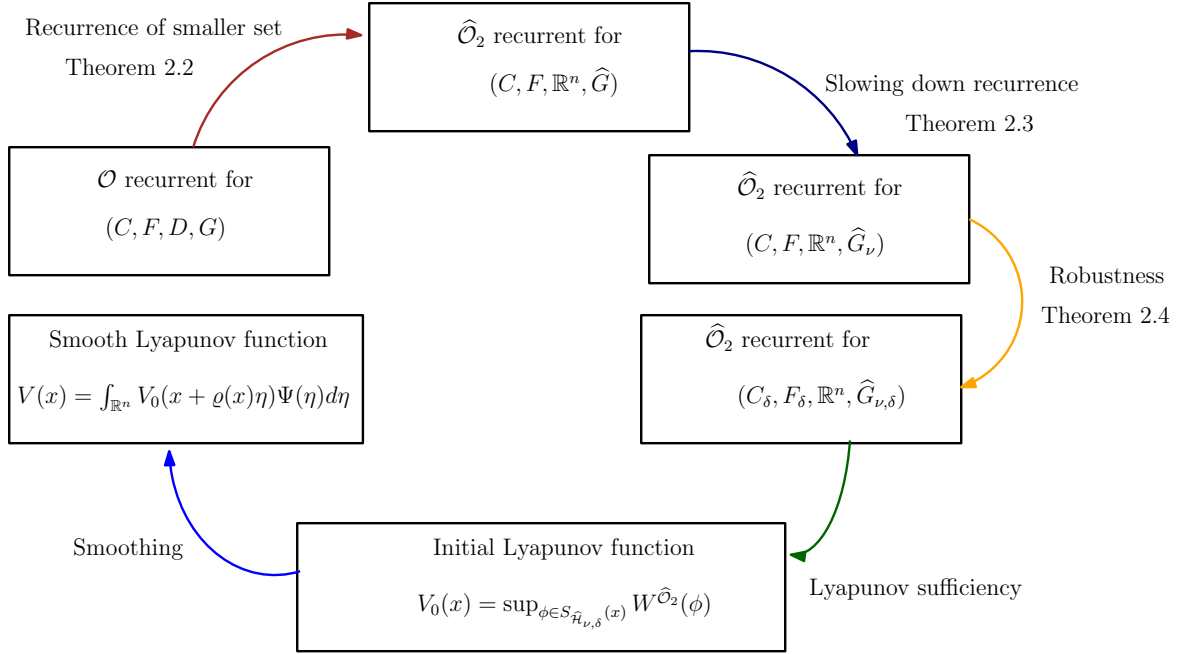


Figure 2.4: Converse Lyapunov theorem for recurrence

We now show that for every compact set  $K$ , there exists a  $T > 0$  such that for every  $\phi \in \mathcal{S}_{\mathcal{H}}(K)$ , either  $t + j < T$  for all  $(t, j) \in \text{dom } \phi$  or there exists  $(t, j)$  such that  $\phi(t, j) \in \mathcal{O}$  and  $t + j \leq T$ . Let  $V^* = \max_{x \in K} V(x)$ . We show that the condition holds with  $T := 2 + V^*$ . Suppose not, then there exists a  $\phi \in \mathcal{S}_{\mathcal{H}}(K)$  such that  $\text{length}(\text{dom } \phi) > T$  and  $\phi(t, j) \notin \mathcal{O}$  for  $t + j \leq T$ . Then, it follows from the Lyapunov inequalities that for all  $(t, j) \in \text{dom}(\phi)$

$$V(\phi(t, j)) \leq V(\phi(0, 0)) - (t + j) \leq V^* - (t + j). \quad (2.14)$$

Then, pick  $(t, j) \in \text{dom}(\phi)$  such that  $V^* \leq t + j \leq T$ . This implies that  $V(\phi(t, j)) < 0$  which is a contradiction since  $V(x) \geq 0$  for all  $x \in \mathbb{R}^n$ . This establishes uniform global recurrence of  $\mathcal{O}$ . A summary of the results on recurrence of open, bounded sets for (3.1) is in Figure 2.5.



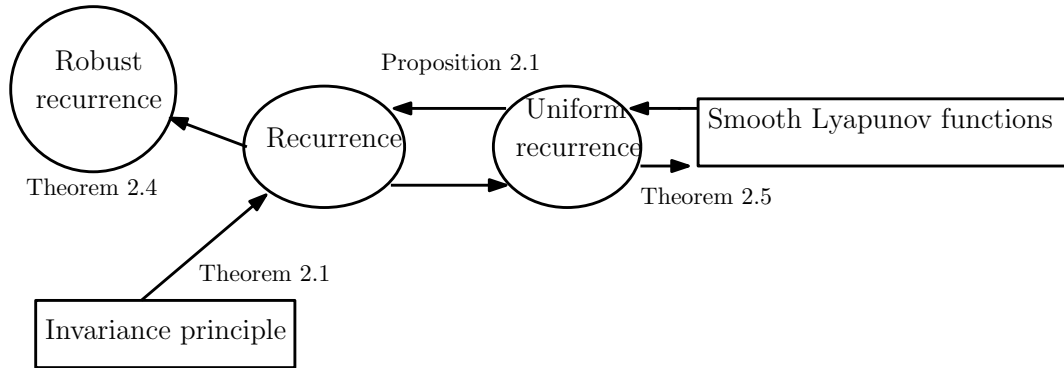


Figure 2.5: Summary of recurrence results for (2.1)

## 2.9 Summary of results for global asymptotic stability

In this section we summarize the existing results on robustness, Lyapunov function based necessary and sufficient conditions and weak sufficient conditions for global asymptotic stability of compact sets. The results presented in this section will be extended to a larger class of stochastic systems in the subsequent chapters. We refer the reader to [14, Chapters 3,7,8], [48] and [46] for more details.

**Definition 2.8** A smooth function  $V : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$  is said to be a Lyapunov function relative to a compact set  $\mathcal{A} \subset \mathbb{R}^n$  for (2.1) if it is radially unbounded,  $V \in \mathcal{PD}(\mathcal{A})$ , and there exists a continuous function  $\varrho \in \mathcal{PD}(\mathcal{A})$  such that

$$\langle \nabla V(x), f \rangle \leq -\varrho(x), x \in C, f \in F(x)$$

$$\max_{g \in G(x)} V(g) \leq V(x) - \varrho(x), x \in D.$$

**Theorem 2.6** The compact set  $\mathcal{A} \subset \mathbb{R}^n$  is globally asymptotically stable for (2.1) if and only if there exists a smooth Lyapunov function relative to  $\mathcal{A}$  for (2.1).

We now state the corresponding results on robustness of asymptotic stability and the invariance principle. The following system data

$$C_\delta := \{x \in \mathbb{R}^n : (x + \delta(x)\mathbb{B}) \cap C \neq \emptyset\} \quad (2.15a)$$

$$D_\delta := \{x \in \mathbb{R}^n : (x + \delta(x)\mathbb{B}) \cap D \neq \emptyset\} \quad (2.15b)$$

$$F_\delta := \overline{\text{con}F}((x + \delta(x)\mathbb{B}) \cap C) + \delta(x)\mathbb{B} \quad (2.15c)$$

$$G_\delta := \{v \in \mathbb{R}^n : v \in g + \delta(g)\mathbb{B}, g \in G(x + \delta(x)\mathbb{B})\} \quad (2.15d)$$

can be viewed a perturbation of the system data in (2.1).

**Theorem 2.7** *Let the compact set  $\mathcal{A} \subset \mathbb{R}^n$  be globally asymptotically stable for (2.1). Then, there exists a continuous function  $\delta \in \mathcal{PD}(\mathcal{A})$  such that set  $\mathcal{A}$  is globally asymptotically stable for (2.15).*

Let  $\mathcal{S}_C^F(x_0)$  refer to solutions of the constrained system  $\dot{x} \in F(x), x \in C$  from initial condition  $x_0$ .

**Definition 2.9** *A continuous function  $\widehat{V} : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$  is a weak-Lyapunov function relative to the compact set  $\mathcal{A} \subset \mathbb{R}^n$  for the system (2.1) if  $\widehat{V}$  is radially unbounded,  $\widehat{V} \in \mathcal{PD}(\mathcal{A})$  and satisfies*

$$\widehat{V}(\phi(t)) \leq \widehat{V}(x_0), \forall x_0 \in C, t \in \text{dom}(\phi), \phi \in \mathcal{S}_C^F(x_0) \quad (2.16)$$

$$\max_{g \in G(x_0)} \widehat{V}(g) \leq \widehat{V}(x_0), \forall x_0 \in D. \quad (2.17)$$

**Theorem 2.8** *Let  $\widehat{V}$  be a weak-Lyapunov function relative to a compact set  $\mathcal{A} \subset \mathbb{R}^n$  for the system (2.1). Then,  $\mathcal{A}$  is globally asymptotically stable if and only if for every  $c > 0$  there does not exist a complete solution that remains in the set  $L_{\widehat{V}}(c)$ .*

# Chapter 3

## Stochastic difference inclusions

### 3.1 Introduction

Stochastic systems are a class of systems for which randomness can affect the system dynamics. The randomness can be due to external noise or a part of the uncertainty in the description of the system model. Stochastic systems analysis is an important aspect in areas related to biological systems ([49], [50]), estimation theory ([51]), financial systems ([52]) and control systems ([53], [54]).

The main goal of this chapter is to introduce the reader to a class of discrete-time stochastic systems modeled by set-valued mappings. For this class of systems, we study stability properties like recurrence and asymptotic stability in probability. In particular, we establish Lyapunov function based sufficient conditions, weak sufficient conditions using the invariance principle and Matrosov functions, robust stability conditions and converse Lyapunov theorems. The results in this chapter are established in [55], [42] and [56].

We now present a brief discussion of the literature on set-valued stochastic systems. The notion of set-valued transition probabilities (Markov transition correspondence) is

introduced in [57] and the problem of existence of invariant measures is studied. In [58] further extensions of the results in [57] related to Markov transition correspondence are established. In [59] and [60] a class of stochastic differential equations modeled by set-valued mappings are analyzed and its application in stochastic optimal control problems is studied. In this chapter, we illustrate the importance of studying set-valued stochastic systems in the context of developing analysis tools related to stability theory.

## 3.2 Preliminaries on difference inclusions with random inputs

The mathematical framework used in this chapter is from [61] and [42]. We consider a set-valued mapping  $G : \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n$  and a discrete-time stochastic system with state  $x \in \mathbb{R}^n$  and a random input  $v \in \mathbb{R}^m$  written formally as

$$x^+ \in G(x, v) . \quad (3.1)$$

The following regularity conditions will be assumed throughout the chapter.

**Standing Assumption 3.1** *The set-valued mapping  $G$  in (3.1) satisfies the following properties:*

1. *For each  $v \in \mathbb{R}^m$  the mapping  $x \mapsto G(x, v)$  is outer semicontinuous.*
2. *The mapping  $G$  is locally bounded.*
3. *The mapping  $v \mapsto \text{graph}(G(\cdot, v)) := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n : y \in G(x, v)\}$  is measurable.*

The regularity conditions in Standing Assumption 3.1 will be used to guarantee robustness of stochastic stability notions and to generate solutions to (3.1) which are ran-

dom processes. In particular, conditions 1-2 are motivated by existing results for deterministic discrete-time systems to guarantee robustness of the stability property and condition 3 is used to guarantee the existence of maximal pre-random solutions to (3.1) described below. A consequence of conditions 1-2 is that  $G(x, v)$  is compact (though possibly empty) for every  $(x, v) \in \mathbb{R}^n \times \mathbb{R}^m$ . Condition 3 implies that the mapping  $v \mapsto G(x, v)$  is measurable.

### 3.2.1 Solution concept

Let  $\mathcal{S}_{c,m}(x)$  denote the set of maximal pre-random solutions to (3.1) starting at  $x$  that are causal, measurable functions of the inputs. That is,  $\phi \in \mathcal{S}_{c,m}(x)$  if  $\phi$  comprises a sequence of measurable functions  $\phi_i : \text{dom } \phi_i \subset (\mathbb{R}^m)^i \rightarrow \mathbb{R}^n$ ,  $i \in \mathbb{Z}_{\geq 0}$ , with  $\phi_0 = x$  such that  $\phi_{i+1}(v_0, \dots, v_i) \in G(\phi_i(v_0, \dots, v_{i-1}), v_i)$  for all  $i \in \mathbb{Z}_{\geq 0}$  and all  $(v_0, \dots, v_i) \in \text{dom } \phi_{i+1}$  with the property that  $\text{dom } \phi_{i+1} = \{(v_0, \dots, v_{i-1}, v_i) \in \text{dom } \phi_i \times \mathbb{R}^m : G(\phi_i(v_0, \dots, v_{i-1}), v_i) \neq \emptyset\}$ . Under Standing Assumption 3.1, it is established in [61, Lemma 3] that  $\mathcal{S}_{c,m}(x)$  is non-empty for each  $x \in \mathbb{R}^n$ .

A probability structure is now added to define random solutions to (3.1). Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space. For  $i \in \mathbb{Z}_{\geq 0}$ , let  $\mathbf{v}_i : \Omega \rightarrow \mathbb{R}^m$  be a sequence of independent, identically distributed (i.i.d.) random variables. Hence  $\mathbf{v}_i^{-1}(F) := \{\omega \in \Omega : \mathbf{v}_i(\omega) \in F\} \in \mathcal{F}$  for each  $F \in \mathbf{B}(\mathbb{R}^m)$ . We denote by  $\mathcal{F}_i$  the natural filtration of  $\mathcal{F}$  with respect to the random variables  $\{\mathbf{v}_i\}_{i=0}^{\infty}$ , where  $\mathcal{F}_i := \sigma\{\mathbf{v}_j^{-1}(A) | j \in \mathbb{Z}_{\geq 1}, j \leq i, A \in \mathcal{F}\}$ . Hence, the natural filtration is the smallest  $\sigma$ -algebra on  $(\Omega, \mathcal{F})$  that contains the pre-images of  $\mathbf{B}(\mathbb{R}^m)$ -measurable subsets on  $\mathbb{R}^m$  for times up to  $i$ . It follows from the i.i.d. property that each random variable has the same probability measure  $\mu : \mathbf{B}(\mathbb{R}^m) \rightarrow [0, 1]$  defined

as  $\mu(F) := \mathbb{P}\{\omega \in \Omega : \mathbf{v}_i(\omega) \in F\}$  and for almost all  $\omega \in \Omega$ ,

$$\mathbb{E}[f(\mathbf{v}_0, \dots, \mathbf{v}_i, \mathbf{v}_{i+1}) | \mathcal{F}_i](\omega) = \int_{\mathbb{R}^m} f(\mathbf{v}_0(\omega), \dots, \mathbf{v}_i(\omega), v) \mu(dv)$$

for each  $i \in \mathbb{Z}_{\geq 0}$  and each measurable  $f : (\mathbb{R}^m)^{i+2} \rightarrow \mathbb{R}$ .

As in [61], [42] a *random process*  $\mathbf{x}$  from  $x \in \mathbb{R}^n$  is a sequence of random variables  $\mathbf{x}_i : \text{dom } \mathbf{x}_i \subset \Omega \rightarrow \mathbb{R}^n$ ,  $i \in \mathbb{Z}_{\geq 0}$ , with  $\mathbf{x}_0 = x$  for all  $\omega \in \Omega$  and  $\text{dom } \mathbf{x}_{i+1} \subset \text{dom } \mathbf{x}_i$ . A random process  $\mathbf{x}$  is *adapted to the natural filtration* of  $\mathbf{v}$  if  $\mathbf{x}_{i+1}$  is  $\mathcal{F}_i$ -measurable for each  $i \in \mathbb{Z}_{\geq 0}$ . That is,  $\mathbf{x}_{i+1}^{-1}(F) \subset \mathcal{F}_i$  for each  $F \in \mathbf{B}(\mathbb{R}^n)$ . A random process  $\mathbf{x}$  from  $x \in \mathbb{R}^n$ , that is adapted to the natural filtration of  $\mathbf{v}$  together with a random variable  $\mathbf{J}_\mathbf{x} : \Omega \rightarrow \mathbb{Z}_{\geq 0} \cup \{\infty\}$  (which denotes the number of elements in the sequence  $\mathbf{x}$ ), is a *random solution* of (3.1) if  $\mathbf{x}_{i+1}(\omega) \in G(\mathbf{x}_i(\omega), \mathbf{v}_i(\omega))$  for all  $\omega \in \text{dom } \mathbf{x}_{i+1} := \{\omega \in \Omega : i+1 < \mathbf{J}_\mathbf{x}(\omega)\}$  and  $i \in \mathbb{Z}_{\geq 0}$ . A random solution  $(\mathbf{x}, \mathbf{J}_\mathbf{x})$  from  $x \in \mathbb{R}^n$  is said to be *maximal* if it cannot be extended, i.e., there does not exist another random solution  $(\mathbf{y}, \mathbf{J}_\mathbf{y})$  from  $x$  such that  $\text{dom } \mathbf{x}_i \subset \text{dom } \mathbf{y}_i$  for all  $i \in \mathbb{Z}_{\geq 0}$ ,  $\mathbf{y}_i(\omega) = \mathbf{x}_i(\omega)$  for all  $\omega \in \text{dom } \mathbf{x}_i$  and all  $i \in \mathbb{Z}_{\geq 0}$ , and  $\text{dom } \mathbf{x}_i \neq \text{dom } \mathbf{y}_i$  for some  $i \in \mathbb{Z}_{\geq 0}$ . We use  $\mathcal{S}_r(x)$  to denote the set of maximal random solutions of (3.1) from  $x \in \mathbb{R}^n$  and write  $\mathbf{x} \in \mathcal{S}_r(x)$ , suppressing the associated random variable  $\mathbf{J}_\mathbf{x}$ .

In essence, the random solution  $\mathbf{x}$  satisfies a measurability and causality condition. The measurability property is required to discuss the behavior of random solutions in terms of the associated probabilities. The causality property imposes a condition on how the random solution  $\mathbf{x}$  can depend on the i.i.d random variables  $\{\mathbf{v}_i\}_{i=0}^\infty$ . The causality condition plays a crucial role in stability analysis and will be discussed later in detail.

It follows from [42, Prop. 1] that, there exists  $\mathbf{x} \in \mathcal{S}_r(x)$  if and only if there exists

$\phi \in \mathcal{S}_{c,m}(x)$  such that, for each  $i \in \mathbb{Z}_{\geq 0}$ ,

$$\begin{aligned} \text{dom } \mathbf{x}_i &= \{\omega \in \Omega : (\mathbf{v}_0(\omega), \dots, \mathbf{v}_{i-1}(\omega)) \in \text{dom } \phi_i\} \\ \mathbf{x}_i(\omega) &= \phi_i(\mathbf{v}_0(\omega), \dots, \mathbf{v}_{i-1}(\omega)) \quad \forall \omega \in \text{dom } \mathbf{x}_i. \end{aligned} \tag{3.2}$$

For  $\mathbf{x} \in \mathcal{S}_r(x)$  we use the convention that  $\mathbb{I}_S(\mathbf{x}_i(\omega)) = 0$  for  $\omega \notin \text{dom } \mathbf{x}_i$  and we define

$$\text{graph}(\mathbf{x}(\omega)) := \bigcup_{i \in \mathbb{Z}_{\geq 0}} (\{i\} \times \mathbf{x}_i(\omega)).$$

### 3.3 Recurrence and asymptotic stability in probability

As noted in [17], there are many different notions of stability that are studied for stochastic systems. The primary reason for such a variety of stability notions is due to the different convergence criteria that are available for a sequence of random variables. In particular, stability notions can be defined in a mean square sense, in an almost sure sense, in the probability sense and in distribution. In this dissertation, we restrict our focus to stochastic stability notions like asymptotic stability in probability and recurrence. We refer the reader to the Appendix for detailed definitions of the various stability properties studied in the literature.

**Definition 3.1** *An open, bounded set  $\mathcal{O} \subset \mathbb{R}^n$  is said to be globally recurrent for (3.1) if for every  $x \in \mathbb{R}^n$  and  $\mathbf{x} \in \mathcal{S}_r(x)$ ,  $\mathbb{E} \left[ \prod_{i \in \mathbb{Z}_{\geq 1}} \mathbb{I}_{\mathbb{R}^n \setminus \mathcal{O}}(\mathbf{x}_i) \right] = 0$ .*

An equivalent characterization of recurrence is the condition that, for each  $x \in \mathbb{R}^n$  and each  $\mathbf{x} \in \mathcal{S}_r(x)$ ,

$$\lim_{k \rightarrow \infty} \mathbb{P} \left( (\text{graph}(\mathbf{x}) \subset (\mathbb{Z}_{<k} \times \mathbb{R}^n)) \vee (\text{graph}(\mathbf{x}) \cap (\mathbb{Z}_{\leq k} \times \mathcal{O})) \neq \emptyset \right) = 1 \quad (3.3)$$

where  $\vee$  denotes the logical “or” operation. Loosely speaking, the recurrence condition requires that for every random solution, the sample paths of the random solution either stop or hit the set  $\mathcal{O}$ . For stochastic systems convergence of solutions to compact sets with probability one is a strong requirement and in the absence of such convergence properties it is useful to consider the weaker notion of recurrence.

**Example 3.1** Consider the dynamical system  $x^+ = vx$ , where  $v \in \{0, 1\}$  with  $\mu(\{0\}) = \mu(\{1\}) = 0.5$ . For this system, we claim that for any  $\varepsilon > 0$ , the open set  $\mathcal{O} = (-\varepsilon, \varepsilon)$  is globally recurrent. Let the initial condition  $x \in \mathbb{R}^n$ . It can be observed that any infinite sequence  $\{\mathbf{v}_i(\omega)\}_{i=0}^{\infty}$ , the input 0 appears atleast once in an almost sure sense. Hence, for almost every  $\omega \in \Omega$ , there exists  $k \in \mathbb{Z}_{\geq 0}$  such that  $\mathbf{x}_{k+1}(\omega) = 0$ . This establishes global recurrence of the set  $\mathcal{O} = (-\varepsilon, \varepsilon)$ .

**Definition 3.2** An open, bounded set  $\mathcal{O} \subset \mathbb{R}^n$  is said to be uniformly globally recurrent for (3.1) if for every compact set  $K \subset \mathbb{R}^n$  and  $\rho > 0$  there exists  $J \in \mathbb{Z}_{\geq 1}$  such that  $\mathbb{E} \left[ \prod_{i=1}^J \mathbb{I}_{\mathbb{R}^n \setminus \mathcal{O}}(\mathbf{x}_i) \right] \leq \rho$  for every  $\mathbf{x} \in \mathcal{S}_r(K)$ .

Next, we define the notion of global asymptotic stability in probability. This property is a straightforward extension of the classical global asymptotic stability definition of compact sets for deterministic systems studied in [7].

**Definition 3.3** A compact set  $\mathcal{A} \subset \mathbb{R}^n$  is globally asymptotically stable in probability for (3.1) if



1. For every  $\varepsilon > 0$  and  $\rho > 0$  there exists a  $\delta > 0$  such that for all  $\mathbf{x} \in \mathcal{S}_r(\mathcal{A} + \delta\mathbb{B})$ ,

$$\mathbb{P}(\text{graph}(\mathbf{x}) \subset (\mathbb{Z}_{\geq 0} \times (\mathcal{A} + \varepsilon\mathbb{B}^\circ))) \geq 1 - \rho.$$

2. For every  $\mathbf{x} \in \mathcal{S}_r(\mathbb{R}^n)$ ,  $\lim_{i \rightarrow \infty} |\mathbf{x}_i(\omega)|_{\mathcal{A}} = 0$  for almost every  $\omega \in \Omega$ .

**Definition 3.4** A compact set  $\mathcal{A} \subset \mathbb{R}^n$  is uniformly globally asymptotically stable in probability for (3.1) if

1. For every  $\varepsilon > 0$  and  $\rho > 0$  there exists a  $\delta > 0$  such that for all  $\mathbf{x} \in \mathcal{S}_r(\mathcal{A} + \delta\mathbb{B})$ ,

$$\mathbb{P}(\text{graph}(\mathbf{x}) \subset (\mathbb{Z}_{\geq 0} \times (\mathcal{A} + \varepsilon\mathbb{B}^\circ))) \geq 1 - \rho.$$

2. For every  $\delta > 0$  and  $\rho > 0$  there exists a  $\varepsilon > 0$  such that for all  $\mathbf{x} \in \mathcal{S}_r(\mathcal{A} + \delta\mathbb{B})$ ,

$$\mathbb{P}(\text{graph}(\mathbf{x}) \subset (\mathbb{Z}_{\geq 0} \times (\mathcal{A} + \varepsilon\mathbb{B}^\circ))) \geq 1 - \rho.$$

3. For every  $\Delta > 0$ ,  $\delta > 0$  and  $\rho > 0$ , there exists  $J \in \mathbb{Z}_{\geq 0}$  such that for every  $\mathbf{x} \in \mathcal{S}_r(\mathcal{A} + \Delta\mathbb{B})$ ,

$$\mathbb{P}(\text{graph}(\mathbf{x}) \cap (\mathbb{Z}_{\geq J} \times \mathbb{R}^n) \subset (\mathbb{Z}_{\geq 0} \times (\mathcal{A} + \delta\mathbb{B}))) \geq 1 - \rho.$$

The equivalence between global recurrence and uniform global recurrence, global asymptotic stability in probability and uniform global asymptotic stability in probability will be established in the following sections and the proof relies primarily on the conditions in Standing Assumption 3.1.

### 3.3.1 The role of causality in stability analysis

We now explain in detail the importance of the causality condition in the definition of a random solution  $\mathbf{x}$  with respect to stability analysis. We illustrate the role of causality through the following example.

**Example 3.2** Consider the stochastic difference inclusion with state  $x = [x_1 \ x_2]$  satisfying

$$\begin{aligned} x_1^+ &\in \{-0.6, 0.6\} \\ x_2^+ &= (x_1 + v)x_2 \end{aligned}$$

where,  $v \in \{-0.6, 0.6\}$  with  $\mu(\{0.6\}) = \mu(\{-0.6\}) = 0.5$ . For this system, we analyze the behavior of the state  $x_2$ . It can be easily observed that for any causal selection of the  $\mathbf{x}_1$  random solution, the solution  $\mathbf{x}_2$  converges to the origin almost surely. This can also be verified using a Lyapunov function approach that will be discussed later in the chapter. However, consider the non-causal selection  $\mathbf{x}_{1,k}(\omega) = \mathbf{v}_k(\omega)$ . For this selection, the solution  $\mathbf{x}_2$  satisfies  $|\mathbf{x}_{2,k+1}(\omega)| = 1.2|\mathbf{x}_{2,k}(\omega)|$  and consequently  $\mathbf{x}_2$  diverges away from the origin almost surely.

The above example illustrates that non-causal selections can be adversarial in nature and hence can lead to unstable behavior. Set-valued mappings can generate solutions that are non-causal and hence for the purpose of stability analysis it is important to impose a causality assumption in the solution definition. The one-step Lyapunov function based conditions for certifying stability properties discussed later in this chapter can only be used to analyze the behavior of causal solutions and in general cannot account for the behavior of non-causal solutions.

## 3.4 Stability in terms of probability functions

In Chapter 2, the notion of worst first hitting times of solutions to open sets is used to aid the development of converse Lyapunov theorems for a class of non-stochastic hybrid systems. In this section, we review analysis tools that are more suitable for the study of stochastic systems. In particular, we focus on viability and reachability probabilities introduced in [61].

### 3.4.1 Weak Viability

For any random solution  $\mathbf{x} \in \mathcal{S}_r(x)$  the condition

$$\left( \text{graph}(\mathbf{x}) \cap (\mathbb{Z}_{\geq J} \times \mathbb{R}^n) \neq \emptyset \right) \wedge \left( (\text{graph}(\mathbf{x}) \cap (\mathbb{Z}_{\leq J} \times \mathbb{R}^n)) \subset (\mathbb{Z}_{\geq 0} \times S) \right)$$

where  $\wedge$  denotes the logical “and” operation asks that for  $i \in \{0, \dots, J\}$ ,  $\mathbf{x}_i$  is not empty and is contained in the set  $S$ . This condition is the complement of the condition used to characterize recurrence in (3.3) when  $S = \mathbb{R}^n \setminus \mathcal{O}$  and  $J \rightarrow \infty$ . The following integrals that are independent of the solution  $\mathbf{x}$  are used to bound the largest probability of this condition. The weak viability probabilities for a closed set  $S \subset \mathbb{R}^n$  and  $(i, x) \in \mathbb{Z}_{\geq 0} \times \mathbb{R}^n$  are defined as

$$\begin{aligned} m_{\subset S}(0, x) &:= 1 \\ m_{\subset S}(i+1, x) &:= \int_{\mathbb{R}^m} \max_{g \in G(x, v)} \mathbb{I}_S(g) m_{\subset S}(i, g) \mu(dv). \end{aligned} \quad (3.4)$$

The following result from [42, Prop. 4] establishes that the function  $m_{\subset S}(k, x)$  is related to the largest probability of solutions staying in the set  $S$  for  $k$  steps.

**Proposition 3.1** *Let  $S \subset \mathbb{R}^n$  be closed. For each  $x \in \mathbb{R}^n$  and  $k \in \mathbb{Z}_{\geq 1}$  there exists*

$\mathbf{x} \in \mathcal{S}_r(x)$  such that

$$m_{\subset S}(k, x) = \mathbb{E} \left[ \prod_{i=1}^k \mathbb{I}_S(\mathbf{x}_i) \right] = \sup_{\mathbf{z} \in \mathcal{S}_r(x)} \mathbb{E} \left[ \prod_{i=1}^k \mathbb{I}_S(\mathbf{z}_i) \right].$$

The weak viability probabilities  $m_{\subset S}(k, x)$  provide an upper bound over all random solutions from  $x$  for the probability of staying in the set  $S$  for  $k$  time steps. According to [61, Lemma 3] the mapping  $x \mapsto m_{\subset S}(i, x) \in [0, 1]$  is well defined, upper semicontinuous, and  $m_{\subset S}(i+1, x) \leq m_{\subset S}(i, x)$  for each  $(i, x) \in \mathbb{Z}_{\geq 0} \times \mathbb{R}^n$ . The monotonicity and boundedness implies that  $\widehat{m}_{\subset S}(x) := \lim_{i \rightarrow \infty} m_{\subset S}(i, x)$  is well defined for each  $x \in \mathbb{R}^n$ . For closed sets  $S_1, S_2$  such that  $S_1 \subset S_2$  we have  $m_{\subset S_1}(i, x) \leq m_{\subset S_2}(i, x)$  for all  $(i, x) \in \mathbb{Z}_{\geq 0} \times \mathbb{R}^n$ . The next result follows from [42, Prop. 5] and provides an equivalent characterization for the recurrence property.

**Proposition 3.2** *Let  $\mathcal{O} \subset \mathbb{R}^n$  be an open, bounded set. The following statements are then equivalent.*

1.  $\mathcal{O}$  is globally recurrent for (3.1).
2.  $\mathcal{O}$  is uniformly globally recurrent for (3.1).
3. For each compact set  $K \subset \mathbb{R}^n$  and  $\rho > 0$  there exists  $\ell \in \mathbb{Z}_{\geq 0}$  such that  $m_{\subset \mathbb{R}^n \setminus \mathcal{O}}(\ell, x) \leq \rho$  for all  $x \in K$ .
4. For each  $x \in \mathbb{R}^n$ ,  $\widehat{m}_{\subset \mathbb{R}^n \setminus \mathcal{O}}(x) = 0$ .
5. There exists  $\gamma \in [0, 1)$  such that, for each  $x \in \mathbb{R}^n$ ,  $\widehat{m}_{\subset \mathbb{R}^n \setminus \mathcal{O}}(x) \leq \gamma$ .

### 3.4.2 Weak Reachability

For any random solution  $\mathbf{x} \in \mathcal{S}_r(x)$  the condition  $\text{graph}(\mathbf{x}) \cap (\mathbb{Z}_{\leq J} \times S) \neq \emptyset$  asks that  $\mathbf{x}$  reaches the set  $S$  within  $J$  time steps. The following integrals that are independent

of the solution  $\mathbf{x}$  are used to bound the largest probability of this condition. The weak reachability probabilities for a closed set  $S \subset \mathbb{R}^n$  and  $(i, x) \in \mathbb{Z}_{\geq 0} \times \mathbb{R}^n$  are defined as

$$\begin{aligned} m_{\cap S}(0, x) &:= 0 \\ m_{\cap S}(i+1, x) &:= \int_{\mathbb{R}^m} \max_{g \in G(x, v)} \max\{\mathbb{I}_S(g), m_{\cap S}(i, g)\} \mu(dv). \end{aligned} \quad (3.5)$$

The following result is from [42, Prop. 6] and establishes that the function  $m_{\cap S}(k, x)$  is related to the largest probability of solutions reaching the set  $S$  within  $k$  steps.

**Proposition 3.3** *Let  $S \subset \mathbb{R}^n$  be closed. For each  $x \in \mathbb{R}^n$  and  $k \in \mathbb{Z}_{\geq 1}$  there exists  $\mathbf{x} \in \mathcal{S}_r(x)$  such that*

$$m_{\cap S}(k, x) = \mathbb{E} \left[ \max_{i \in \{1, \dots, k\}} \mathbb{I}_S(\mathbf{x}_i) \right] = \sup_{\mathbf{z} \in \mathcal{S}_r(x)} \mathbb{E} \left[ \max_{i \in \{1, \dots, k\}} \mathbb{I}_S(\mathbf{z}_i) \right].$$

The reachability probabilities  $m_{\cap S}(k, x)$  provide an upper bound over all random process solutions from  $x$  for the probability of reaching the set  $S$  within  $k$  time steps. Due to [61, Lemma 2], the functions  $x \mapsto m_{\cap S}(i, x) \in [0, 1]$  are well defined, upper semicontinuous, and  $m_{\cap S}(i, x) \leq m_{\cap S}(i+1, x)$  for each  $(i, x) \in \mathbb{Z}_{\geq 0} \times \mathbb{R}^n$ . Hence  $\lim_{i \rightarrow \infty} m_{\cap S}(i, x)$  is well defined for each  $x \in \mathbb{R}^n$ . For closed sets  $S_1, S_2$  such that  $S_1 \subset S_2$  we have  $m_{\cap S_1}(i, x) \leq m_{\cap S_2}(i, x)$  for all  $(i, x) \in \mathbb{Z}_{\geq 0} \times \mathbb{R}^n$ .

The next result follows from [42, Prop. 5, 7, 8] and provides an equivalent characterization for the asymptotic stability in probability property.

**Proposition 3.4** *Let  $\mathcal{A} \subset \mathbb{R}^n$  be a compact set. The following statements are then equivalent.*

1.  $\mathcal{A}$  is globally asymptotically stable in probability for (3.1).

2.  $\mathcal{A}$  is uniformly globally asymptotically stable in probability for (3.1).

3. The following conditions hold:

- For every  $\varepsilon > 0, \rho > 0$ , there exists  $\delta > 0$  such that  $\lim_{i \rightarrow \infty} m_{\mathbb{R}^n \setminus (\mathcal{A} + \varepsilon \mathbb{B}^o)}(i, x) \leq \rho$  for  $x \in \mathcal{A} + \delta \mathbb{B}$ .
- For each  $\varepsilon > 0$  and  $x \in \mathbb{R}^n$ ,  $\widehat{m}_{\mathbb{R}^n \setminus (\mathcal{A} + \varepsilon \mathbb{B}^o)}(x) = 0$ .

### 3.4.3 Preliminary bounds on viability and reachability probabilities

In this section we present some preliminary bounds related to viability and reachability probabilities that will be used to establish the main results of this chapter. The proof of the bounds are established for a larger of class of stochastic hybrid systems in a subsequent chapter and hence we only state the results in this section. The following result relates the viability and reachability probabilities.

**Lemma 3.1** *Let  $S \subset \mathbb{R}^n$  be closed and suppose  $S = S_1 \cup S_2$  where  $S_1$  and  $S_2$  are closed. Then, for every  $(i, x) \in \mathbb{Z}_{\geq 0} \times \mathbb{R}^n$ ,  $m_{\subset S}(i, x) \leq m_{\subset S_1}(i, x) + m_{\subset S_2}(i, x)$ .*

The next result relates the weak viability probabilities of two systems where there is an appropriate containment between them. We use subscripts “a” and “b” for probabilities associated with  $x^+ \in G_a(x, v)$  and  $x^+ \in G_b(x, v)$  respectively.

**Lemma 3.2** *Let  $S \subset \mathbb{R}^n$  be a closed set and suppose there exists  $F \in \mathbf{B}(\mathbb{R}^m)$  with  $\mu(F) = 0$  such that  $G_a(x, v) = G_b(x, v)$  for  $(x, v) \in S \times (\mathbb{R}^m \setminus F)$ . Then  $m_{a, \subset S}(i, x) = m_{b, \subset S}(i, x)$  for all  $(i, x) \in \mathbb{Z}_{\geq 0} \times S$ .*

The following result will be used to upper bound the viability probabilities and it can be viewed as semi-group type property for discrete-time stochastic systems.

**Lemma 3.3** *For closed sets  $S_0, S_1 \subset \mathbb{R}^n$  satisfying  $S_1 \subseteq S_0$  and  $(k, j, x) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \times \mathbb{R}^n$ ,*

$$m_{\subset S_0}(k + j, x) \leq m_{\subset S_1}(k, x) + \sup_{\xi \in \overline{\mathbb{R}^n \setminus S_1}} m_{\subset S_0}(j, \xi). \quad (3.6)$$

## 3.5 Global recurrence

In this section, we develop analysis tools related to global recurrence of open, bounded sets for (3.1). In particular, we present a Lyapunov function based characterization for recurrence, establish robustness of the recurrence property, and state relaxed conditions for certifying the recurrence property which rely on Lyapunov-like functions satisfying only a non-increasing on average property.

### 3.5.1 Sufficient conditions for global recurrence

Lyapunov-like criteria for certifying recurrence in both continuous and discrete-time stochastic systems are in [62], [63] and [22]. In this section, we establish sufficient conditions using Lyapunov functions for a class of discrete-time systems modeled by set-valued mappings.

**Definition 3.5** *An upper semicontinuous function  $V : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$  is said to be a sufficient recurrence-Lyapunov function<sup>1</sup> relative to  $\mathcal{O}$  for (3.1) if it is radially unbounded and there exists a continuous function  $\varrho : \mathbb{R}^n \rightarrow \mathbb{R}_{> 0}$  such that for all  $x \in \mathbb{R}^n \setminus \mathcal{O}$ ,*

$$\int_{\mathbb{R}^m} \max_{g \in G(x, v)} V(g) \mu(dv) \leq V(x) - \varrho(x). \quad (3.7)$$

<sup>1</sup>If  $\phi$  is upper semicontinuous and  $G(x, v) \neq \emptyset$ , then there exists  $g^* \in G(x, v)$  such that  $\sup_{g \in G(x, v)} \phi(g) = \phi(g^*)$ . Hence, we use  $\max_{g \in G(x, v)} V(g)$  for  $\sup_{g \in G(x, v)} V(g)$ . We also use  $\max_{g \in G(x, v)} \phi(g) = 0$  when  $G(x, v) = \emptyset$  and  $\phi : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ .

**Theorem 3.1** *Let  $\mathcal{O} \subset \mathbb{R}^n$  be an open bounded set. If there exists a sufficient recurrence-Lyapunov function relative to  $\mathcal{O}$  for (3.1) then the set  $\mathcal{O}$  is globally recurrent for (3.1).*

In essence, the sufficient recurrence-Lyapunov function  $V$  decreases strictly on average along solutions outside the set  $\mathcal{O}$ . We observe that for initial conditions from the set  $\mathcal{O}$  the function  $V$  need not satisfy any decrease properties. This can be attributed to the fact that recurrent sets are not necessarily invariant in a probabilistic sense and when the solutions hit the set  $\mathcal{O}$ , the solutions can leave the set causing an increase in  $V$  on average. We now define a stronger form of the sufficient recurrence-Lyapunov function in (3.7) and it characterizes the behavior of the Lyapunov function along solutions from the set  $\mathcal{O}$ .

**Definition 3.6** *An upper semicontinuous function  $V : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$  is said to be a Lyapunov function relative to  $\mathcal{O}$  for (3.1) if it is radially unbounded and there exists a continuous function  $\varrho : \mathbb{R}^n \rightarrow \mathbb{R}_{> 0}$  such that for all  $x \in \mathbb{R}^n$ ,*

$$\int_{\mathbb{R}^m} \max_{g \in G(x,v)} V(g) \mu(dv) \leq V(x) - \varrho(x) + \mathbb{I}_{\mathcal{O}}(x). \quad (3.8)$$

The decrease condition in (3.8) is stronger than (3.7) as it uniformly bounds the expected value of the Lyapunov function for solutions starting from  $\mathcal{O}$ . Hence, it follows that a Lyapunov function relative to  $\mathcal{O}$  for (3.1) is also a sufficient recurrence-Lyapunov function relative to  $\mathcal{O}$  for (3.1). However, the following example shows that the converse is not necessarily true.

**Example 3.3** We consider the system  $x^+ = g(x)v$  where  $v \sim \text{Cauchy}(0, 1)$ ,  $g : \mathbb{R} \rightarrow [0, 1]$



is continuous and satisfies

$$g(x) = \begin{cases} 0 & \text{for } |x| \geq 2 \\ 1 & \text{for } |x| \leq 1. \end{cases}$$

Let  $\mathcal{O} = (-2, 2)$ ,  $f(v) = \frac{1}{\pi(1+v^2)}$  and  $V(x) = |x|$ . Then for  $|x| \geq 2$  we have

$$\int_{\mathbb{R}} V(g)\mu(dv) = \int_{-\infty}^{\infty} V(g)f(v)dv = 0 = V(x) - |x|.$$

This bound implies that  $V$  is a sufficient recurrence-Lyapunov function relative to  $\mathcal{O}$ .

Now for  $|x| \leq 1$ ,

$$\int_{-\infty}^{\infty} V(g)f(v)dv = \int_{-\infty}^{\infty} |v|f(v)dv = \int_1^{\infty} \frac{1}{\pi u} du.$$

Then it follows that the expected value is not bounded for some solutions starting from  $\mathcal{O}$ . Hence  $V$  is not a Lyapunov function relative to  $\mathcal{O}$  for this system.

The following result provides a more explicit relation between the two Lyapunov functions.

**Proposition 3.5** *Let  $\mathcal{O} \subset \mathbb{R}^n$  be an open, bounded set. If  $V$  is a sufficient recurrence-Lyapunov function relative to  $\mathcal{O}$  for (3.1), then there exists a concave,  $\mathcal{K}_{\infty}$  function  $\kappa$  such that  $\kappa(V)$  is a Lyapunov function relative to  $\mathcal{O}$  for (3.1).*

### 3.5.2 Robust global recurrence

In this section we establish robustness of recurrence property to various state dependent perturbations similar to Chapter 2. Robustness of global recurrence is a key property that will be used in developing a converse Lyapunov theorem. The proofs of

the results in this section are presented in the appendix.

Similar to the results regarding robustness of recurrence for non-stochastic hybrid systems in Chapter 2, we consider robustness to three different types of perturbations. We establish that recurrence of a set implies the existence of a smaller recurrent set, slowing down quantities related to the average value of the worst first hitting times to the set  $\mathcal{O}$  can still preserve recurrence and recurrence is robust to sufficiently small perturbations in the system data.

We begin by asserting that if  $\mathcal{O}$  is globally recurrent for (3.1) then there exists a subset of  $\mathcal{O}$  that also preserves the same property.

**Theorem 3.2** *If an open bounded set  $\mathcal{O} \subset \mathbb{R}^n$  is globally recurrent for (3.1) then there exists an open bounded set  $\widehat{\mathcal{O}}$  and  $\varepsilon > 0$  such that  $\widehat{\mathcal{O}} + \varepsilon\mathbb{B} \subset \mathcal{O}$  and  $\widehat{\mathcal{O}}$  is globally recurrent for (3.1).*

In order to construct a smooth Lyapunov function for (3.1) that satisfies (3.8) we initially build a Lyapunov function from solutions to a system that is an inflation of (3.1). The first inflation that we consider is

$$x^+ \in G_\nu(x, v) := G(x, v) \cup M_\nu(x) \quad (3.9)$$

where  $M_\nu(x) = \{x_0\} + \nu(|x - x_0|)\mathbb{B}$ , where  $\nu \in \mathcal{K}_\infty$  and  $x_0 \in \widehat{\mathcal{O}}$ . This inflation will be used to guarantee radial unboundedness of the constructed Lyapunov function and it slows down the recurrence property. The next result follows from [42, Prop. 2].

**Proposition 3.6** *For each  $\nu \in \mathcal{K}_\infty$ , the set-valued mapping  $G_\nu$  defined in (3.9) satisfies the conditions of Standing Assumption 3.1.*

If the open, bounded set  $\widehat{\mathcal{O}}$  is globally recurrent for (3.1), we would like to assert that  $\widehat{\mathcal{O}}$  will be globally recurrent for (3.9) by picking  $\nu$  small enough. The next result

establishes that it is possible to find such a function  $\nu$  in order to slow down the recurrence property.

**Theorem 3.3** *If the open, bounded set  $\widehat{\mathcal{O}} \subset \mathbb{R}^n$  is globally recurrent for (3.1) then there exists  $\nu \in \mathcal{K}_\infty$  such that  $\widehat{\mathcal{O}}$  is globally recurrent for (3.9).*

The second inflation relative to (3.9) is used to guarantee smoothness of the constructed Lyapunov function in the converse Lyapunov theorem and has the form

$$x^+ \in G_{\rho,\nu}(x, v) := \{w \in \mathbb{R}^n : w \in g + \rho(g)\mathbb{B}, g \in G_\nu(x + \rho(x)\mathbb{B}, v)\} \quad (3.10)$$

where  $\rho : \mathbb{R}^n \rightarrow \mathbb{R}_{>0}$  is continuous.

**Proposition 3.7** *For each  $\nu \in \mathcal{K}_\infty$  and each continuous function  $\rho : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ , the set-valued mapping  $G_{\rho,\nu}$  defined in (3.10) satisfies the conditions of Standing Assumption 3.1.*

The previous result follows from [42, Prop. 3] and the next result states that global recurrence is robust to sufficiently small state dependent perturbations.

**Theorem 3.4** *If the open bounded set  $\widehat{\mathcal{O}} \subset \mathbb{R}^n$  is globally recurrent for (3.9) then there exists  $\rho : \mathbb{R}^n \rightarrow \mathbb{R}_{>0}$  continuous such that  $\widehat{\mathcal{O}}$  is globally recurrent for (3.10).*

### 3.5.3 Necessary condition for global recurrence

Converse Lyapunov theorems for stochastic systems appear in [20], [64], [65] and [22]. In this section, we establish a converse Lyapunov theorem for the recurrence property for a class of stochastic difference inclusions in (3.1). One of the fundamental ways in which the main result in this section differs from other converse theorems in the literature is

that we establish the existence of a smooth Lyapunov function as a necessary condition for recurrence.

**Theorem 3.5** *The open bounded set  $\mathcal{O} \subset \mathbb{R}^n$  is strongly globally recurrent for (3.1) if and only if there exists a smooth Lyapunov function relative to  $\mathcal{O}$  for (3.1).*

We now prove the above theorem. All probabilities in this proof are generated from (3.10) with  $\widehat{\mathcal{O}} \subset \mathcal{O}$ ,  $x_0 \in \widehat{\mathcal{O}}$ ,  $\nu \in \mathcal{K}_\infty$  and  $\rho : \mathbb{R}^n \rightarrow \mathbb{R}_{>0}$  continuous chosen according to Theorems 3.2, 3.3 and 3.4 so that the open, bounded set  $\widehat{\mathcal{O}} \subset \mathbb{R}^n$  is globally recurrent for (3.10).

Let  $\tau \in \mathcal{K}_\infty$ . Then, for all  $x \in \mathbb{R}^n$  define

$$W(x) := \sum_{i=1}^{\infty} \tau(i) M_{\widehat{\mathcal{O}}}(i, x) \mathbb{I}_{\mathbb{R}^n \setminus \widehat{\mathcal{O}}}(x). \quad (3.11)$$

where for all  $(i, x) \in \mathbb{Z}_{\geq 1} \times \mathbb{R}^n$ ,

$$M_{\widehat{\mathcal{O}}}(i, x) := (m_{\mathbb{C}\mathbb{R}^n \setminus \widehat{\mathcal{O}}}(i-1, x) - m_{\mathbb{C}\mathbb{R}^n \setminus \widehat{\mathcal{O}}}(i, x)). \quad (3.12)$$

**Proposition 3.8** *There exists  $\tau \in \mathcal{K}_\infty$  such that  $\widehat{W}(x) := \sum_{i=1}^{\infty} \tau(i) M_{\widehat{\mathcal{O}}}(i, x)$  is well defined, locally bounded and upper semicontinuous.*

*Proof:* Let  $K \subset \mathbb{R}^n$  be a compact set. Since the set  $\widehat{\mathcal{O}}$  is uniformly globally recurrent for (3.10) we can uniformly bound  $m_{\mathbb{C}\mathbb{R}^n \setminus \widehat{\mathcal{O}}}(i, x)$  for all  $(i, x) \in \mathbb{Z}_{\geq 0} \times K$  by a function  $\sigma_K \in \mathcal{L}$  such that  $m_{\mathbb{C}\mathbb{R}^n \setminus \widehat{\mathcal{O}}}(i, x) \leq \sigma_K(i)$  for all  $i \in \mathbb{Z}_{\geq 0}$ . It follows from global recurrence of  $\widehat{\mathcal{O}}$  and (3.12) that for all  $(j, x) \in \mathbb{Z}_{\geq 0} \times K$

$$\begin{aligned} \sum_{i=j+1}^{\infty} M_{\widehat{\mathcal{O}}}(i, x) &= \sum_{i=j+1}^{\infty} \left( m_{\mathbb{C}\mathbb{R}^n \setminus \widehat{\mathcal{O}}}(i-1, x) - m_{\mathbb{C}\mathbb{R}^n \setminus \widehat{\mathcal{O}}}(i, x) \right) \\ &= m_{\mathbb{C}\mathbb{R}^n \setminus \widehat{\mathcal{O}}}(j, x) \leq \sigma_K(j). \end{aligned}$$

Without loss of generality we can assume that  $\sigma_{2^i\mathbb{B}}(j) \leq \sigma_{2^{i+1}\mathbb{B}}(j) \forall (i, j) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ . Let  $\ell : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{Z}_{\geq 0}$  be a strictly increasing unbounded mapping satisfying  $\ell(0) = 0$  and  $\sigma_{2^i\mathbb{B}}(\ell(i)) \leq 2^{-i}$  for all  $i \in \mathbb{Z}_{\geq 1}$ . Let  $\tilde{\ell} \in \mathcal{K}_\infty$  satisfy  $\tilde{\ell}(i) = \ell(i)$  for each  $i \in \mathbb{Z}_{\geq 0}$ . Define  $\tau(s) := \tilde{\ell}^{-1}(s)$  for all  $s \geq 0$ . Given  $x \in \mathbb{R}^n$ , let  $k \in \mathbb{Z}_{\geq 1}$  be such that  $x \in 2^{k-1}\mathbb{B}$ . Then

$$\begin{aligned}
\sum_{i=1}^{\infty} \tau(i) M_{\widehat{\mathcal{O}}}(i, x) &= \sum_{j=1}^{\infty} \left( \sum_{i=\ell(j-1)+1}^{\ell(j)} \tau(i) M_{\widehat{\mathcal{O}}}(i, x) \right) \\
&\leq \sum_{j=1}^{\infty} \left( \sum_{i=\ell(j-1)+1}^{\ell(j)} j M_{\widehat{\mathcal{O}}}(i, x) \right) \\
&\leq \sum_{j=1}^{\infty} j \left( \sum_{i=\ell(j-1)+1}^{\infty} M_{\widehat{\mathcal{O}}}(i, x) \right) \leq \sum_{j=1}^{\infty} j \sigma_{2^{k-1}\mathbb{B}}(\ell(j-1)) \\
&\leq \sum_{j=1}^{k-1} j \sigma_{2^{k-1}\mathbb{B}}(\ell(j-1)) + \sum_{j=k}^{\infty} j \sigma_{2^{k-1}\mathbb{B}}(\ell(j-1)) \\
&\leq \sum_{j=1}^{k-1} j \sigma_{2^{k-1}\mathbb{B}}(0) + \sum_{j=k}^{\infty} j \sigma_{2^{j-1}\mathbb{B}}(\ell(j-1)) \\
&\leq \frac{k(k-1)}{2} \sigma_{2^{k-1}\mathbb{B}}(0) + \sum_{j=k}^{\infty} j 2^{-(j-1)}.
\end{aligned}$$

Since  $\sum_{j=1}^{\infty} j 2^{-(j-1)} < \infty$  it follows that  $x \mapsto \widehat{W}(x)$  is well defined and bounded on compact sets. From (3.12) we also have that for all  $x \in \mathbb{R}^n$ ,

$$\begin{aligned}
\widehat{W}(x) &= \sum_{i=1}^{\infty} \tau(i) (m_{\mathbb{C}\mathbb{R}^n \setminus \widehat{\mathcal{O}}}(i-1, x) - m_{\mathbb{C}\mathbb{R}^n \setminus \widehat{\mathcal{O}}}(i, x)) \\
&= \sum_{i=1}^{\infty} \tau(i) m_{\mathbb{C}\mathbb{R}^n \setminus \widehat{\mathcal{O}}}(i-1, x) - \sum_{i=1}^{\infty} \tau(i-1) m_{\mathbb{C}\mathbb{R}^n \setminus \widehat{\mathcal{O}}}(i-1, x) \\
&= \sum_{i=1}^{\infty} (\tau(i) - \tau(i-1)) m_{\mathbb{C}\mathbb{R}^n \setminus \widehat{\mathcal{O}}}(i-1, x). \tag{3.13}
\end{aligned}$$

Then, from the local boundedness of  $\widehat{W}$  we have that for every  $x \in \mathbb{R}^n$ ,  $\delta > 0$  and  $\gamma > 0$  there exists  $i^* \in \mathbb{Z}_{\geq 1}$  such that  $\widehat{W}(z) \leq \sum_{i=1}^{i^*} (\tau(i) - \tau(i-1)) m_{\mathbb{C}\mathbb{R}^n \setminus \widehat{\mathcal{O}}}(i-1, z) + \gamma$  for

all  $z \in x + \delta\mathbb{B}$ . Since  $\tau \in \mathcal{K}_\infty$ , we have  $\tau(i) - \tau(i-1) > 0 \forall i \in \mathbb{Z}_{\geq 1}$ . Let  $\{x_j\}_{j=1}^\infty$  be a sequence of points converging to  $x$ . Then, from the upper semicontinuity of the viability probabilities we have that

$$\begin{aligned} \limsup_{j \rightarrow \infty} \widehat{W}(x_j) &\leq \limsup_{j \rightarrow \infty} \left( \sum_{i=1}^{i^*} (\tau(i) - \tau(i-1)) m_{\mathbb{C}\mathbb{R}^n \setminus \widehat{\mathcal{O}}}(i-1, x_j) \right) + \gamma \\ &\leq \sum_{i=1}^{i^*} (\tau(i) - \tau(i-1)) m_{\mathbb{C}\mathbb{R}^n \setminus \widehat{\mathcal{O}}}(i-1, x) + \gamma \\ &\leq \widehat{W}(x) + \gamma. \end{aligned}$$

Since  $\gamma > 0$  is arbitrary,  $\widehat{W}$  is upper semicontinuous. ■

**Proposition 3.9** *There exists  $\tau \in \mathcal{K}_\infty$  such that  $W$  in (3.11) is well defined, locally bounded and upper semicontinuous.*

*Proof:* Let  $\tau \in \mathcal{K}_\infty$  be chosen according to 3.8 so that  $\widehat{W}(\cdot)$  is well defined. Since  $W(x) = \widehat{W}(x) \mathbb{I}_{\mathbb{R}^n \setminus \widehat{\mathcal{O}}}(x)$  for all  $x \in \mathbb{R}^n$ , it follows that the mapping  $x \mapsto W(x)$  is well defined and bounded on compact sets. Then, from the upper semicontinuity of  $\widehat{W}(\cdot)$  and  $\mathbb{I}_{\mathbb{R}^n \setminus \widehat{\mathcal{O}}}(\cdot)$  it follows that the product  $W(\cdot)$  is upper semicontinuous. ■

**Proposition 3.10** *There exists  $\bar{\varrho} : \mathbb{R}^n \rightarrow \mathbb{R}_{>0}$  continuous and  $\lambda > 0$  such that for all  $x \in \mathbb{R}^n$ ,  $\int_{\mathbb{R}^m} \max_{g \in G_{\rho, \nu}(x, v)} W(g) \mu(dv) \leq W(x) - \bar{\varrho}(x) + \lambda \mathbb{I}_{\widehat{\mathcal{O}}}(x)$ .*

*Proof:* Let  $\kappa(i) = \tau(i) - \tau(i-1)$  for all  $i \in \mathbb{Z}_{\geq 1}$ . Then, from the definition of  $\widehat{W}$  in (3.13) it follows that

$$\begin{aligned}
\int_{\mathbb{R}^m} \max_{g \in G_{\rho, \nu}(x, v)} W(g) \mu(dv) &\leq \int_{\mathbb{R}^m} \sum_{i=1}^{\infty} \max_{g \in G_{\rho, \nu}(x, v)} \kappa(i) m_{\mathbb{C}\mathbb{R}^n \setminus \widehat{\mathcal{O}}}(i-1, g) \mathbb{I}_{\mathbb{R}^n \setminus \widehat{\mathcal{O}}}(g) \mu(dv) \\
&= \sum_{i=1}^{\infty} \kappa(i) \int_{\mathbb{R}^m} \max_{g \in G_{\rho, \nu}(x, v)} \mathbb{I}_{\mathbb{R}^n \setminus \widehat{\mathcal{O}}}(g) m_{\mathbb{C}\mathbb{R}^n \setminus \widehat{\mathcal{O}}}(i-1, g) \mu(dv) \\
&= \sum_{i=1}^{\infty} \kappa(i) m_{\mathbb{C}\mathbb{R}^n \setminus \widehat{\mathcal{O}}}(i-1, x) - \sum_{i=1}^{\infty} \kappa(i) (m_{\mathbb{C}\mathbb{R}^n \setminus \widehat{\mathcal{O}}}(i-1, x) - m_{\mathbb{C}\mathbb{R}^n \setminus \widehat{\mathcal{O}}}(i, x)) \\
&= \sum_{i=1}^{\infty} \kappa(i) m_{\mathbb{C}\mathbb{R}^n \setminus \widehat{\mathcal{O}}}(i-1, x) (\mathbb{I}_{\mathbb{R}^n \setminus \widehat{\mathcal{O}}}(x) + \mathbb{I}_{\widehat{\mathcal{O}}}(x)) - \tilde{\varrho}(x) \\
&\leq W(x) - \tilde{\varrho}(x) + \lambda \mathbb{I}_{\widehat{\mathcal{O}}}(x)
\end{aligned}$$

where,

$$\begin{aligned}
\lambda &:= \sup_{x \in \widehat{\mathcal{O}}} \sum_{i=1}^{\infty} (\tau(i) - \tau(i-1)) m_{\mathbb{C}\mathbb{R}^n \setminus \widehat{\mathcal{O}}}(i-1, x) \\
\tilde{\varrho}(x) &:= \sum_{i=1}^{\infty} \left( \tau(i) - \tau(i-1) \right) \left( m_{\mathbb{C}\mathbb{R}^n \setminus \widehat{\mathcal{O}}}(i-1, x) - m_{\mathbb{C}\mathbb{R}^n \setminus \widehat{\mathcal{O}}}(i, x) \right).
\end{aligned}$$

From (3.12) we have that  $\lambda = \sup_{x \in \widehat{\mathcal{O}}} \sum_{i=1}^{\infty} \tau(i) M_{\widehat{\mathcal{O}}}(i, x) = \sup_{x \in \widehat{\mathcal{O}}} \widehat{W}(x)$  and  $\tilde{\varrho}(x) \leq \sum_{i=1}^{\infty} (\tau(i) - \tau(i-1)) m_{\mathbb{C}\mathbb{R}^n \setminus \widehat{\mathcal{O}}}(i-1, x) = \sum_{i=1}^{\infty} \tau(i) M_{\widehat{\mathcal{O}}}(i, x) \leq \widehat{W}(x)$ . Then, it follows from the proof of Proposition 3.8 that  $\lambda$  is finite and  $\tilde{\varrho}$  is bounded. Also from the definition we have that  $\lambda \geq \kappa(1) > 0$ .

Now we prove that  $\tilde{\varrho}$  is bounded away from zero on compact sets. Let  $R > 0$ . Choose  $\ell \in \mathbb{Z}_{\geq 1}$  such that  $\sigma_{R\mathbb{B}}(\ell) \leq 0.5$ , where  $\sigma_{R\mathbb{B}} \in \mathcal{L}$  is such that  $m_{\mathbb{R}^n \setminus \widehat{\mathcal{O}}}(i, x) \leq \sigma_{R\mathbb{B}}(i)$  for all  $(i, x) \in \mathbb{Z}_{\geq 0} \times R\mathbb{B}$ . Such a function exists because of the uniform strong global recurrence

of the set  $\widehat{\mathcal{O}}$ . Then, for  $x \in R\mathbb{B}$ ,

$$\begin{aligned}
\tilde{\varrho}(x) &\geq \sum_{i=1}^{\ell} (\tau(i) - \tau(i-1)) (m_{\mathbb{C}\mathbb{R}^n \setminus \widehat{\mathcal{O}}}(i-1, x) - m_{\mathbb{C}\mathbb{R}^n \setminus \widehat{\mathcal{O}}}(i, x)) \\
&\geq \min_{k \in \{1, \dots, \ell\}} (\tau(k) - \tau(k-1)) \sum_{i=1}^{\ell} (m_{\mathbb{C}\mathbb{R}^n \setminus \widehat{\mathcal{O}}}(i-1, x) - m_{\mathbb{C}\mathbb{R}^n \setminus \widehat{\mathcal{O}}}(i, x)) \\
&\geq \min_{k \in \{1, \dots, \ell\}} (\tau(k) - \tau(k-1)) (1 - m_{\mathbb{C}\mathbb{R}^n \setminus \widehat{\mathcal{O}}}(\ell, x)) \\
&\geq \min_{k \in \{1, \dots, \ell\}} (\tau(k) - \tau(k-1)) (1 - \sigma_{R\mathbb{B}}(\ell)) \\
&\geq 0.5 \min_{k \in \{1, \dots, \ell\}} (\tau(k) - \tau(k-1)).
\end{aligned}$$

Since  $R$  is arbitrary and  $\tau \in \mathcal{K}_{\infty}$ , it follows that  $\tilde{\varrho}$  is bounded away from zero on compact subsets of  $\mathbb{R}^n$ . Then, let  $\bar{\varrho} : \mathbb{R}^n \rightarrow \mathbb{R}_{>0}$  be such that it is continuous and satisfies  $\bar{\varrho}(x) \leq \tilde{\varrho}(x)$  for all  $x \in \mathbb{R}^n$ . Then, it follows that for all  $x \in \mathbb{R}^n$ ,

$$\int_{\mathbb{R}^m} \max_{g \in G_{\rho, \nu}(x, v)} W(g) \mu(dv) \leq W(x) - \bar{\varrho}(x) + \lambda \mathbb{I}_{\widehat{\mathcal{O}}}(x)$$

which establishes the result. ■

The next result is used to prove radial unboundedness of  $W$  by preventing big jumps to the set  $\widehat{\mathcal{O}}$  by some solutions starting from large initial conditions.

**Proposition 3.11** *For the system (3.10), for each  $\nu \in \mathcal{K}_{\infty}$ ,  $x_0 \in \widehat{\mathcal{O}}$ ,  $\rho : \mathbb{R}^n \rightarrow \mathbb{R}_{>0}$  continuous and  $k \in \mathbb{Z}_{\geq 0}$  there exists  $R > 0$  such that  $m_{\mathbb{C}\mathbb{R}^n \setminus \widehat{\mathcal{O}}}(k, x) = 1$  for all  $x \in \mathbb{R}^n \setminus (\widehat{\mathcal{O}} + R\mathbb{B}^{\circ})$ .*

*Proof:* Let  $x_0 \in \widehat{\mathcal{O}}$  and  $\widehat{R} > 0$  be such that  $\widehat{\mathcal{O}} \subset \{x_0\} + \widehat{R}\mathbb{B}^{\circ}$ . Since  $\widehat{\mathcal{O}}$  is bounded,  $\widehat{R}$  exists. Define  $\widehat{S} := \{x_0\} + \widehat{R}\mathbb{B}^{\circ}$  and  $S := \mathbb{R}^n \setminus \widehat{S}$ . From the definition of  $G_{\nu}$  in (3.9) and  $G_{\rho, \nu}$  in (3.10) we have that  $\{x_0\} + \nu(|x - x_0|)\mathbb{B} \subset G_{\rho, \nu}(x, v)$  for all  $(x, v) \in S \times \mathbb{R}^m$



and  $x_0 \in \widehat{\mathcal{O}}$ . We claim that for the system (3.10) and for all  $(k, x) \in \mathbb{Z}_{\geq 0} \times S$ ,

$$m_{\subset S}(k, x) \geq \mathbb{I}_{[\widehat{R}, \infty)} \left( \min_{i \in \{1, \dots, k\}} \nu^i(|x - x_0|) \right). \quad (3.14)$$

The bound holds by definition for  $k = 0$ . Suppose it holds for some  $k \in \mathbb{Z}_{\geq 0}$  and every  $x \in S$ . Then

$$\begin{aligned} m_{\subset S}(k+1, x) &= \int_{\mathbb{R}^m} \max_{g \in G_{\rho, \nu}(x, v)} \mathbb{I}_S(g) m_{\subset S}(k, g) \mu(dv) \\ &\geq \int_{\mathbb{R}^m} \max_{g \in \{x_0\} + \nu(|x - x_0|)\mathbb{B}} \mathbb{I}_S(g) \mathbb{I}_{[\widehat{R}, \infty)} \left( \min_{i \in \{1, \dots, k\}} \nu^i(|g - x_0|) \right) \mu(dv) \\ &= \max_{g \in \{x_0\} + \nu(|x - x_0|)\mathbb{B}} \mathbb{I}_S(g) \mathbb{I}_{[\widehat{R}, \infty)} \left( \min_{i \in \{1, \dots, k\}} \nu^i(|g - x_0|) \right) \\ &= \mathbb{I}_{[\widehat{R}, \infty)}(\nu(|x - x_0|)) \mathbb{I}_{[\widehat{R}, \infty)} \left( \min_{i \in \{1, \dots, k\}} \nu^{i+1}(|x - x_0|) \right) \\ &= \mathbb{I}_{[\widehat{R}, \infty)} \left( \min_{i \in \{1, \dots, k+1\}} \nu^i(|x - x_0|) \right). \end{aligned}$$

By induction (3.14) holds for all  $k \in \mathbb{Z}_{\geq 0}$ . Now let  $k \in \mathbb{Z}_{\geq 0}$  be given. Let  $\tilde{R} = |x - x_0|$ . Pick  $\tilde{R} > 0$  such that  $\min_{i \in \{1, \dots, k\}} \nu^i(\tilde{R}) \geq \widehat{R}$ , which can be achieved since  $\nu^i \in \mathcal{K}_\infty$  for each  $i \in \{1, \dots, k\}$ . Now pick  $R > 0$  such that  $\{x_0\} + \tilde{R}\mathbb{B}^o \subset \widehat{\mathcal{O}} + R\mathbb{B}^o$ . With this choice, it follows from (3.14) that  $m_{\cap S}(k, x) = 1$  for all  $x \in \mathbb{R}^n \setminus (\widehat{\mathcal{O}} + R\mathbb{B}^o)$ . The result of the proposition now holds as  $S \subset \mathbb{R}^n \setminus \widehat{\mathcal{O}}$ .  $\blacksquare$

**Corollary 3.1** *When constructed from the system (3.10) with  $\nu \in \mathcal{K}_\infty$  and  $\rho : \mathbb{R}^n \rightarrow \mathbb{R}_{>0}$  continuous, such that  $\widehat{\mathcal{O}}$  is globally recurrent then the function  $W$  in (3.11) is radially unbounded.*

*Proof:* Using Proposition 3.11, given  $i^* \in \mathbb{Z}_{>0}$ , let  $R > 0$  be such that  $m_{\mathbb{C}\mathbb{R}^n \setminus \widehat{\mathcal{O}}}(i^* - 1, x) = 1$  for all  $x \in \mathbb{R}^n \setminus (\widehat{\mathcal{O}} + R\mathbb{B}^o)$ . Then, for  $x \in \mathbb{R}^n \setminus (\widehat{\mathcal{O}} + R\mathbb{B}^o)$ ,

$$\begin{aligned} W(x) &= \sum_{i=1}^{\infty} \tau(i) M_{\widehat{\mathcal{O}}}(i, x) \geq \tau(i^*) \sum_{i=i^*}^{\infty} M_{\widehat{\mathcal{O}}}(i, x) \\ &= \tau(i^*) m_{\mathbb{C}\mathbb{R}^n \setminus \widehat{\mathcal{O}}}(i^* - 1, x) = \tau(i^*). \end{aligned}$$

Since  $\tau \in \mathcal{K}_{\infty}$  and  $i^*$  is arbitrary, the corollary follows. ■

Now define  $V(x) := W(x)/\lambda$  and  $\varrho(x) := \bar{\varrho}(x)/\lambda$  for all  $x \in \mathbb{R}^n$ . Then, it follows that  $V$  is upper semicontinuous, radially unbounded and satisfies (3.8). Now we smooth  $V$  to get the results of Theorem 3.5.

Let  $\sigma^* > 0$  be such that  $\widehat{\mathcal{O}} + \sigma^*\mathbb{B}^o \subset \mathcal{O}$ . Such a  $\sigma^*$  exists because of Theorem 3.2. Following [66], define  $V_s(x) := \int_{\mathbb{R}^n} V(x + \sigma(x)\xi)\psi(\xi)d\xi$  and  $\varrho_s(x) := \int_{\mathbb{R}^n} \varrho(x + \sigma(x)\xi)\psi(\xi)d\xi$  for all  $x \in \mathbb{R}^n$ , where  $\psi : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$  is smooth with support on  $\mathbb{B}$  and  $\sigma : \mathbb{R}^n \rightarrow \mathbb{R}_{>0}$  is smooth on  $\mathbb{R}^n$ . It follows that  $\varrho_s$  is continuous and positive for all  $x \in \mathbb{R}^n$ . We pick  $\sigma$  to satisfy  $\sigma(x) \leq \min\{|x| + c, \sigma^*\}$  for some  $c > 0$ . Then we have that  $V_s(x) \geq \inf_{\xi \in \mathbb{B}} V(x + \sigma^*\xi)$ . Since  $V$  is radially unbounded, it follows that  $V_s$  is radially unbounded. As in [42] we also choose  $\sigma$  sufficiently small so that

$$\sigma(x) \leq 0.5\rho(x) \leq \rho(x + \sigma(x)\xi) \quad \forall (x, \xi) \in \mathbb{R}^n \times \mathbb{B}. \quad (3.15)$$

It follows from (3.15) that

$$x \in \{x + \sigma(x)\xi\} + \rho(x + \sigma(x)\mathbb{B}) \quad \forall (x, \xi) \in \mathbb{R}^n \times \mathbb{B}. \quad (3.16)$$

It follows from (3.15), (3.16) that

$$g \in G(x, v), \tilde{g} = g + \sigma(g)\xi, \xi \in \mathbb{B} \Big\} \Rightarrow \tilde{g} \in G_{\rho, \nu}(x + \sigma(x)\xi, v).$$

Define  $\widehat{\mathcal{O}}_\sigma := \widehat{\mathcal{O}} + \sup_{x \in \mathbb{R}^n} \sigma(x)\mathbb{B}^o$ . Then, from the definition of  $\sigma$ , it follows that  $\widehat{\mathcal{O}}_\sigma \subset \widehat{\mathcal{O}}_{\sigma^*} \subset \mathcal{O}$ . Then, we have that

$$\int_{\mathbb{R}^n} \mathbb{I}_{\widehat{\mathcal{O}}}(x + \sigma(x)\xi)\psi(\xi)d\xi \leq \mathbb{I}_{\widehat{\mathcal{O}}_{\sigma^*}}(x) \leq \mathbb{I}_{\mathcal{O}}(x).$$

Then, from the above conditions it follows that for all  $x \in \mathbb{R}^n$ ,

$$\begin{aligned} \int_{\mathbb{R}^m} \max_{g \in G(x, v)} V_s(g)\mu(dv) &= \int_{\mathbb{R}^m} \max_{g \in G(x, v)} \left( \int_{\mathbb{R}^n} V(g + \sigma(g)\xi)\psi(\xi)d\xi \right) \mu(dv) \\ &\leq \int_{\mathbb{R}^m} \left( \int_{\mathbb{R}^n} \max_{g \in G(x, v)} V(g + \sigma(g)\xi)\psi(\xi)d\xi \right) \mu(dv) \\ &= \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^m} \max_{g \in G(x, v)} V(g + \sigma(g)\xi)\mu(dv) \right) \psi(\xi)d\xi \\ &\leq \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^m} \max_{g \in G_{\rho, \nu}(x + \sigma(x)\xi, v)} V(g)\mu(dv) \right) \psi(\xi)d\xi \\ &\leq \int_{\mathbb{R}^n} \left( V(x + \sigma(x)\xi) - \varrho(x + \sigma(x)\xi) + \mathbb{I}_{\widehat{\mathcal{O}}}(x + \sigma(x)\xi) \right) \psi(\xi)d\xi \\ &\leq V_s(x) - \varrho_s(x) + \mathbb{I}_{\mathcal{O}}(x). \end{aligned}$$

Then, it follows that  $V_s$  is a smooth Lyapunov function relative to  $\mathcal{O}$  for (3.1).

### 3.5.4 Weak sufficient conditions for global recurrence

In this section, we focus on relaxed sufficient conditions for certifying global recurrence. In particular, we do not rely on the existence of Lyapunov functions satisfying strict decrease conditions on average along solutions outside the recurrent set. We present two approaches to establish weak sufficient conditions for recurrence. The first approach

is through the well studied concept of invariance principle. The second approach is through the use of Matrosovs functions.

### Invariance principle

The invariance principle is an important tool to establish weak sufficient conditions for stability properties in the absence of Lyapunov functions satisfying strict decrease conditions along solutions. Typically, the invariance principle uses a Lyapunov function satisfying non-strict decrease conditions along with the knowledge of behavior of solutions on certain level sets of the Lyapunov function to conclude asymptotic stability properties. For non-stochastic hybrid systems an invariance principle is established in [28] for global asymptotic stability and in Chapter 2 for global recurrence. We now extend the results to a class of stochastic difference inclusions.

**Definition 3.7** *A continuous function  $\widehat{V} : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$  is a weak-Lyapunov function relative to an open, bounded set  $\mathcal{O} \subset \mathbb{R}^n$  for the system (3.1) if  $\widehat{V}$  is radially unbounded and satisfies*

$$\int_{\mathbb{R}^m} \max_{g \in G(x,v) \cap (\mathbb{R}^n \setminus \mathcal{O})} \widehat{V}(g) \mu(dv) \leq \widehat{V}(x), \forall x \in \mathbb{R}^n \setminus \mathcal{O}. \quad (3.17)$$

A random solution  $\mathbf{x}$  is almost surely complete if for almost all  $\omega \in \Omega$ ,  $\mathbf{x}(\omega)$  is complete. The proof of the next result will be presented in a subsequent chapter for a larger class of stochastic hybrid systems and hence we only state the result here.

**Theorem 3.6** *Let  $\widehat{V}$  be a weak-Lyapunov function relative to an open, bounded set  $\mathcal{O} \subset \mathbb{R}^n$  for the system (3.1). Then,  $\mathcal{O}$  is globally recurrent if and only if for every  $c \geq 0$  for which  $L_{\widehat{V}}(c) \cap (\mathbb{R}^n \setminus \mathcal{O})$  is non-empty there does not exist an almost surely complete random solution  $\mathbf{x}$  that remains in the set  $L_{\widehat{V}}(c) \cap (\mathbb{R}^n \setminus \mathcal{O})$  almost surely.*

### Matrosov Theorem

The invariance principles developed in [67], [68] relaxes the typical Lyapunov sufficient conditions required to establish global asymptotic stability for time-invariant differential equations. In [4] Matrosov established sufficient conditions for uniform global asymptotic stability in time-varying systems by using the notion of multiple Lyapunov-like functions with definitely non zero derivatives when the derivative of a Lyapunov function satisfying a weak decrease condition is zero. Matrosov's theorem in [4] used only one auxiliary function, but this has been extended to the case of multiple auxiliary functions in [5], [69]. Unlike the invariance principle, Matrosov function approach does not require the knowledge of behavior of solutions to conclude asymptotic stability. Also, the Matrosov function approach is applicable to time-varying systems. As illustrated in the example in [70], the invariance principle cannot be used to analyze global asymptotic stability for time-varying systems. In this section we present a Matrosov theorem for characterizing global recurrence of open, bounded sets for (3.1). The proof of the result is in the appendix.

**Theorem 3.7** *An open, bounded set  $\mathcal{O} \subset \mathbb{R}^n$  is globally recurrent for (3.1) if the following conditions hold.*

1. *There exists an upper semicontinuous, radially unbounded function  $V : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$  such that,*

$$\int_{\mathbb{R}^m} \max_{g \in G(x,v) \cap (\mathbb{R}^n \setminus \mathcal{O})} V(g) \mu(dv) \leq V(x) \quad x \in \mathbb{R}^n \setminus \mathcal{O}. \quad (3.18)$$

2. *For each  $R > 0$  there exists  $N \in \mathbb{Z}_{\geq 1}$ , upper semicontinuous functions  $W_i : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$  and continuous functions  $Y_i : \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $i \in \{1, \dots, N\}$  such that for all  $x \in$*

$R\mathbb{B} \setminus \mathcal{O}$ ,

$$\int_{\mathbb{R}^m} \max_{g \in G(x,v) \cap (\mathbb{R}^n \setminus \mathcal{O}) \cap R\mathbb{B}} W_i(g) \mu(dv) - W_i(x) \leq Y_i(x), \quad (3.19)$$

and with the definitions,  $Y_0(x) := 0$  for all  $x \in \mathbb{R}^n$  and  $Y_{N+1}(x) := 1$  for all  $x \in \mathbb{R}^n$ , we have the following property for each  $j \in \{0, \dots, N\}$  : if  $x \in (\mathbb{R}^n \setminus \mathcal{O}) \cap R\mathbb{B}$  and  $Y_i(x) = 0$  for all  $i \in \{0, \dots, j\}$  then  $Y_{j+1}(x) \leq 0$ .

### 3.5.5 Recurrence in stochastic systems vs non-stochastic systems

In this section we highlight some of the ways in which the recurrence property for stochastic systems and non-stochastic systems differ. In Chapter 2, it is established that recurrence of an open, bounded set for a non-stochastic hybrid system implies the solutions are uniformly ultimately bounded and the existence of compact set that is uniformly globally asymptotically stable. However, these implications are not true for stochastic systems. The following example illustrates these issues.

**Example 3.4** Consider the discrete-time stochastic system  $x^+ = \max\{0, x+v\}$  with  $x \in \mathbb{Z}_{\geq 0}$ , and the random variable  $v \in \{-1, 1\}$  with a distribution  $\mu$  satisfying  $\mu(\{-1\}) = 0.6$  and  $\mu(\{1\}) = 0.4$ . For this system  $V(x) = |x|$  is a Lyapunov function that guarantees global recurrence of the set  $\mathcal{O} = (-1, 1)$  since  $V$  is radially unbounded, and for  $x \in \mathbb{Z}_{\geq 1}$ ,

$$\mathbb{E}[V(x^+)] = 0.4|x+1| + 0.6|x-1| = V(x) - 0.2.$$

Then, it follows that  $V$  guarantees global recurrence of the set  $\mathcal{O}$ . We also have from [22, Thm 8.1.2] that every set of the form  $\mathcal{O}_r = (r, r+2)$  is recurrent for  $r \in \mathbb{Z}_{\geq 1}$ . This implies that the reachable set from any such  $\mathcal{O}_r$  is not bounded in a probabilistic sense

as solutions return to arbitrarily large sets infinitely often with probability one. It can be observed that the set  $\mathcal{O}$  does not have any invariance-like property or stability-like property. Finally, no compact set  $\mathcal{A} \subset \mathbb{Z}_{\geq 0}$  can be asymptotically stable in probability, since solutions starting from the set  $\mathcal{A}$  can leave with positive probability. Hence, the example highlights the differences in the recurrence property for stochastic and non-stochastic systems.

A summary of the results on global recurrence for open, bounded sets for (3.1) is in Figure 3.1.

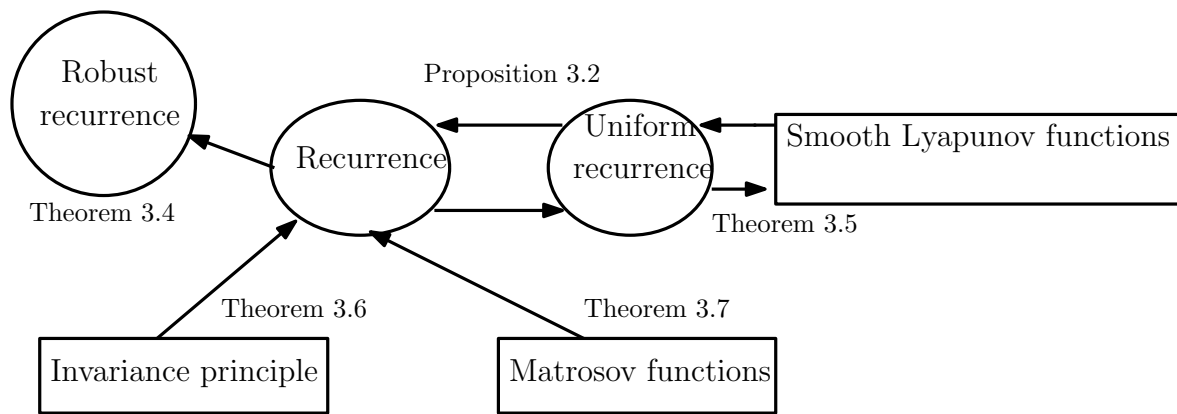


Figure 3.1: Summary of recurrence results for (3.1)

### 3.6 Global asymptotic stability in probability

In this section, we state results related to analysis tools for the global asymptotic stability in probability property. We refer the reader to [42] for the proofs.

**Definition 3.8** *An upper semicontinuous function  $V : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$  is said to be a Lyapunov function relative to the compact set  $\mathcal{A} \subset \mathbb{R}^n$  for (3.1) if it is radially unbounded,*

$V \in \mathcal{PD}(\mathcal{A})$ , and there exists a continuous function  $\varrho \in \mathcal{PD}(\mathcal{A})$  such that for all  $x \in \mathbb{R}^n$ ,

$$\int_{\mathbb{R}^m} \max_{g \in G(x,v)} V(g) \mu(dv) \leq V(x) - \varrho(x). \quad (3.20)$$

**Theorem 3.8** *Let  $\mathcal{A} \subset \mathbb{R}^n$  be a compact set. If there exists a Lyapunov function relative to  $\mathcal{A}$  for (3.1) then the set  $\mathcal{A}$  is globally asymptotically stable in probability for (3.1).*

**Example 3.5** Consider the stochastic difference inclusion in Example 3.2 with state  $x = [x_1 \ x_2]$  satisfying

$$\begin{aligned} x_1^+ &\in \{-0.6, 0.6\} \\ x_2^+ &= (x_1 + v)x_2 \end{aligned}$$

We claim that the set  $\mathcal{A} := \{-0.6, 0.6\} \times \{0\}$  is globally asymptotically stable in probability. Consider the Lyapunov function  $V(x) = x_2^2$ . Then,

$$\mathbb{E} \left[ \max_{g \in G(x,\cdot)} V(x^+) \right] = x_2^2 - 0.28x_2^2. \quad (3.21)$$

Since  $V$  is radially unbounded and positive definite with respect to  $\mathcal{A}$ , it follows that  $\mathcal{A}$  is globally asymptotically stable in probability. We note that the Lyapunov function approach can only be used to analyze the behavior of causal random solutions. As explained in Example 3.2, the non-causal selection  $\mathbf{x}_{1,k}(\omega) = \mathbf{v}_k(\omega)$  leads to unstable behavior.

We now state the corresponding results on robustness of asymptotic stability in probability, converse Lyapunov theorem and the invariance principle. For a continuous function



$\rho : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ , define

$$x^+ \in G_\rho(x, v) := \{w \in \mathbb{R}^n : w \in g + \rho(g)\mathbb{B}, g \in G(x + \rho(x)\mathbb{B}, v)\}. \quad (3.22)$$

**Theorem 3.9** *Let the compact set  $\mathcal{A} \subset \mathbb{R}^n$  be globally asymptotically stable in probability for (3.1). Then, there exists a continuous function  $\rho \in \mathcal{PD}(\mathcal{A})$  such that set  $\mathcal{A}$  is globally asymptotically stable in probability for (3.22).*

**Theorem 3.10** *The compact set  $\mathcal{A} \subset \mathbb{R}^n$  is globally asymptotically stable in probability for (3.1) if and only if there exists a smooth Lyapunov function relative to  $\mathcal{A}$  for (3.1).*

**Definition 3.9** *A continuous function  $\widehat{V} : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$  is a weak-Lyapunov function relative to a compact set  $\mathcal{A} \subset \mathbb{R}^n$  for the system (3.1) if  $\widehat{V}$  is radially unbounded,  $\widehat{V} \in \mathcal{PD}(\mathcal{A})$  and satisfies*

$$\int_{\mathbb{R}^m} \max_{g \in G(x, v)} \widehat{V}(g) \mu(dv) \leq \widehat{V}(x), \forall x \in \mathbb{R}^n. \quad (3.23)$$

**Theorem 3.11** *Let  $\widehat{V}$  be a weak-Lyapunov function relative to relative to a compact set  $\mathcal{A} \subset \mathbb{R}^n$  for the system (3.1). Then,  $\mathcal{A}$  is globally asymptotically stable in probability if and only if for every  $c > 0$  there does not exist an almost surely complete random solution  $\mathbf{x}$  that remains in the set  $L_{\widehat{V}}(c)$  almost surely.*

We note that weak sufficient conditions using Matrosov functions for asymptotic stability in probability are established in [71]. A summary of the results on global asymptotic stability in probability of compact sets for (3.1) is in Figure 3.2.

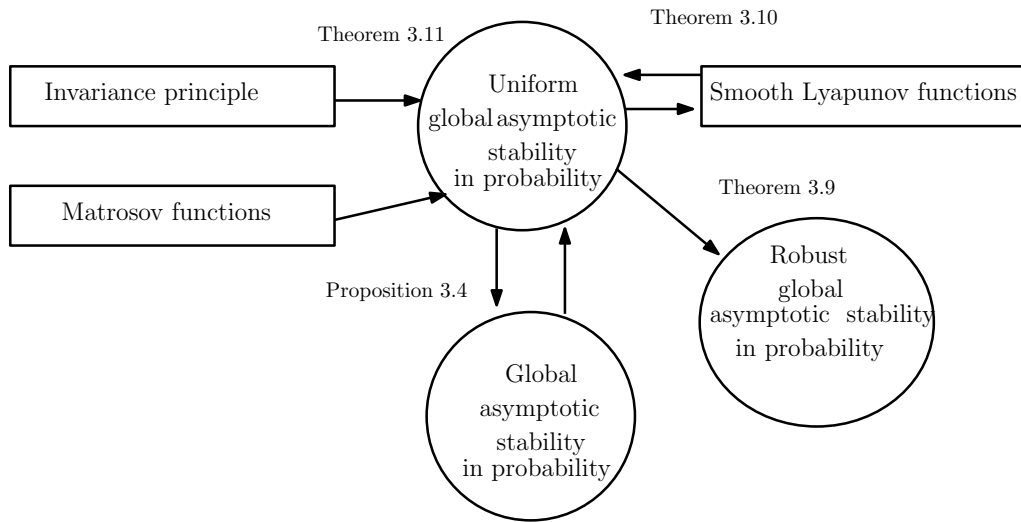


Figure 3.2: Summary of stability results for (3.1)

# Chapter 4

## Robust stochastic stability under discontinuous stabilization

### 4.1 Introduction

The aim of this chapter is to study the robustness of global asymptotic stability in probability for a class of constrained discrete-time stochastic systems under the action of discontinuous control laws. In the previous chapters, the class of systems for which robustness is studied satisfied good regularity properties. In this chapter, we focus on stochastic systems stabilized by discontinuous feedback laws for which the robustness results from Chapter 3 are not applicable. In particular, the closed loop stochastic system under the action of a discontinuous control law need not satisfy the regularity conditions listed in the Standing assumption from Chapter 3. The results of this chapter are from [72] and the proofs are in the appendix.

Discontinuous control laws arise from control synthesis methodologies sometimes out of necessity since there are controllable systems that are not continuously stabilizable.

The discrete-time cubic integrator with state  $x = (x_1, x_2)$  and control input  $u$  satisfying

$$\begin{aligned}x_1^+ &= x_1 + u \\x_2^+ &= x_2 + u^3\end{aligned}$$

is a system for which there does not exist any continuous control law to stabilize the origin. See [73] for details. Discontinuous control laws also arise frequently in the context of systems stabilized by model predictive control due to the presence of state and terminal constraints. See [74] and [75] for details. In the case of stochastic model predictive control, the control policies are assumed only to be a measurable function of the state and not necessarily continuous. See [76] for details. Studying robustness under discontinuous stabilization is important as there are examples where the discontinuous control law can stabilize the closed loop system, but the stability need not be robust. In particular, arbitrarily small perturbations can prevent convergence of the state to the desired attractor. Consider a system with state  $x = (x_1, x_2)$  and control input  $u$  satisfying

$$\begin{aligned}x_1^+ &= -\frac{(x_1^2 + x_2^2) + x_1}{(1 + (x_1^2 + x_2^2)u^2 - 2x_1u)} \\x_2^+ &= \frac{x_2}{(1 + (x_1^2 + x_2^2)u^2 - 2x_1u)}.\end{aligned}$$

For this system, under the constraints  $|u| \leq 1, x \in D := \{x \in \mathbb{R}^2, |x_1| \leq c\}$  for some  $c \in (0, 1)$ , a MPC control law that asymptotically stabilizes the origin is proposed in [77]. In [77, Proposition 14] it is further established that the asymptotic stability of the origin under the proposed MPC law is not robust with respect to measurement errors or additive disturbances.

## 4.2 Constrained stochastic systems with control inputs

In this section, we explain the class of constrained stochastic systems with control inputs considered in the rest of this chapter and state the basic assumptions we impose on the closed loop system.

Consider a function  $f : \mathcal{X} \times \mathcal{U} \times \mathcal{V} \rightarrow \mathcal{X}$ , where  $\mathcal{X} \subseteq \mathbb{R}^n$  and  $\mathcal{U} \subseteq \mathbb{R}^m$  are closed sets,  $\mathcal{V} \subseteq \mathbb{R}^p$  is measurable, and a stochastic controlled difference equation

$$x^+ = f(x, u, v) \tag{4.1}$$

with state variable  $x \in \mathcal{X}$ , control input  $u \in \mathcal{U}$ , and random input  $v \in \mathcal{V}$  (eventually specified as a random variable from a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  to  $\mathcal{V}$ ). The random variables  $\mathbf{v}_i : \Omega \rightarrow \mathcal{V}$ , for  $i \in \mathbb{Z}_{\geq 0}$ , are independent and identically distributed (i.i.d.) with a distribution function  $\mu : \mathbf{B}(\mathcal{V}) \rightarrow [0, 1]$  defined as  $\mu(F) := \mathbb{P}(\{\omega \in \Omega \mid \mathbf{v}_i(\omega) \in F\})$ .

We consider the following regularity conditions throughout this chapter.

**Standing Assumption 4.1** *The function  $f$  satisfies the following properties:*

1.  $f$  is locally bounded;
2. for any  $v \in \mathcal{V}$ , the mapping  $(x, u) \mapsto f(x, u, v)$  is continuous;
3. for any  $(x, u) \in \mathcal{X} \times \mathcal{U}$ , the mapping  $v \mapsto f(x, u, v)$  is measurable.

Given a stochastic difference equation of the kind

$$x^+ = g(x, v) \tag{4.2}$$

with  $g : \mathcal{X} \times \mathcal{V} \rightarrow \mathcal{X}$  locally bounded, and  $v \mapsto g(x, v)$  measurable for all  $x \in \mathcal{X}$ , we recall the notion of a Lyapunov function.

**Definition 4.1** *An upper semicontinuous function  $V : \mathcal{X} \rightarrow \mathbb{R}_{\geq 0}$  is a Lyapunov function relative to the compact set  $\mathcal{A} \subset \mathbb{R}^n$  for (4.2) if there exist functions  $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$  and  $\rho \in \mathcal{PD}(\mathcal{A})$  such that for all  $x \in \mathcal{X}$  we have  $\alpha_1(|x|_{\mathcal{A}}) \leq V(x) \leq \alpha_2(|x|_{\mathcal{A}})$  and*

$$\int_{\mathcal{V}} V(g(x, v)) \mu(dv) \leq V(x) - \rho(x). \quad (4.3)$$

We will now assume that there exists a locally bounded, possibly discontinuous, state-feedback control law, associated with a continuous Lyapunov function as follows.

**Assumption 4.1** *The function  $\kappa : \mathcal{X} \rightarrow \mathcal{U}$  is a locally bounded control law such that  $V : \mathcal{X} \rightarrow \mathbb{R}_{\geq 0}$  is a continuous Lyapunov function relative to the compact set  $\mathcal{A} \subset \mathcal{X}$  for the closed-loop stochastic difference equation*

$$x^+ = f(x, \kappa(x), v). \quad (4.4)$$

The main goal of this chapter is to analyze the robustness of the Lyapunov condition 4.3 for the system 4.4 and consequently make observations related to robustness of the global asymptotic stability in probability property. In particular, when the control law is discontinuous we study if the Lyapunov conditions in 4.3 are preserved under arbitrarily small perturbations.

### 4.3 Continuous Lyapunov function implies robustness

In this section, we analyze the robustness properties of the closed loop system (4.4) under the conditions of Standing Assumption 4.1 and Assumption 4.1. We first highlight some of the differences that arise when this problem is studied for stochastic systems instead of deterministic systems.

For deterministic systems, the existence of a continuous Lyapunov function  $V$  for the closed loop system under assumptions similar to Standing Assumption 4.1 implies that  $V$  is also a Lyapunov function for a perturbed version of the closed loop system if the perturbation is sufficiently small. For stochastic systems, it is not necessarily true that the Lyapunov function from Assumption 4.1 also works for a perturbed version of the closed loop system (4.4). Finally, in stochastic systems, causality plays an important role in the type of perturbations for which robustness can be achieved for (4.4). For deterministic systems, the issue of causality does not arise in the robustness analysis. We explain in more detail the above issues through examples later in this section.

Given a continuous Lyapunov function  $V$  relative to the compact attractor  $\mathcal{A}$  for the nominal closed-loop system (4.4), we first establish that there exists a concave function  $\Gamma \in \mathcal{K}_\infty$  such that the function  $\Gamma(V)$  is a continuous stochastic Lyapunov function relative to  $\mathcal{A}$  for a perturbed closed-loop system. The following results are a consequence of the proof of Proposition 3.5. We also refer the reader to [72] for a proof.

**Lemma 4.1** *For any measurable function  $\Phi : \mathbb{R}^p \rightarrow \mathbb{R}_{\geq 0}$ , there exists a concave function  $\Gamma \in \mathcal{K}_\infty$  such that  $\int_{\mathbb{R}^p} \Gamma(\Phi(v))\mu(dv) < \infty$ .*

**Lemma 4.2** *If Assumption 4.1 holds, then for any concave  $\Gamma \in \mathcal{K}_\infty$ , we have that  $\Gamma(V)$  is a Lyapunov function relative to  $\mathcal{A}$  for (4.4).*

Let us first consider the smallest set-valued inflation of the control law  $\kappa$ , i.e. its regularization  $K : \mathcal{X} \rightrightarrows \mathcal{U}$  defined as

$$K(x) := \bigcap_{\rho > 0} \overline{\kappa(\{x\} + \rho\mathbb{B}) \cap \mathcal{X}}, \quad (4.5)$$

which is locally bounded and outer semicontinuous [14, Lemma 5.16], even if  $\kappa$  is a discontinuous function.

The following result shows some robustness of the Lyapunov condition for (4.4).

**Proposition 4.1** *If Assumption 4.1 holds, then there exists a concave  $\Gamma \in \mathcal{K}_\infty$  and  $\varrho \in \mathcal{PD}(\mathcal{A})$  such that for all  $x \in \mathcal{X}$  we have*

$$\max_{u \in K(x)} \int_{\mathcal{V}} \Gamma(V(f(x, u, v))) \mu(dv) \leq \Gamma(V(x)) - \varrho(x). \quad (4.6)$$

The following example highlights the role of causality in robustness analysis and shows that we cannot derive a Lyapunov condition with the selection “ $\max_{u \in K(x)}$ ” inside the integral in (4.6).

**Example 4.1** Consider the stochastic controlled difference equation

$$x^+ = f(x, u, v) = (u + v)x$$

where  $\mathcal{V} = \{-0.6, 0.6\}$ , and  $\mu(\{-0.6\}) = \mu(\{0.6\}) = 0.5$ . The state-feedback control law

$$\kappa(x) = \begin{cases} 0.6 & \text{if } x \in \mathbb{Q} \\ -0.6 & \text{otherwise,} \end{cases}$$

being  $\mathbb{Q}$  the set of rational numbers, makes  $V(x) = |x|$  a continuous Lyapunov function relative to  $\mathcal{A} = \{0\}$ , as for all  $x \in \mathbb{R}$  we have  $\int_{\mathcal{V}} |x^+| \mu(dv) \leq 0.6|x|$ .



We now consider the smallest perturbation of the control law  $\kappa$ , namely the controller regularization  $K$  defined in (4.5). With the selection “ $\max_{u \in K(x)}$ ” inside the integral we get  $\int_{\mathcal{V}} \max_{u \in K(x)} |x^+| \mu(dv) = \int_{\mathcal{V}} \max_{u \in K(x)} |(u+v)x| \mu(dv) = 1.2|x| > |x|$ , therefore the Lyapunov condition does not hold. The primary reason why the Lyapunov conditions fail is that, “ $\max_{u \in K(x)}$ ” inside the integral allows for the possibility of non-causal selections. In particular, the selection  $u = v$  is now admissible and it can be observed that the selection is non-causal and adversarial.

In the following example we show that, in general, the use of a suitable concave  $\Gamma \in \mathcal{K}_\infty$  is strictly necessary in Proposition 4.1, because arbitrarily small perturbations can induce the integral in (4.3) to be unbounded.

**Example 4.2** Consider the stochastic difference equation

$$x^+ = f(x, u, v) = x - u + |x/2 - u| \Phi(v)$$

where  $\Phi : \mathcal{V} \rightarrow \mathbb{R}_{\geq 0}$  is locally bounded and measurable, but such that  $\int_{\mathcal{V}} \Phi(v) \mu(dv) = \infty$ . The control law  $u(x) = x/2$  induces the closed-loop system to be  $x^+ = x/2$  for which  $V(x) = |x|$  is a Lyapunov function relative to  $\mathcal{A} = \{0\}$ . However, for any  $\delta > 0$ , the control law  $u(x) = \delta + x/2$  induces the closed-loop to be  $x^+ = x/2 - \delta + \delta \Phi(v)$ . Then, from Jensen’s inequality we get

$$\begin{aligned} \int_{\mathcal{V}} |x/2 - \delta + \delta \Phi(v)| \mu(dv) &\geq \left| \int_{\mathcal{V}} (x/2 - \delta + \delta \Phi(v)) \mu(dv) \right| \\ &= \left| \int_{\mathcal{V}} (x/2 - \delta) \mu(dv) + \delta \int_{\mathcal{V}} \Phi(v) \mu(dv) \right| \\ &\geq \delta \left| \int_{\mathcal{V}} \Phi(v) \mu(dv) \right| = \infty. \end{aligned}$$

Therefore, for any  $\delta > 0$ , we have

$$\max_{w \in \{u(x)\} + \delta \mathbb{B}} \int_{\mathcal{V}} V(f(x, w, v)) \mu(dv) = \infty,$$

even if  $V$  is a continuous Lyapunov function.

We now state one of the main result of this chapter. Under Standing Assumption 4.1, if there exists a continuous Lyapunov function relative to the compact attractor  $\mathcal{A} \subset \mathcal{X}$ , then the Lyapunov condition (4.6) is robust to sufficiently small, state-dependent, strictly causal, worst-case perturbations  $\delta \in \mathcal{PD}(\mathcal{A})$ . We indeed consider the following set-valued inflations  $K_\delta : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ , with  $\text{dom} K_\delta = \mathcal{X}$ , and  $f_\delta : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \rightrightarrows \mathbb{R}^n$ , with  $\text{dom} f_\delta = \mathcal{X} \times \mathcal{U} \times \mathcal{V}$ , respectively of the mapping  $K$  and of the function  $f$ .

$$K_\delta(x) := (K(\{x\} + \delta(x)\mathbb{B}) \cap \mathcal{X}) + \delta(x)\mathbb{B}, \forall x \in \text{dom} K_\delta \quad (4.7)$$

$$f_\delta(x, u, v) := (f(\{x\} + \delta(x)\mathbb{B}) \cap \mathcal{X}, u, v) + \delta(x)\mathbb{B} \cap \mathcal{X}, \forall (x, u, v) \in \text{dom} f_\delta. \quad (4.8)$$

**Theorem 4.1** *If Assumption 4.1 holds, then there exist  $\delta \in \mathcal{PD}(\mathcal{A})$ , a concave  $\Gamma \in \mathcal{K}_\infty$  and  $\varrho \in \mathcal{PD}(\mathcal{A})$  such that for all  $x \in \mathcal{X}$  we have*

$$\max_{u \in K_\delta(x)} \int_{\mathcal{V}} \max_{\varphi \in f_\delta(x, u, v)} \Gamma(V(\varphi)) \mu(dv) \leq \Gamma(V(x)) - \varrho(x). \quad (4.9)$$

*If there exists a compact set  $\mathcal{C} \subseteq \mathcal{V}$  such that  $\mu(\mathcal{C}) = 1$ , then (4.9) holds with  $\Gamma := \text{Id}$ .*

## 4.4 Strictly causal generalized random solutions

Now we study how Lyapunov conditions predict the stochastic stability properties for random solutions associated with the stochastic difference equation  $x^+ = f(x, \kappa(x), v)$

(4.4). We could consider random solutions of system (4.4) directly, but there are the following two issues. First, since in Assumption 4.1 we have not assumed that the control law  $\kappa : \mathcal{X} \rightarrow \mathcal{U}$  is a measurable function, there is no guarantee that the iteration

$$\mathbf{x}_{i+1}(\omega) := f(\mathbf{x}_i(\omega), \kappa(\mathbf{x}_i(\omega)), \mathbf{v}_i(\omega)), \quad \text{for } i \in \mathbb{Z}_{\geq 0}, \quad (4.10)$$

$\mathbf{x}_0(\omega) := \xi_0 \in \mathcal{X}$ , yields measurable functions  $\mathbf{x}_i : \Omega \rightarrow \mathcal{X}$ , for  $i \in \mathbb{Z}_{\geq 0}$ . Secondly, even when the function  $\kappa$  is measurable, the behavior of the random solution that is generated by the iteration (4.10) may not accurately predict the behavior of the system in the presence of small, random or worst-case, perturbations. For these reasons, we choose to define a notion of *generalized random solution*. Generalized random solutions do not require the control law  $\kappa$  to be measurable and, as we will see, their behavior predicts the behavior of the system under small, random or worst-case, strictly causal perturbations.

This latter feature is also present for generalized solutions to non-stochastic difference inclusions as introduced in [7]. In the case of non-stochastic difference equations  $x^+ = f(x, \kappa(x))$ , generalized solutions are the solutions of the difference inclusion  $x^+ \in f(x, K(x))$ , with  $K$  being the controller regularization as defined in (4.5). It follows from [7] that the existence of a continuous Lyapunov function for  $x^+ = f(x, \kappa(x))$  implies the existence of a continuous Lyapunov function for  $x^+ \in f(x, K(x))$  and even for an inflation of this latter system. However, Example 4.1 suggests that this result does not hold for  $x^+ \in f(x, K(x), v)$  (4.4) in the stochastic case. This fact and the results of the previous section motivate an alternative definition of generalized solutions in the stochastic case, that turns out to generate the same solutions as  $x^+ \in f(x, K(x))$  in the non-stochastic case and yet yields a robust Lyapunov result in the stochastic case.

Our strictly causal generalized random solutions are random solutions to the stochas-

tic difference inclusion

$$\begin{aligned} \begin{pmatrix} x \\ u \end{pmatrix}^+ \in G_0(x, u, v) := & \left\{ \begin{pmatrix} f(x, u, v) \\ K(f(x, u, v)) \end{pmatrix} \right\} \\ & \text{if } ((x, u), v) \in \text{graph}(K) \times \mathcal{V}; \emptyset \text{ otherwise.} \end{aligned} \quad (4.11)$$

The types of perturbations to which the behaviors of the solutions will be robust are strictly causal perturbations that appear in the stochastic difference inclusion

$$\begin{aligned} \begin{pmatrix} x \\ u \end{pmatrix}^+ \in G_\delta(x, u, v) := & \left\{ \begin{pmatrix} \varphi \\ K_\delta(\varphi) \end{pmatrix} \mid \varphi \in f_\delta(x, u, v) \right\} \\ & \text{if } ((x, u), v) \in \text{graph}(K_\delta) \times \mathcal{V}; \emptyset \text{ otherwise,} \end{aligned} \quad (4.12)$$

where  $f_\delta$  and  $K_\delta$  are the inflations of  $f$  and  $K$  respectively, as defined in (4.7), (4.8).

The motivation for considering the above inclusions is that the selection  $u \in K_\delta(x)$  “does not depend” on the current random input  $v$ . This property is what we call strict causality. We notice that, for each  $(x, u) \in \mathcal{X} \times \mathcal{U}$ , if  $u \in K_\delta(x)$ , then we have  $u^+ \in K_\delta(x^+)$ .

Let us first assert certain regularity properties of the set-valued mapping  $G_\delta : \mathcal{X} \times \mathcal{U} \times \mathcal{V} \rightrightarrows \mathcal{X} \times \mathcal{U}$  in (4.12), by exploiting Standing Assumption 4.1. The same regularity properties hold for  $G_0$  defined in (4.11).

**Proposition 4.2** *For all continuous functions  $\delta : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ , the set-valued mapping  $G_\delta$  defined in (4.12) satisfies the following regularity conditions:*

1. for any  $v \in \mathcal{V}$  the mapping  $(x, u) \mapsto G_\delta(x, u, v)$  is outer semicontinuous;
2. the mapping  $v \mapsto \text{graph}(G_\delta(\cdot, \cdot, v)) := \{(x, u, y) \in \mathcal{X} \times \mathcal{U} \times (\mathcal{X} \times \mathcal{U}) \mid y \in G_\delta(x, u, v)\}$

is measurable;

3. the mapping  $G_\delta$  is locally bounded.

Since Proposition 4.2 shows that  $G_0$  in (4.11) and  $G_\delta$  in (4.12) have the same regularity conditions given in [42, Standing Assumption 1] and Standing Assumption 3.1, we can define the notion of solutions for the stochastic difference inclusion (4.12) having (extended) state variable  $z := \begin{pmatrix} x \\ u \end{pmatrix} \in (\mathcal{X} \times \mathcal{U})$ . We also define *generalized random solutions* to (4.4) as the solutions for the regularized stochastic difference inclusion (4.11).

We now show that (4.9) established in Theorem 4.1 is closely related to a Lyapunov condition for the extended stochastic difference inclusion (4.12), with Lyapunov function  $\bar{V} : \mathcal{X} \times \mathcal{U} \rightarrow \mathbb{R}_{\geq 0}$  relative to the compact attractor  $\bar{\mathcal{A}} \subset \mathcal{X} \times \mathcal{U}$  explicitly defined in the following preliminary result.

**Lemma 4.3** *For any  $\delta \in \mathcal{PD}(\mathcal{A})$ , the function  $W : \mathcal{X} \times \mathcal{U} \rightarrow \mathbb{R}_{\geq 0}$  defined as*

$$W(x, u) := |(x, u)|_{\text{graph}(K_\delta)} \quad (4.13)$$

is such that  $W(x, u) = 0 \iff u \in K_\delta(x)$ . The set

$$\bar{\mathcal{A}} := \{(x, u) \in \mathcal{X} \times \mathcal{U} \mid x \in \mathcal{A}, (x, u) \in \text{graph}(K)\} \subseteq \mathcal{X} \times \mathcal{U} \quad (4.14)$$

is compact. For any  $\delta \in \mathcal{PD}(\mathcal{A})$ ,  $\Gamma \in \mathcal{K}_\infty$  and  $V : \mathcal{X} \rightarrow \mathbb{R}_{\geq 0}$  upper semicontinuous (respectively, continuous), the function  $\bar{V} : \mathcal{X} \times \mathcal{U} \rightarrow \mathbb{R}_{\geq 0}$  defined as

$$\bar{V}(x, u) := \Gamma(V(x)) + W(x, u) \quad (4.15)$$

is upper semicontinuous (respectively, continuous). If there exist  $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$  such that  $\alpha_1(|x|_{\mathcal{A}}) \leq V(x) \leq \alpha_2(|x|_{\mathcal{A}})$  for all  $x \in \mathcal{X}$ , then there exist  $\bar{\alpha}_1, \bar{\alpha}_2 \in \mathcal{K}_\infty$  such that

$\bar{\alpha}_1(|(x, u)|_{\bar{\mathcal{A}}}) \leq \bar{V}(x, u) \leq \bar{\alpha}_2(|(x, u)|_{\bar{\mathcal{A}}})$  for all  $(x, u) \in (\mathcal{X} \times \mathcal{U})$ .

We now state the main result of this chapter. Under the conditions of Standing Assumption 4.1, we can establish that the Lyapunov condition in Assumption 4.1 is robust to sufficiently small strictly causal perturbations. In particular, the next result establishes that the Lyapunov conditions in Assumption 4.1 implies the existence of a Lyapunov function for a perturbed version of (4.4).

**Theorem 4.2** *If Assumption 4.1 holds, then  $\delta \in \mathcal{PD}(\mathcal{A})$ ,  $\Gamma \in \mathcal{K}_\infty$  and  $\varrho \in \mathcal{PD}(\mathcal{A})$  satisfying (4.9),  $W$  in (4.13),  $\bar{\mathcal{A}}$  in (4.14), and  $\bar{V}$  in (4.15) are such that for all  $(x, u) \in \mathcal{X} \times \mathcal{U}$  we have*

$$\int_{\mathcal{V}} \max_{g \in G_\delta(x, u, v)} \bar{V}(g) \mu(dv) \leq \bar{V}(x, u) - \bar{\varrho}(x, u), \quad (4.16)$$

with  $\bar{\varrho} \in \mathcal{PD}(\bar{\mathcal{A}})$  defined as  $\bar{\varrho}(x, u) := W(x, u)/2 + \varrho(x)$ .

*Proof:* With  $\bar{V} := \Gamma(V) + W$  as in (4.15), which is such that  $\bar{V} \in \mathcal{PD}(\bar{\mathcal{A}})$  according to Lemma 4.3, the Lyapunov condition (4.16) reads as

$$\int_{\mathcal{V}} \max_{\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \in G_\delta(x, u, v)} (\Gamma(V(g_1)) + W(g_1, g_2)) \mu(dv) \leq \Gamma(V(x)) + W(x, u) - \bar{\varrho}(x, u). \quad (4.17)$$

We notice that for any  $\delta \in \mathcal{PD}(\mathcal{A})$ , we have  $\bar{\mathcal{A}} = \{(x, u) \in \mathcal{X} \times \mathcal{U} \mid x \in \mathcal{A}, (x, u) \in \text{graph}(K)\} = \{(x, u) \in \mathcal{X} \times \mathcal{U} \mid x \in \mathcal{A}, (x, u) \in \text{graph}(K_\delta)\}$ . Now, for any  $\delta \in \mathcal{PD}(\mathcal{A})$ , if  $u \notin K_\delta(x)$  then by definition (4.12), we get  $G_\delta(x, u, v) = \emptyset$ , so that  $\max_{\begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \in \emptyset} \Gamma(V(g_1)) + W(g_1, g_2) = 0$ . Then (4.17) can be trivially satisfied by choosing  $\bar{\varrho}(x, u) := W(x, u)/2 + \varrho(x)$ , so that we get  $\Gamma(V(x)) - \varrho(x) + W(x, u)/2 \geq W(x, u)/2 \geq 0$ . We notice that  $\bar{\varrho} \in \mathcal{PD}(\bar{\mathcal{A}})$ . While if  $u \in K_\delta(x)$ , then  $W(x, u) = 0$  in view of Lemma 4.3 and, according to (4.12),  $g_2 \in K_\delta(g_1)$ , and hence  $W(g_1, g_2) = 0$  also in view of Lemma 4.3. Therefore we

get  $\max_{u \in K_\delta(x)} \int_{\mathcal{V}} \max_{g_1 \in f_\delta(x, u, v)} \Gamma(V(g_1)) \mu(dv) \leq \Gamma(V(x)) - \varrho(x)$ , which is equivalent to (4.9). ■

It follows from the regularity properties of  $G_\delta$  established in Proposition 4.2, the inequality (4.16) in Proposition 4.2 and the definition of Lyapunov function that  $\bar{V}$  (4.15) is an Lyapunov function relative to  $\bar{\mathcal{A}}$  defined in (4.14) for (4.12). In essence, the above result establishes the robustness of global asymptotic stability in probability property even under the action of a discontinuous control law for the closed loop stochastic system, provided the perturbation is sufficiently small and strictly causal. Similar results for the recurrence property also exist and we refer the reader to [72] for more details.

# Chapter 5

## Stochastic hybrid systems

### 5.1 Introduction

Stochastic hybrid systems (SHS) allow continuous-time evolution of the states, discrete-time events and probabilistic behavior. In SHS, randomness can affect the continuous-time dynamics, the discrete-time dynamics or the transition between the dynamics. Consequently, SHS models with varying degrees of complexity are studied in the literature. Frameworks for modeling SHS are in [23], [25], [24] and [78]. SHS models arise frequently in the context of complex systems like air traffic management systems, networked control systems and systems biology. See [79], [80], [81] and [26] for more details. The recent survey paper [27] presents a unified modeling framework for the various SHS representations in the literature and addresses stability related issues. In particular, important topics that are well studied in the case of non-stochastic hybrid systems like sufficient conditions for stability, weak sufficient conditions for stability, invariance principle, robust stability conditions and converse Lyapunov theorems are analyzed in detail in [27] for stochastic hybrid systems that produce unique solutions.

In this chapter, the class of systems we study are stochastic hybrid systems modeled by



set-valued mappings for which the randomness is restricted to the discrete-time dynamics. The system model we study can account for spontaneous transitions, forced transitions and probabilistic resets. We adopt the framework for modeling SHS with non-unique solutions proposed in [25] and [82]. This class of systems covers other frameworks such as piecewise-deterministic Markov processes (PDMP) and Markov jump systems.

The main goal of this chapter is to introduce the reader to a class of stochastic hybrid systems modeled by set-valued mappings and develop results related to the invariance principle. We use the invariance principle to develop weak sufficient conditions for stability and recurrence. As a consequence of the invariance principle we also establish sufficient conditions for stochastic stability properties that rely on Lyapunov-like functions satisfying strict decrease properties. The results in this chapter are from [83]. Other aspects related to stability theory like converse theorems and robustness are studied in detail in a subsequent chapter.

## 5.2 Preliminaries on stochastic hybrid systems

We consider a class of stochastic hybrid systems introduced in [25] with a state  $x \in \mathbb{R}^n$  and random input  $v \in \mathbb{R}^m$  written formally as

$$\dot{x} \in F(x), x \in C \tag{5.1a}$$

$$x^+ \in G(x, v^+), x \in D \tag{5.1b}$$

$$v \sim \mu(\cdot) \tag{5.1c}$$

where  $C, D \subset \mathbb{R}^n$  represent the flow and jump sets (where continuous and discrete evolution of the state is permitted) respectively and  $F, G$  represent the flow and jump maps respectively. The continuous-time dynamics is modeled by a differential inclusion and

the discrete-time dynamics is modeled by a stochastic difference inclusion.

The distribution function  $\mu$  is derived from the probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  and a sequence of independent, identically distributed (i.i.d.) input random variables  $\mathbf{v}_i : \Omega \rightarrow \mathbb{R}^m$  defined on  $(\Omega, \mathcal{F}, \mathbb{P})$  for  $i \in \mathbb{Z}_{\geq 1}$ . Then  $\mu$  is defined as  $\mu(A) = \mathbb{P}(\omega \in \Omega : \mathbf{v}_i(\omega) \in A)$  for every  $A \in \mathbf{B}(\mathbb{R}^m)$ . We denote by  $\mathcal{F}_i$  the collection of sets  $\{\omega : (\mathbf{v}_1(\omega), \dots, \mathbf{v}_i(\omega)) \in A\}$ ,  $A \in \mathbf{B}((\mathbb{R}^m)^i)$  which are the sub- $\sigma$  fields of  $\mathcal{F}$  that form the natural filtration of  $\mathbf{v} = \{\mathbf{v}_i\}_{i=1}^{\infty}$ . We refer to the stochastic hybrid system in (5.1) by the notation  $\mathcal{H}$ . For simplicity we will refer to the stochastic hybrid system through its data as

$$\mathcal{H} := (C, F, D, G, \mu). \quad (5.2)$$

We now define the notion of random solution to (5.1) under the following basic assumptions that is a combination of Standing Assumptions 2.1 and 3.1.

**Standing Assumption 5.1** *The data of the stochastic hybrid system  $\mathcal{H}$  satisfies the following conditions:*

1. *The sets  $C, D \subset \mathbb{R}^n$  are closed;*
2. *The mapping  $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$  is outer-semicontinuous, locally bounded with nonempty convex values on  $C$ ;*
3. *The mapping  $G : \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n$  is locally bounded and the mapping  $v \mapsto \text{graph}(G(\cdot, v)) := \{(x, y) \in \mathbb{R}^{2n} : y \in G(x, v)\}$  is measurable with closed values.*

### 5.2.1 Solution concept

Let  $(\Omega, \mathcal{F})$  be a measurable space. A stochastic hybrid arc is a mapping  $\mathbf{x}$  defined on  $\Omega$  such that  $\mathbf{x}(\omega)$  is a hybrid arc for each  $\omega \in \Omega$  and the set-valued mapping from  $\Omega$  to

$\mathbb{R}^{n+2}$  defined by

$$\omega \mapsto \text{graph}(\mathbf{x}(\omega)) := \{(t, j, z) : \phi = \mathbf{x}(\omega), (t, j) \in \text{dom}(\phi), z = \phi(t, j)\}$$

is  $\mathcal{F}$ -measurable with closed values. We define  $\text{graph}(\mathbf{x}(\omega))_{\leq j} := \text{graph}(\mathbf{x}(\omega)) \cap (\mathbb{R}_{\geq 0} \times \{0, \dots, j\} \times \mathbb{R}^n)$ . An  $\{\mathcal{F}_j\}_{j=0}^{\infty}$  adapted stochastic hybrid arc is a stochastic hybrid arc  $\mathbf{x}$  such that the mapping

$$\omega \mapsto \text{graph}(\mathbf{x}(\omega))_{\leq j} := \text{graph}(\mathbf{x}(\omega)) \cap (\mathbb{R}_{\geq 0} \times \{0, \dots, j\} \times \mathbb{R}^n$$

is  $\mathcal{F}_j$  measurable for each  $j \in \mathbb{Z}_{\geq 0}$ . An adapted stochastic hybrid arc  $\mathbf{x}$  is a solution starting from  $x$  denoted  $\mathbf{x} \in \mathcal{S}_r(x)$  if  $\mathbf{x}(\omega)$  is a solution to (5.1) with inputs  $\{\mathbf{v}_i(\omega)\}_{i=1}^{\infty}$ ; that is with  $\phi_\omega := \mathbf{x}(\omega)$  we have

1.  $\phi_\omega(0, 0) = x$ ;
2. if  $(t_1, j), (t_2, j) \in \text{dom}(\phi_\omega)$  with  $t_1 < t_2$  then, for almost every  $t \in [t_1, t_2]$ ,  $\phi_\omega(t, j) \in C$  and  $\dot{\phi}_\omega(t, j) \in F(\phi_\omega(t, j))$ ;
3. if  $(t, j), (t, j+1) \in \text{dom}(\phi_\omega)$  then  $\phi_\omega(t, j) \in D$  and  $\phi_\omega(t, j+1) \in G(\phi_\omega(t, j), \mathbf{v}_{j+1}(\omega))$ .

We observe that the set of hybrid arcs with closed graphs can be thought of as a subset in the space of not-identically empty-valued outer semicontinuous set-valued mappings from  $\mathbb{R}^2$  to  $\mathbb{R}^n$ . It follows from [84, Theorem 5.50], equipped with the metric of graph distance, this space is a separable, locally compact, complete (and  $\sigma$ -compact) metric space, which we denote  $(\mathcal{X}, \mathbf{d})$ .

The data  $(C, F, D, G, \mu)$  of the stochastic hybrid system  $\mathcal{H}$  are assumed to satisfy the conditions of Standing Assumption 5.1 throughout the rest of this chapter and the subsequent chapter. The main motivations for imposing such conditions are as follows.

Firstly, the sequential compactness of solutions of SHS established in [85] uses Standing Assumption 5.1, and are crucial to developing the invariance principle for SHS. Secondly, under the conditions of Standing Assumption 5.1, it is established in [25] that the system generates non-trivial random solutions. Finally, the equivalence between uniform and non-uniform versions of stability and recurrence which holds under the conditions of Assumption Standing Assumption 5.1 will be required to establish weak sufficient conditions for asymptotic stability in probability and recurrence.

### 5.3 Weak total recurrence

The rest of this chapter will be devoted to establishing a result similar to the invariance principle for the class of stochastic hybrid systems satisfying Standing Assumption 5.1. In this section we introduce the reader to the concept of “weak total recurrence in probability” of compact sets, explain the importance of this concept and present motivations for considering this concept over the more frequently studied “invariant” set concept.

A notion of weak total recurrence of sets is presented in [61] for stochastic difference inclusions. We note that we use a weaker definition of weak total recurrence introduced in [86] that utilizes probabilities of certain events as opposed to expected values used in [61]. The main reason for adopting the definitions in [86] is that the weaker version of the definition is sufficient for establishing the main results of the chapter.

We now recall the notion of weak total recurrence introduced in [86]. Given compact sets  $S, K \subset \mathbb{R}^n$ ,  $0 \leq \tau_1 < \tau_2$  and  $\phi$  a solution to the non-stochastic hybrid system

$(K, F, K, K)$  (with  $C = K$ ,  $D = K$  and  $G(x) = K$ ) starting from  $K$  we define

$$\begin{aligned} \varphi_{\tau_1, \tau_2, S}(\phi) := & \max_{(t_1, j_1), (t_2, j_2) \in \text{dom}\phi} \int_{t_1}^{t_2} \mathbb{I}_S(\phi(s, j(s))) ds + \sum_{i=j_1+1}^{j_2} \mathbb{I}_S(\phi(t(i), i-1)) \\ & \tau_1 \leq t_1 + j_1 \leq t_2 + j_2 \leq \tau_2 \end{aligned}$$

where  $j(s)$  is the smallest index  $j$  such that  $(s, j) \in \text{dom}(\phi)$  and  $t(i)$  is the smallest time  $t$  such that  $(t, i) \in \text{dom}(\phi)$ . For the case when  $\tau_1 = 0$  and  $\tau_2 = \tau$ , we refer to the mapping by  $\varphi_{\tau, S}(\cdot)$ . The mapping  $\varphi_{\tau, S}(\cdot)$  refers to the total amount of time that a hybrid arc spends in the set  $S$  within hybrid time  $\tau$ . For the case when  $\phi \in \mathcal{X}$  is not in the set of solutions generated by  $(K, F, K, K)$  we define  $\varphi_{\tau, S}(\phi) = 0$ . It can be observed that if  $S_1 \subset S_2$  then  $\varphi_{\tau, S_1}(\phi) \leq \varphi_{\tau, S_2}(\phi)$ . More generally,

$$S \subset \bigcup_{i=1}^n S_i \implies \varphi_{\tau, S}(\phi) \leq \sum_{i=1}^n \varphi_{\tau, S_i}(\phi). \quad (5.3)$$

The next result is proved in the appendix and it establishes that the function  $\varphi_{\tau_1, \tau_2, S}$  is upper semicontinuous with respect to the hybrid arcs generated by the system  $(K, F, K, K)$ .

**Lemma 5.1** *Let  $K, S \subset \mathbb{R}^n$  be compact. For each  $0 \leq \tau_1 < \tau_2$ , and a sequence of solutions  $\phi_i$  converging to a solution  $\phi$ , we have  $\limsup_{i \rightarrow \infty} \varphi_{\tau_1, \tau_2, S}(\phi_i) \leq \varphi_{\tau_1, \tau_2, S}(\phi)$ .*

Let  $\Psi \subset \mathbb{R}^n$  be compact. For each  $\varepsilon > 0$  and compact set  $K \subset \mathbb{R}^n$ , let  $\mathcal{S}_r^\varepsilon(K)$  denote the solutions of  $(C_\varepsilon, F, D_\varepsilon, G_\varepsilon)$  from the set  $K$  where

$$C_\varepsilon := C \cap (\Psi + \varepsilon\mathbb{B}) \quad (5.4a)$$

$$D_\varepsilon := D \cap (\Psi + \varepsilon\mathbb{B}) \quad (5.4b)$$

$$G_\varepsilon(x, v) := G(x, v) \cap (\Psi + \varepsilon\mathbb{B}). \quad (5.4c)$$

**Definition 5.1** A point  $x \in \Psi$  is said to be weakly recurrent in probability relative to  $\Psi$  for  $\mathcal{H}$  if, for each  $\varepsilon > 0$  there exists  $\varrho > 0$  and for each  $\Delta > 0$  there exist  $\tau > 0$  and  $\mathbf{x} \in \mathcal{S}_r^\varepsilon(\Psi + \varepsilon\mathbb{B})$  such that, with the definitions  $S_\varepsilon := \{x\} + \varepsilon\mathbb{B}$  and (5.4),

$$\mathbb{P}(\Delta \leq \varphi_{\tau, S_\varepsilon}(\mathbf{x})) \geq \varrho. \quad (5.5)$$

In other words, a point is weakly recurrent relative to  $\Psi$  if, for every neighborhood of the point, there exists a random solution visiting the neighborhood for arbitrarily large times with positive probability while staying close to the  $\Psi$ . A compact set  $\Psi \subset \mathbb{R}^n$  is said to be *weakly totally recurrent in probability for  $\mathcal{H}$*  if each point in  $\Psi$  is weakly recurrent in probability relative to  $\Psi$  for  $\mathcal{H}$ . The mapping  $\omega \mapsto \varphi_{\tau, S_\varepsilon}(\mathbf{x}(\omega))$  is measurable due to Lemma 5.1 and the mapping  $\omega \mapsto \mathbf{x}(\omega)$  being measurable from [85, Section II.B]. Hence, the event  $\{\omega : \Delta \leq \varphi_{\tau, S_\varepsilon}(\mathbf{x}(\omega))\}$  is measurable.

The motivation for considering the concept of weak total recurrence as opposed to the concept of invariance is that even for non-stochastic hybrid systems weakly totally recurrent sets are typically smaller than weakly invariant (forward and backward) sets and hence, establishing convergence to weakly totally recurrent sets provides a sharper characterization. For stochastic hybrid systems, similar connections between weakly totally recurrent in probability sets, weakly forward invariant sets (in an almost sure sense) and an intermediary quasi-invariance property are studied. Also, as observed in [86], weak backward invariance does not seem to be a natural concept to study for stochastic hybrid systems and hence only minimal observations regarding the property are presented in this chapter.

For any compact set  $K$ , the union of all subsets of  $K$  that are weakly totally recurrent in probability provides the largest set in  $K$  that is weakly totally recurrent in probability. The main results in this chapter are stated in terms of such sets. The next result precisely

establishes the notion of largest weakly totally recurrent in probability set inside compact sets.

**Lemma 5.2** *Let  $K \subset \mathbb{R}^n$  be compact and let  $\mathcal{R}$  be a collection of subsets of  $K$  that are weakly totally recurrent in probability for  $\mathcal{H}$ . Then the set  $\widehat{\Psi} := \overline{\bigcup_{\Psi \in \mathcal{R}} \Psi}$  is a compact subset of  $K$  that is weakly totally recurrent in probability for  $\mathcal{H}$ .*

*Proof:* The containment  $\widehat{\Psi} \subset K$  is a result of  $K$  being compact and  $\Psi \subset K$  for each  $\Psi \in \mathcal{R}$ . Let  $\zeta \in \widehat{\Psi}$  and  $\varepsilon > 0$  be arbitrary. From the definition of  $\widehat{\Psi}$ , there exist  $\Psi \in \mathcal{R}$ ,  $\eta \in \Psi$  and  $\varepsilon_1 \in (0, \varepsilon)$  satisfying  $S_{\varepsilon_1} := \{\eta\} + \varepsilon_1\mathbb{B} \subset \{\zeta\} + \varepsilon\mathbb{B} =: S_\varepsilon$ . In turn, it follows from the weak total recurrence in probability of  $\Psi$  that there exist  $\varrho > 0$  and for each  $\Delta > 0$  there exist  $\mathbf{x} \in \mathcal{S}_r^{\varepsilon_1}(\Psi + \varepsilon_1\mathbb{B})$  and  $\tau > 0$  such that (5.5) holds with  $S_{\varepsilon_1}$  in place of  $S_\varepsilon$  and, since  $S_{\varepsilon_1} \subset S_\varepsilon$ , (5.5) holds with  $S_\varepsilon$  not replaced by  $S_{\varepsilon_1}$  as well. Since  $\Psi + \varepsilon_1\mathbb{B} \subset \widehat{\Psi} + \varepsilon\mathbb{B}$ , it follows that  $\mathbf{x} \in \mathcal{S}_r^\varepsilon(\widehat{\Psi} + \varepsilon\mathbb{B})$ , where the solutions  $\mathcal{S}_r^\varepsilon$  come from the data (5.4) with  $\widehat{\Psi}$  in place of  $\Psi$ . It follows that  $\zeta$  is weakly totally recurrent in probability relative to  $\widehat{\Psi}$ . ■

## 5.4 The recurrence principles

In this section, we present the main results of this chapter related to the sets to which bounded random solutions converge. Since we characterize convergence to sets that are weakly totally recurrent as opposed to weakly invariant, we refer to our results as “recurrence principles”. The proofs of the main results are presented in the Appendix.

### 5.4.1 Limit sets of random solutions

For the stochastic hybrid system in (5.1), we now define the notion of a limit set of a bounded random solution. Given a compact set  $K \subset \mathbb{R}^n$ , a random solution  $\mathbf{x}$  is said

to be *almost surely contained in  $K$*  if  $\text{graph}(\mathbf{x}(\omega)) \subset \mathbb{R}^2 \times K$  for almost all  $\omega \in \Omega$ . A random solution  $\mathbf{x}$  is said to be *complete with positive probability* if there exists  $\rho > 0$  such that  $\mathbb{P}(\text{dom } \mathbf{x} \cap \Gamma_{\geq i} \neq \emptyset \quad \forall i \in \mathbb{Z}_{\geq 0}) \geq \rho$ .

**Definition 5.2** *For a random solution  $\mathbf{z}$  that is almost surely contained in a compact set and complete with positive probability, we define its recurrent in probability set, denoted  $\Psi(\mathbf{z})$ , to be the set of points  $\zeta \in \mathbb{R}^n$  such that, for each  $\varepsilon > 0$  there exists  $\varrho > 0$  and for each  $\Delta > 0$  there exists  $\tau > 0$  such that, with  $S_\varepsilon := \{\zeta\} + \varepsilon\mathbb{B}$ ,  $\mathbb{P}(\Delta \leq \varphi_{\tau, S_\varepsilon}(\mathbf{z})) \geq \varrho$ .*

In other words, the set  $\Psi(\mathbf{z})$  denotes the set of points such that the solution  $\mathbf{z}$  visits every neighborhood of the set for arbitrarily large times with a positive probability.

For non-stochastic hybrid systems (2.1) in Chapter 2, under Standing Assumption 3.1, it is established in [14, Prop 6.21] that a complete, bounded solution of (2.1) converges to its  $\Omega$ -limit set which is non-empty, compact and satisfies a weak invariance property. The first main result of this chapter establishes a similar characterization of the behavior of a random solution  $\mathbf{z}$  that is almost surely bounded and complete with positive probability. In particular, we establish convergence properties with respect to the limit set  $\Psi(\mathbf{z})$  and prove that  $\Psi(\mathbf{z})$  is non-empty, compact and satisfies a weak total recurrence property

**Theorem 5.1** *Let  $K_\infty \subset K \subset \mathbb{R}^n$  be compact and  $\mathbf{z}$  be almost surely contained in  $K$ , complete with positive probability, and such that almost every complete sample path converges to  $K_\infty$ . Then  $\Psi(\mathbf{z})$  is nonempty, compact, contained in  $K_\infty$ , weakly totally recurrent in probability, and almost every complete sample path of  $\mathbf{z}$  converges to  $\Psi(\mathbf{z})$ .*

### 5.4.2 Krasovskii-LaSalle functions

In this section we describe Lyapunov-like functions that are non-increasing during flows and non-increasing on average during jumps.



Let  $K \subset \Lambda \subset \mathbb{R}^n$  be compact sets. A continuous function  $V : \Lambda \rightarrow \mathbb{R}_{\geq 0}$  is a *stochastic Krasovskii-LaSalle* function relative to  $(K, \Lambda)$  if

$$V(\phi(t)) \leq V(x) - \int_0^t \kappa(\phi(s)) ds, t \in \text{dom}(\phi), \phi \in \mathcal{S}_{C \cap \Lambda}^F(x)$$

$$\int_{\mathbb{R}^m} \max_{g \in G(x,v) \cap \Lambda} V(g) \mu(dv) \leq V(x) - \kappa(x), \forall x \in D \cap \Lambda,$$

where  $\kappa : \Lambda \rightarrow \mathbb{R}_{\geq 0}$  is continuous and  $\kappa(x) > 0$  when  $x \in \Lambda \setminus K$  and  $\mathcal{S}_{C \cap \Lambda}^F(x)$  refers to the solutions of (5.1a) starting at  $x$  with the flow set  $C \cap \Lambda$ . Since  $\Lambda$  is compact and  $V$  is continuous there exist  $0 \leq c_1 < c_2$  such that  $V(\Lambda) \in [c_1, c_2]$ . Under the existence of Krasovskii-LaSalle functions, we will refine the sets to which bounded random solutions converge.

For non-stochastic hybrid systems (2.1) in Chapter 2, under Standing Assumption 3.1 and the existence of a non-increasing Lyapunov function, it is established in [14, Thm 8.2] that complete, bounded solutions of (2.1) converge to the largest weakly invariant set within the level set of the Lyapunov function. The second main result of this chapter establishes a similar characterization for the complete sample paths of a bounded random solution  $\mathbf{x}$  in the presence of a non-increasing on average Lyapunov-like function. In particular, we establish that almost every complete path of  $\mathbf{x}$  converges to the largest weakly totally recurrent set within the level set of the Lyapunov-like function.

**Theorem 5.2** *Let  $V$  be a stochastic Krasovskii-LaSalle function relative to  $(K, \Lambda)$ . Then, for every random solution  $\mathbf{x}$  generated from the data  $(C \cap \Lambda, F, D \cap \Lambda, G \cap \Lambda, \mu)$  almost every complete sample path  $\mathbf{x}(\omega)$  converges to the largest weakly totally recurrent in probability set contained in  $K \cap L_V(c(\omega))$  for some  $c(\omega) \in [c_1, c_2]$ .*

## 5.5 Corollaries of the recurrence principle

In this section we present some important corollaries of Theorem 5.1 and also make connections to the recurrence principle established for stochastic difference inclusions in [61].

**Corollary 5.1** *Let  $K \subset \mathbb{R}^n$  be compact, let  $\mathbf{z}$  be a solution that is almost surely contained in  $K$  and let  $K_\infty$  be such that, for each  $\varepsilon > 0$  and  $\varrho > 0$  there exists  $\Delta > 0$  such that, with  $S_\varepsilon := K \setminus (K_\infty + \varepsilon\mathbb{B}^\circ)$ , we have*

$$\mathbb{P}(\Delta \leq \varphi_{\tau, S_\varepsilon}(\mathbf{z})) \leq \varrho \quad \forall \tau \geq 0. \quad (5.6)$$

*Under these conditions, almost every complete sample path of  $\mathbf{z}$  converges to the largest weakly totally recurrent set contained in  $K_\infty$ .*

*Proof:* We claim that, under the conditions of the corollary, almost every complete sample path converges to  $K_\infty$ . Indeed, if this is not the case then there exists  $\varepsilon > 0$  and  $\varrho > 0$  and for each  $\Delta > 0$  there exists  $\tau$  such that

$$\mathbb{P}(\Delta \leq \varphi_{\tau, S_\varepsilon}(\mathbf{z})) > \varrho. \quad (5.7)$$

But this contradicts the assumption of the Corollary. Now the result follows from Theorem 5.1. ■

**Corollary 5.2** *Let  $K \subset \mathbb{R}^n$  be compact, let  $\mathbf{z}$  be a solution that is almost surely contained in  $K$  and let  $K_\infty$  be such that, for each  $\varepsilon > 0$  there exists  $\Delta > 0$  such that, with  $S_\varepsilon := K \setminus (K_\infty + \varepsilon\mathbb{B}^\circ)$ , we have*

$$\mathbb{E}[\varphi_{\tau, S_\varepsilon}(\mathbf{z})] \leq \Delta \quad \forall \tau \geq 0. \quad (5.8)$$

Under these conditions, almost every complete sample path of  $\mathbf{z}$  converges to the largest weakly totally recurrent set contained in  $K_\infty$ .

*Proof:* We claim that (5.8) implies (5.6). Indeed, suppose (5.8) holds but (5.6) does not hold, i.e., there exists  $\varepsilon > 0$  and  $\varrho > 0$  such that for  $\hat{\Delta} > \Delta/\varrho$  there exists  $\tau > 0$  such that

$$\mathbb{P}\left(\hat{\Delta} \leq \varphi_{\tau, S_\varepsilon}(\mathbf{z})\right) \geq \varrho. \quad (5.9)$$

Then  $\mathbb{E}[\varphi_{\tau, S_\varepsilon}(\mathbf{z})] \geq \hat{\Delta}\varrho > \Delta$ , which contradicts the bound (5.8) and establishes the result.  $\blacksquare$

Given a compact set  $K \subset \mathbb{R}^n$ , lower semicontinuous functions  $\kappa_1, \kappa_2 : K \rightarrow \mathbb{R}_{\geq 0}$ , and  $\tau > 0$ , for each  $\phi$  that is a solution of  $(K, F, K, K)$  we define

$$\varphi_{\tau, \kappa_1, \kappa_2}(\phi) := \max_{(t, j) \in \text{dom}\phi, t+j \leq \tau} \left( \int_0^t \kappa_1(\phi(s, j(s))) ds + \sum_{i=1}^j \kappa_2(\phi(t(i), i-1)) \right).$$

The following result will be used in the proof of Theorem 5.2 and is similar to the result in [48, Corollary 5.6] for non-stochastic hybrid systems.

**Corollary 5.3** *Let  $K \subset \mathbb{R}^n$  be compact,  $\mathbf{z}$  be a solution that is almost surely contained in  $K$ , and  $\kappa_1, \kappa_2 : K \rightarrow \mathbb{R}_{\geq 0}$  be lower semicontinuous functions such that, for some  $\Delta > 0$ ,*

$$\mathbb{E}[\varphi_{i, \kappa_1, \kappa_2}(\mathbf{z})] \leq \Delta \quad \forall i \in \mathbb{Z}_{\geq 0}. \quad (5.10)$$

*Then almost every complete sample path of  $\mathbf{z}$  converges to the largest weakly totally recurrent set contained in the union of the zero-level sets of  $\kappa_1$  and  $\kappa_2$ .*

*Proof:* We first note that the measurability of the mapping  $\omega \mapsto \varphi_{i, \kappa_1, \kappa_2}(\mathbf{x}(\omega))$  follows from induction due to the measurability of  $\omega \mapsto \mathbf{x}(\omega)$  and the lower semicontinuity

of  $\kappa_1$  and  $\kappa_2$ . Define  $K_\infty := \{x \in K : \kappa_1(x)\kappa_2(x) = 0\}$ . For each  $\varepsilon > 0$ , define the compact set  $S_\varepsilon := K \setminus (K_\infty + \varepsilon\mathbb{B}^\circ)$  and

$$\underline{\kappa}_\varepsilon := \min_{i \in \{1,2\}} \inf_{z \in S_\varepsilon} \kappa_i(z). \quad (5.11)$$

Since  $\kappa_1$  and  $\kappa_2$  are lower semicontinuous, it follows that  $\underline{\kappa}_\varepsilon > 0$ . Indeed, if  $\underline{\kappa}_\varepsilon = 0$  then there exist  $j \in \{1, 2\}$  and a sequence  $z_i \in S_\varepsilon$  converging to some  $z \in S_\varepsilon$  with  $\kappa_j(z_i) \rightarrow 0$  as  $i \rightarrow \infty$ . Then by lower semicontinuity  $\kappa_j(z) \leq \liminf_{i \rightarrow \infty} \kappa_j(z_i) = 0$ , which contradicts  $z \in S_\varepsilon$ . Now the result follows from the bound

$$\mathbb{I}_{S_\varepsilon}(z) \leq \kappa_j(z)/\underline{\kappa}_\varepsilon \quad \forall z \in K, j \in \{1, 2\} \quad (5.12)$$

which gives that without loss of generality (5.10) implies (5.8) with  $\Delta/\underline{\kappa}_\varepsilon$  in place of  $\Delta$ . ■

The main difference between the next theorem and Theorem 5.1 is the assumption of the random solution being contained almost surely in a compact set. In particular, the following result focuses only on convergence of sample paths of the random solution that remains bounded. A similar result is established in [61, Thm 6] for stochastic difference inclusions. The proof is presented in the appendix.

**Theorem 5.3** *Let  $K_\infty \subset \mathbb{R}^n$  be compact. Let  $\mathbf{x}$  be a random solution and  $\Omega_{K,\infty}$  be the set of all  $\omega \in \Omega$  such that  $\mathbf{x}(\omega)$  is complete and converges to  $K_\infty$ . Then, for almost every  $\omega \in \Omega_{K,\infty}$ ,  $\mathbf{x}(\omega)$  converges to the largest weakly totally recurrent in probability set contained within  $K_\infty$ .*

## 5.6 Comparison to invariance properties

In this section, we will compare the weak total recurrence in probability concept to well known invariance concepts. In particular, we establish that

1. Each compact set that is weakly forward invariant almost surely contains a weakly totally recurrent in probability set
2. Each compact set that is weakly totally recurrent in probability is weakly forward invariant almost surely.

Hence, our motivation to study weakly totally recurrent sets as opposed to weakly forward invariant sets is justified since the former is usually smaller and provides a sharper characterization when describing solution behavior. We will also describe an intermediary invariance property introduced in [86] called “weak quasi-return invariance” and the proof for establishing the relationship between weak forward invariance and weak total recurrence relies on this intermediate property.

### 5.6.1 Weak quasi-return invariance

A compact set  $\Psi \subset \mathbb{R}^n$  is *weakly long-time quasi-return-invariant in probability* for  $\mathcal{H}$  if, for each  $x \in \Psi$ ,  $\tau > 0$ , and  $\varepsilon > 0$ , and with the definition  $S_\varepsilon := \{x\} + \varepsilon\mathbb{B}$ , there exists  $\mathbf{x} \in \mathcal{S}_r^\varepsilon(S_\varepsilon)$  (where, as before,  $\mathcal{S}_r^\varepsilon$  denotes solutions of  $(C_\varepsilon, F, D_\varepsilon, G_\varepsilon, \mu)$  defined via (5.1a) and (5.4)) such that

$$\mathbb{P}(\text{graph}(\mathbf{x}) \cap (\Gamma_{\geq \tau} \times S_\varepsilon) \neq \emptyset) \geq 1 - \varepsilon. \quad (5.13)$$

In essence a set is weakly long-time quasi-return invariant in probability if for every neighborhood of every point in the set there exists a random solution such that the

probability of visiting the neighborhood after arbitrarily large times while staying close to the set can be made arbitrarily close to one.

**Theorem 5.4** *If a compact set is weakly totally recurrent in probability for  $\mathcal{H}$  then it is weakly long-time quasi-return-invariant in probability for  $\mathcal{H}$ .*

The following lemma is used to prove Theorem 5.4. It establishes that a set that is not weakly long-time quasi-return-invariant in probability for  $\mathcal{H}$  is also not weakly totally recurrent in probability for  $\mathcal{H}$ .

**Lemma 5.3** *Suppose  $x \in \Psi$ ,  $\tau > 0$  and  $\varepsilon > 0$  are such that, with the definitions  $S_\varepsilon := \{x\} + \varepsilon\mathbb{B}$  and (5.4), for each solution  $\mathbf{y} \in \mathcal{S}_\tau^\varepsilon(S_\varepsilon)$ ,*

$$\mathbb{P}(\text{graph}(\mathbf{y}) \cap (\Gamma_{\geq \tau} \times S_\varepsilon) \neq \emptyset) \leq 1 - \varepsilon. \quad (5.14)$$

*Under these conditions, for each  $\varepsilon_1 \in (0, \varepsilon)$  and with the definitions  $S_{\varepsilon_1} := \{x\} + \varepsilon_1\mathbb{B}$  and (5.4), for each  $\mathbf{x} \in \mathcal{S}_\tau^{\varepsilon_1}(\Psi + \varepsilon_1\mathbb{B})$ ,*

$$\mathbb{E} [\varphi_{j\tau, S_{\varepsilon_1}}(\mathbf{x})] \leq \tau \left( 1 + \sum_{i=0}^{j-1} (1 - \varepsilon)^i \right) \quad \forall j \in \mathbb{Z}_{\geq 1} \quad (5.15)$$

so that

$$\mathbb{E} [\varphi_{\hat{\tau}, S_{\varepsilon_1}}(\mathbf{x})] \leq \tau (1 + \varepsilon^{-1}) \quad \forall \hat{\tau} > 0. \quad (5.16)$$

*In particular,  $x$  is not weakly recurrent in probability with respect to  $\Psi$  for  $\mathcal{H}$ .*

*Proof:* We use the notation  $\mathbf{x}_\omega := \mathbf{x}(\omega)$ . We define a sequence of hitting times as follows:  $(\mathbf{T}_0(\omega), \mathbf{J}_0(\omega)) := (0, 0)$  and, for each  $i \in \mathbb{Z}_{\geq 0}$ ,  $(\mathbf{T}_{i+1}(\omega), \mathbf{J}_{i+1}(\omega))$  is the infimum

over  $(t, j) \in \text{dom } \mathbf{x}_\omega$  such that  $t + j \geq \tau + \mathbf{T}_i(\omega) + \mathbf{J}_i(\omega)$  and  $\mathbf{x}_\omega(t, j) \in \{x\} + \varepsilon \mathbb{B}^\circ$ . By this construction,

$$j\tau \leq \mathbf{T}_j(\omega) + \mathbf{J}_j(\omega) \quad \forall (j, \omega) \in \mathbb{Z}_{\geq 0} \times \Omega \quad (5.17)$$

and the amount of hybrid time that a trajectory  $\mathbf{x}_\omega$  spends in the set  $S_{\varepsilon_1}$  between  $(\mathbf{T}_i(\omega), \mathbf{J}_i(\omega))$  and  $(\mathbf{T}_{i+1}(\omega), \mathbf{J}_{i+1}(\omega))$  is bounded by  $\tau$ . Let  $\Omega_{\mathbf{T}_i} := \{\omega : \mathbf{T}_i(\omega) \neq \emptyset\}$ . Due to the assumption of the lemma,

$$\mathbb{P}(\Omega_{\mathbf{T}_i}) \leq (1 - \varepsilon)^{i-1} \quad \forall i \in \mathbb{Z}_{\geq 1}. \quad (5.18)$$

It follows from these observations that, for each  $j \in \mathbb{Z}_{\geq 1}$ ,

$$\begin{aligned} \mathbb{E} [\varphi_{j\tau, S_{\varepsilon_1}}(\mathbf{x})] &\leq \tau \sum_{i=0}^j \mathbb{P}(\Omega_{\mathbf{T}_i}) = \tau \left( \mathbb{P}(\Omega_{\mathbf{T}_0}) + \sum_{i=1}^j \mathbb{P}(\Omega_{\mathbf{T}_i}) \right) \\ &= \tau \left( 1 + \sum_{i=1}^j (1 - \varepsilon)^{i-1} \right) = \tau \left( 1 + \sum_{i=0}^{j-1} (1 - \varepsilon)^i \right) \end{aligned}$$

which is (5.15). Then (5.16) follows from the fact that  $\sum_{i=0}^{\infty} (1 - \varepsilon)^i = \varepsilon^{-1}$ . In turn,  $x$  cannot be weakly recurrent in probability with respect to  $\Psi$  for  $\mathcal{H}$  since the condition (5.5) and that the fact that  $\Delta > 0$  can be made arbitrarily large by picking  $\tau$  sufficiently large implies that  $\mathbb{E} [\varphi_{j\tau, S_{\varepsilon_1}}(\mathbf{x})]$  grows unbounded with  $\tau$ .  $\blacksquare$

We also refer the reader to [86, Example 1] which illustrates the gap between weakly totally recurrent sets and weakly quasi invariant sets.

## 5.6.2 Weak forward invariance

A compact set  $\Psi \subset \mathbb{R}^n$  is *weakly forward invariant almost surely for  $\mathcal{H}$*  if, for each  $x \in \Psi$ , there exists  $\mathbf{x} \in \mathcal{S}_r(x)$  such that, for almost every  $\omega \in \Omega$ ,  $\mathbf{x}(\omega)$  is complete and

remains in  $\Psi$  for all time. The next result establishes that a compact set that is weakly forward invariant almost surely for  $\mathcal{H}$  contains a weakly totally recurrent in probability set and is a consequence of the recurrence principle in Theorem 5.1.

**Proposition 5.1** *Each compact set that is weakly forward invariant almost surely for  $\mathcal{H}$  contains a nonempty, compact set that is weakly totally recurrent in probability for  $\mathcal{H}$ .*

*Proof:* Let  $K$  denote the compact set that is weakly forward invariant almost surely. According to this property, there exists a solution  $\mathbf{z}$  that is complete and contained in  $K$  almost surely. Define  $K_\infty := K$ . By Theorem 5.1, the recurrent in probability set for  $\mathbf{z}$  is nonempty, compact, contained in  $K$ , and weakly totally recurrent in probability for  $\mathcal{H}$ , which establishes the result.  $\blacksquare$

The next result relies on a sequential compactness result established [85] for the class of stochastic hybrid systems studied in this chapter.

**Theorem 5.5** *If a compact set is weakly long-time quasi-return-invariant in probability for  $\mathcal{H}$  then it is weakly forward invariant almost surely for  $\mathcal{H}$ .*

*Proof:* Let  $x \in \Psi$ . Using weak long-time quasi-return-invariance in probability for  $\mathcal{H}$ , for each  $i \in \mathbb{Z}_{\geq 1}$  there exists  $\mathbf{x}_i \in \mathcal{S}_r^{i-1}(S_{i-1})$  (with  $S_{i-1} := \{x\} + 1/i\mathbb{B}$ ) such that

$$\mathbb{P}(\text{graph}(\mathbf{x}_i) \cap (\Gamma_{\geq i-1} \times S_{i-1}) \neq \emptyset) \geq 1 - i^{-1}. \quad (5.19)$$

For each  $i \in \mathbb{Z}_{\geq 1} \cup \{\infty\}$ , define  $\varphi_i : \mathcal{X} \rightarrow \mathbb{R}_{\geq 0}$  as

$$\varphi_i(T) := \begin{cases} 1 & \text{graph}(T) \subset (\mathbb{R}^2 \times (\Psi + i^{-1}\mathbb{B})) \\ 0 & \text{otherwise.} \end{cases} \quad (5.20)$$

With  $\varphi := \varphi_\infty$ , it can be shown that [85, Assumption 2] holds. It follows from (5.4), the fact that  $\mathbf{x}_i \in \mathcal{S}_r^{i-1}(S_{i-1})$ , and (5.20) that  $1 = \mathbb{E}[\varphi_i(\mathbf{x}_i)]$  for all  $i \in \mathbb{Z}_{\geq 1}$ . It now follows



from [85, Corollary 1] that for each  $x \in \Psi$  there exists  $\mathbf{x} \in \mathcal{S}_r(x)$  such that  $\mathbf{x}$  is complete and  $1 = \mathbb{E}[\varphi(\mathbf{x})]$ , i.e.,  $\mathbf{x}$  remains in  $\Psi$  almost surely. In other words,  $\Psi$  is weakly forward invariant almost surely for  $\mathcal{H}$ .  $\blacksquare$

**Corollary 5.4** *If a compact set is weakly totally recurrent in probability for  $\mathcal{H}$  then it is weakly forward invariant almost surely for  $\mathcal{H}$ .*

### 5.6.3 Weak backward invariance

A compact set  $\Psi \subset \mathbb{R}^n$  is *weakly backward invariant almost surely for  $\mathcal{H}$*  if, for each  $\zeta \in \Psi$  and  $\tau > 0$  there exists  $\mathbf{x} \in \mathcal{S}_r(\Psi)$  such that, for almost every  $\omega \in \Omega$ ,  $\mathbf{x}(\omega)$  reaches  $\zeta$  after hybrid time  $\tau$  and remains in  $\Psi$  before reaching  $\zeta$ . The next result establishes a connection between long-time quasi-return invariance and weak backward invariance for the specific case of non-stochastic systems.

**Proposition 5.2** *If  $\Psi \subset \mathbb{R}^n$  can be established to be weakly long-time quasi-return-invariant in probability for  $\mathcal{H}$  using solutions that are almost surely constant (as a function  $\omega$ ) then  $\Psi$  is weakly backward invariant almost surely for  $\mathcal{H}$ .*

*Proof:* Let  $\zeta \in \Psi$  and  $\tau > 0$ . Due to the assumption of the proposition, for each  $i \in \mathbb{Z}_{\geq 1}$  there exists  $\mathbf{x}_i \in \mathcal{S}_r^{i-1}(S_{i-1})$  such that

$$\text{graph}(\mathbf{x}_i) \cap (\Gamma_{\geq \tau} \times S_{i-1}) \neq \emptyset \quad a.s. \quad (5.21)$$

where we have used that  $\omega \mapsto \mathbf{x}_i(\omega)$  is almost surely constant and  $S_{i-1} = \{\zeta\} + i^{-1}\mathbb{B}$ . Again using this property and by applying a time shift to  $\mathbf{x}_i(\omega)$  (similar to the proof of [14, Prop 6.21]) we can assume that

$$\text{graph}(\mathbf{x}_i) \cap ((\Gamma_{\geq \tau} \cap \Gamma_{\leq \tau+1}) \times S_{i-1}) \neq \emptyset \quad a.s. \quad (5.22)$$

though we no longer can assume that the solutions start in  $S_{i-1}$ . Now, for  $i \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ , we let  $\varphi_i$  be the indicator function on mappings whose graphs are contained in  $\mathbb{R}^2 \times (\Psi + i^{-1}\mathbb{B})$  and that intersect the compact set  $(\Gamma_{\geq \tau} \cap \Gamma_{\leq \tau+1}) \times S_{i-1}$ . These functions and  $\varphi := \varphi_\infty$  satisfy [85, Assumption 2]. Moreover,  $\varphi_i(\mathbf{x}_i) = 1$  almost surely. It thus follows from [85, Theorem 1] that there exists  $\mathbf{x} \in \mathcal{S}_r^0(\Psi)$  such that  $\mathbb{E}[\varphi(\mathbf{x})] = 1$ . This solution verifies almost sure weak backward invariance. ■

The importance of the assumption regarding the non-stochastic nature of the hybrid system in Proposition 5.2 is illustrated through [86, Example 3]. The example highlights that Proposition 5.2 is not true for general stochastic hybrid systems.

## 5.7 Application to stability theory

The definitions of stochastic stability properties for the class of SHS in (5.1) is stated below, and are adopted from [25].

**Definition 5.3** *The compact set  $\mathcal{A} \subset \mathbb{R}^n$  is uniformly Lyapunov stable in probability for (5.1) if for each  $\varepsilon > 0$  and  $\rho > 0$  there exists a  $\delta > 0$  such that, for  $\xi \in \mathcal{A} + \delta\mathbb{B}$ ,  $\mathbf{x} \in \mathcal{S}_r(\xi)$*

$$\mathbb{P}\left(\text{graph}(\mathbf{x}) \subset (\mathbb{R}^2 \times (\mathcal{A} + \varepsilon\mathbb{B}^o))\right) \geq 1 - \rho. \quad (5.23)$$

**Definition 5.4** *The compact set  $\mathcal{A}$  is uniformly Lagrange stable in probability for (5.1) if for each  $\delta > 0$  and  $\rho > 0$ , there exists  $\varepsilon > 0$  such that the inequality (5.23) holds.*

The set  $\mathcal{A}$  is *uniformly globally stable in probability* for (5.1) if it is both Lyapunov stable and Lagrange stable in probability for (5.1).

**Definition 5.5** *The set  $\mathcal{A}$  is uniformly globally attractive in probability for (5.1) if for*

each  $\varepsilon > 0$ ,  $\rho > 0$  and  $R > 0$  there exists a  $\tau \geq 0$  so that, for  $\mathbf{x} \in \mathcal{S}_r(\mathcal{A} + R\mathbb{B})$ ,

$$\mathbb{P}\left(\text{graph}(\mathbf{x}) \cap (\Gamma_{\geq \tau} \times \mathbb{R}^n) \subset (\mathbb{R}^2 \times (\mathcal{A} + \varepsilon\mathbb{B}^o))\right) \geq 1 - \rho. \quad (5.24)$$

The compact set  $\mathcal{A} \subset \mathbb{R}^n$  is *uniformly globally asymptotically stable in probability* for (5.1) if it is globally stable in probability for (5.1) and uniformly globally attractive in probability for (5.1).

**Definition 5.6** *An open, bounded set  $\mathcal{O} \subset \mathbb{R}^n$  is uniformly globally recurrent for (5.1) if there are no finite escape times for (5.1a) and for each  $\rho > 0$  and  $R > 0$  there exists  $\tau \geq 0$  such that for  $\xi \in R\mathbb{B}$  and  $\mathbf{x} \in \mathcal{S}_r(\xi)$ ,*

$$\mathbb{P}\left(\left(\text{graph}(\mathbf{x}) \subset (\Gamma_{< \tau} \times \mathbb{R}^n)\right) \vee \left(\text{graph}(\mathbf{x}) \cap (\Gamma_{\leq \tau} \times \mathcal{O})\right)\right) \geq 1 - \rho. \quad (5.25)$$

### 5.7.1 Relaxed sufficient conditions

In this section we present weak sufficient conditions for verifying stochastic stability like asymptotic stability in probability and recurrence.

First, we present sufficient conditions for stability and recurrence based on the recurrence principle in Theorem 5.1. Define  $C_\cap := C \cap (\mathbb{R}^n \setminus \mathcal{O})$ ,  $D_\cap := D \cap (\mathbb{R}^n \setminus \mathcal{O})$ ,  $G_\cap(x, v) = G(x, v) \cap (\mathbb{R}^n \setminus \mathcal{O})$  and  $K_{\delta, \Delta} := \{x \in \mathbb{R}^n : |x|_{\mathcal{A}} \in [\delta, \Delta]\}$  for  $0 < \delta < \Delta < \infty$ .

An alternative way to establish uniform global asymptotic stability in probability of a compact set  $\mathcal{A}$  is by proving  $\mathcal{A}$  is uniformly globally stable in probability and for every  $\delta, \Delta > 0$ , the complement of the compact set  $K_{\delta, \Delta}$  is uniformly globally recurrent. See [25, Section 2.3] and [27, Section 4] for more details. The following theorem uses this equivalence to establish relaxed sufficient conditions for UGAS in probability of a compact set  $\mathcal{A}$ .

**Theorem 5.6** *Suppose the compact set  $\mathcal{A}$  is uniformly globally stable in probability for  $\mathcal{H}$ . The set  $\mathcal{A}$  is uniformly globally asymptotically stable in probability for  $\mathcal{H}$  if for each  $0 < \delta < \Delta < \infty$ , the set  $K_{\delta,\Delta}$  contains no compact set that is almost surely weakly forward invariant for  $\mathcal{H}$ .*

*Proof:* We claim that the assumptions of the theorem imply global recurrence of complement of  $K_{\delta,\Delta}$  for every  $0 < \delta < \Delta < \infty$ . The proof of uniformly globally asymptotically stable in probability then follows from [25, Prop 3.1, 2.4, 2.2]. The proof of the claim proceeds by contradiction. Suppose for some  $0 < \delta < \Delta < \infty$ , the complement of  $K_{\delta,\Delta}$  is not recurrent. Then, there exists a random solution  $\mathbf{z}$  that is generated by the system  $(C \cap K_{\delta,\Delta}, F, D \cap K_{\delta,\Delta}, G \cap K_{\delta,\Delta}, \mu)$ , that is almost surely contained in  $K_{\delta,\Delta}$  and complete with positive probability. By definition, every complete sample path of  $\mathbf{z}$  converges to the compact set  $K_{\delta,\Delta}$ . Hence, by the recurrence principle in Theorem 5.1 it converges to the weakly totally recurrent in probability set contained in  $K_{\delta,\Delta}$ . It follows from Proposition 5.1 that the weakly totally recurrent in probability set contains an almost surely weakly forward invariant set. Hence  $K_{\delta,\Delta}$  contains an almost surely weakly forward invariant set. This contradicts the assumption of the theorem and establishes global recurrence of complement of  $K_{\delta,\Delta}$  for every  $0 < \delta < \Delta < \infty$ . ■

Similarly, an alternative way to establish uniform global recurrence of an open, bounded set  $\mathcal{O}$  is by proving  $\overline{\mathcal{O}}$  is uniformly Lagrange stable in probability and for every  $\Delta > 0$ , the complement of the compact set  $(\overline{\mathcal{O}} + \Delta\mathbb{B}) \setminus \mathcal{O}$  is uniformly globally recurrent for the truncated system  $(C_\Delta, F, D_\Delta, G_\Delta, \mu)$ . See [25, Section 2.3] and [27, Section 4] for more details. The following theorem uses this equivalence to establish relaxed sufficient conditions for uniform global recurrence of an open, bounded set  $\mathcal{O}$ .

**Theorem 5.7** *Suppose the compact set  $\overline{\mathcal{O}}$  is uniformly Lagrange stable in probability for  $(C_\Delta, F, D_\Delta, G_\Delta, \mu)$ . Then,  $\mathcal{O}$  is uniformly globally recurrent for  $\mathcal{H}$  if for each  $\Delta > 0$ , there*

does not exist a almost surely weakly forward invariant set contained in the compact set  $(\overline{\mathcal{O}} + \Delta\mathbb{B}) \setminus \mathcal{O}$  for the system  $(C_\cap, F, D_\cap, G_\cap, \mu)$ .

*Proof:* We claim that the assumptions of the theorem imply global recurrence of complement of  $\mathcal{O}_\Delta := (\overline{\mathcal{O}} + \Delta\mathbb{B}) \setminus \mathcal{O}$  for every  $0 < \Delta < \infty$  for the system  $(C_\cap, F, D_\cap, G_\cap, \mu)$ . The proof of uniform global recurrence of  $\mathcal{O}$  follows from [25, Prop 3.1, 2.4, 2.3]. We establish the claim by contradiction. Suppose for some  $0 < \Delta < \infty$ , the complement of  $\mathcal{O}_\Delta$  is not recurrent. Then, there exists a random solution  $\mathbf{z}$  that is generated by the system  $(C \cap \mathcal{O}_\Delta, F, D \cap \mathcal{O}_\Delta, G \cap \mathcal{O}_\Delta, \mu)$ , that is almost surely contained in  $\mathcal{O}_\Delta$  and complete with positive probability. By definition, every complete sample path of  $\mathbf{z}$  converges to the compact set  $\mathcal{O}_\Delta$ . Hence by Theorem 5.1 it converges to the weakly totally recurrent in probability set contained in  $\mathcal{O}_\Delta$ . From Proposition 5.1 it follows that the weakly totally recurrent in probability set contains an almost surely weakly forward invariant set. Hence  $\mathcal{O}_\Delta$  contains an almost surely weakly forward invariant set. This contradicts the assumption and establishes global recurrence of complement of  $\mathcal{O}_\Delta$  for every  $0 < \Delta < \infty$ . ■

It can be observed that Theorems 5.6 and 5.7 do not utilize Lyapunov-like functions that satisfy strict decrease conditions on average. In fact, the uniform globally stability in probability and uniform Lagrange stability in probability assumptions in Theorems 5.6 and 5.7 can be achieved through Lyapunov functions satisfying non-strict decrease conditions on average as established in [25, Thm 4.1, 4.2].

We now present a sharper version of the weak sufficient conditions for stability using the Krasovskii-LaSalle function based recurrence principle from Theorem 5.2. In Theorems 5.6 and 5.7, we need to rule out the presence of almost surely weakly forward invariant sets in certain sets bounded away from the sets  $\mathcal{A}$  and  $\mathcal{O}$  respectively. Using the results in Theorem 5.2, we can refine the results in Theorems 5.6 and 5.7 so that

we need to rule out the presence of almost surely weakly forward invariant sets only in certain level sets of a Lyapunov-like function.

**Definition 5.7** *A continuous function  $\widehat{V} : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$  is a weak-Lyapunov function relative to a compact set  $\mathcal{A} \subset \mathbb{R}^n$  for the system  $(C, F, D, G, \mu)$  if  $\widehat{V}(x) = 0 \iff x \in \mathcal{A}$ ,  $\widehat{V}$  is radially unbounded and satisfies*

$$\begin{aligned} \widehat{V}(\phi(t)) &\leq \widehat{V}(x), \quad \forall x \in C, t \in \text{dom}(\phi), \phi \in \mathcal{S}_C^F(x) \\ \int_{\mathbb{R}^m} \max_{g \in G(x,v)} \widehat{V}(g) \mu(dv) &\leq \widehat{V}(x), \quad \forall x \in D. \end{aligned}$$

A result on weak sufficient conditions for global asymptotic stability using non-increasing Lyapunov-like functions is stated in [14, Thm 8.2] for a class of non-stochastic hybrid systems modeled by (2.1). We now establish weak sufficient conditions for uniformly globally asymptotically stable in probability of compact sets for (5.1) using Theorem 5.2. In particular, we establish uniformly globally asymptotically stable in probability of compact sets using the existence of a Lyapunov function and by ruling out the existence of random solutions  $\mathbf{x}$  that remain in non-zero level sets of the Lyapunov function. The conditions of the theorem are sharper compared to the results from Theorem 5.1 due to the refined convergence results established using the Krasovskii-LaSalle functions.

**Theorem 5.8** *Let  $\widehat{V}$  be a weak-Lyapunov function relative to a compact set  $\mathcal{A} \subset \mathbb{R}^n$  for the system  $\mathcal{H}$ . Then,  $\mathcal{A}$  is uniformly globally asymptotically stable in probability if and only if for every  $c > 0$ , there does not exist an almost surely complete solution  $\mathbf{x}$  that remains in the set  $L_{\widehat{V}}(c)$  almost surely.*

*Proof:*  $\Rightarrow$  Uniform global asymptotic stability in probability of the set  $\mathcal{A}$  implies that there does not exist an almost surely complete solution  $\mathbf{x}$  that remains in a non-

zero level set of the weak-Lyapunov function almost surely since the existence of such a solution would contradict almost sure convergence to the set  $\mathcal{A}$  required by uniform global asymptotic stability in probability.

$\Leftarrow$  The Lyapunov function  $\widehat{V}$  satisfies the conditions of [25, Thm 4.2] from which uniform global stability in probability follows. Since there are no almost surely complete random solutions that remain in a non-zero level set of  $\widehat{V}$  with probability one for all time, we can conclude that no non-zero level set of  $\widehat{V}$  contains an almost surely weakly forward invariant set. Then, it follows from Corollary 5.4 that no non-zero level set of  $\widehat{V}$  contains a weakly totally recurrent in probability set. We now establish that for every  $\mathbf{x}$  almost every complete sample path converges to  $\mathcal{A}$ .

Suppose this is not true. Let  $\mathbf{x}$  be any random solution with  $\mathbf{x} \in \mathcal{S}_r(x)$  for  $x \in \mathcal{A} + \delta\mathbb{B}$  for some  $\delta > 0$  such that  $\mathbb{P}(\Omega_c) \geq \rho_1 > 0$ , where for  $\omega \in \Omega_c$ ,  $\mathbf{x}(\omega)$  is complete and does not converge to  $\mathcal{A}$ . From uniform Lagrange stability in probability, there exists  $\varepsilon > 0$  such that

$$\mathbb{P}\left(\text{graph}(\mathbf{x}(\omega)) \subset (\mathbb{R}^2 \times (\mathcal{A} + \varepsilon\mathbb{B}))\right) \geq 1 - \rho_1/2. \quad (5.26)$$

Let  $\mathbf{x}_\varepsilon$  be a truncated solution of  $\mathbf{x}$  whose sample paths are restricted to the compact set  $\mathcal{A} + \varepsilon\mathbb{B}$ . Since no non-zero level set of  $\widehat{V}$  contains a weakly totally recurrent in probability set, this necessarily means that from Theorem 5.2 complete sample paths  $\mathbf{x}_\varepsilon(\omega)$  converge to the zero level set, which is set  $\mathcal{A}$ . Then, it follows that for almost all  $\omega \in \Omega_c$ ,  $\mathbf{x}(\omega)$  cannot stay in the set  $\mathcal{A} + \varepsilon\mathbb{B}$ . Hence we have

$$\mathbb{P}\left(\text{graph}(\mathbf{x}(\omega)) \cap (\mathbb{R}^2 \times \mathbb{R}^n \setminus (\mathcal{A} + \varepsilon\mathbb{B})) \neq \emptyset\right) \geq \rho_1. \quad (5.27)$$

We also have from (5.26) that

$$\mathbb{P}\left(\text{graph}(\mathbf{x}(\omega)) \cap (\mathbb{R}^2 \times \mathbb{R}^n \setminus (\mathcal{A} + \varepsilon\mathbb{B})) \neq \emptyset\right) \leq \rho_1/2. \quad (5.28)$$

This leads to a contradiction that establishes that  $\rho_1$  must be zero. Hence, every open neighborhood of  $\mathcal{A}$  is globally recurrent. Uniform global recurrence now follows from [85, Thm 6] using sequential compactness results for solutions of (5.1). Then, uniform global asymptotic stability in probability follows from [25, Prop 2.2].  $\blacksquare$

**Example 5.1** Let the state  $z = (x_1, x_2, \tau) \in \mathbb{R}^3$  and  $M \in \mathbb{Z}_{>0}$ . Consider the system  $\mathcal{H} := (C, f, D, G, \mu)$  with state  $z$  and

$$\begin{aligned} C &:= \mathbb{R} \times \mathbb{R} \times [0, M] \\ D &:= \mathbb{R} \times \mathbb{R} \times \{M\} \\ f(z) &:= \begin{bmatrix} x_2 - x_1 \\ -x_1 - x_2^3 \\ 1 \end{bmatrix} \\ G(z, v) &:= \begin{bmatrix} [0, v]x_2 \\ [0, v]x_1 \\ 0 \end{bmatrix} \end{aligned}$$

where  $v$  is a random variable such that  $v \sim \text{Uniform}[0, \sqrt{3}]$ . Then,  $\mathbb{E}[v^2] = 1$ . Consider the Lyapunov function  $V(z) = x_1^2 + x_2^2$  and the compact set  $\mathcal{A} = \{0\} \times \{0\} \times [0, M]$ . Then,  $V$  is radially unbounded, locally bounded and

$$\begin{aligned} \langle \nabla V(z), f \rangle &\leq -x_1^2 - x_2^4 \\ \int \max_{g \in G(z, v)} V(g) \mu(dv) &= V(z). \end{aligned}$$



Hence, during jumps we do not have strict decrease in expected value for the Lyapunov function due to the selection  $g(z, v) = [vx_2, vx_1, 0] \in G(z, v)$ . We can rule out the existence of random solutions that remain in non-zero level sets of  $V$  with probability one since almost every sample path of the random solution flows for  $M$  seconds in between jumps, and  $V$  decreases strictly during flows outside  $\mathcal{A}$ .

Finally, we present a similar result for the recurrence property using Theorem 5.2.

**Definition 5.8** *A continuous function  $\widehat{V} : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$  is a weak-Lyapunov function relative to an open, bounded set  $\mathcal{O} \subset \mathbb{R}^n$  for the system  $\mathcal{H}$  if  $\widehat{V}$  is radially unbounded and satisfies*

$$\begin{aligned} \widehat{V}(\phi(t)) &\leq \widehat{V}(x), \quad \forall x \in C_\cap, t \in \text{dom}(\phi), \phi \in \mathcal{S}_{C_\cap}^F(x) \\ \int_{\mathbb{R}^m} \max_{g \in G(x, v) \cap (\mathbb{R}^n \setminus \mathcal{O})} \widehat{V}(g) \mu(dv) &\leq \widehat{V}(x), \quad \forall x \in D_\cap. \end{aligned}$$

We point that while the Krasovskii- LaSalle functions are defined with respect to compact sets  $(K, \Lambda)$ , the Lyapunov functions used in this section are defined on the set  $C \cup D \cup G(D \times \mathcal{V})$ , where  $\mathcal{V} := \cup_{\omega \in \Omega, i \in \mathbb{Z}_{\geq 0}} \mathbf{V}_i(\omega)$ .

A result on weak sufficient conditions for recurrence using non-increasing Lyapunov-like functions is stated in [31, Thm 1] for a class of non-stochastic hybrid systems modeled by (2.1). We now establish weak sufficient conditions for uniform global recurrence of open, bounded sets for (5.1) using Theorem 5.2. In particular, we establish uniform global recurrence of open, bounded sets using the existence of a Foster function and by ruling out the existence of random solutions  $\mathbf{x}$  that remain in level sets of the Foster function outside the set  $\mathcal{O}$ . The conditions of the following theorem are sharper compared to the conditions in Theorem 5.7

**Theorem 5.9** *Let  $\widehat{V}$  be a weak-Lyapunov function relative to an open, bounded set  $\mathcal{O} \subset \mathbb{R}^n$  for the system  $\mathcal{H}$ . Then,  $\mathcal{O}$  is uniformly globally recurrent if and only if there does not exist an almost surely complete solution  $\mathbf{x}$  that remains almost surely in the set  $L_{\widehat{V}}(c) \cap (\mathbb{R}^n \setminus \mathcal{O})$  for every  $c \geq 0$  for which  $L_{\widehat{V}}(c) \cap (\mathbb{R}^n \setminus \mathcal{O})$  is non-empty.*

*Proof:*  $\Rightarrow$  If there exists an almost surely complete solution  $\mathbf{x}$  that remains almost surely in a level set of the weak-Lyapunov function that is completely contained in the set  $L_{\widehat{V}}(c) \cap (\mathbb{R}^n \setminus \mathcal{O})$  for some  $c \geq 0$ , it contradicts the assumption that  $\mathcal{O}$  is uniformly globally recurrent.

$\Leftarrow$  The Lyapunov function implies Lagrange stability in probability of the set  $\overline{\mathcal{O}}$  for the truncated system  $(C_{\cap}, F, D_{\cap}, G_{\cap}, \mu)$ . Since there are no almost surely complete random solutions that remain in  $L_{\widehat{V}}(c) \cap (\mathbb{R}^n \setminus \mathcal{O})$  for every  $c \geq 0$  for which  $L_{\widehat{V}}(c) \cap (\mathbb{R}^n \setminus \mathcal{O})$  is non-empty, we can conclude that no set of the form  $L_{\widehat{V}}(c) \cap (\mathbb{R}^n \setminus \mathcal{O})$  contains an almost surely weakly forward invariant set. Then, it follows from Corollary 5.4 that such sets do not contain a weakly totally recurrent in probability set.

We will now claim recurrence of  $\mathcal{O}$  for every solution for the truncated system  $(C_{\cap}, F, D_{\cap}, G_{\cap}, \mu)$ . Then, the proof follows from [25, Prop 2.3]. Suppose the claim is not true. Let  $\mathbf{x}$  be any random solution with  $\mathbf{x} \in \mathcal{S}_r(x)$  for  $x \in \overline{\mathcal{O}} + \delta\mathbb{B}$  for some  $\delta > 0$  such that  $\mathbb{P}(\Omega_c) \geq \rho_1 > 0$ , where for  $\omega \in \Omega_c$ ,  $\mathbf{x}(\omega)$  is complete and does not hit the set  $\mathcal{O}$ . From uniform Lagrange stability in probability, there exists  $\varepsilon > 0$  such that

$$\mathbb{P}\left(\text{graph}(\mathbf{x}(\omega)) \subset (\mathbb{R}^2 \times (\overline{\mathcal{O}} + \varepsilon\mathbb{B}))\right) \geq 1 - \rho_1/2. \quad (5.29)$$

Let  $\mathbf{x}_{\varepsilon}$  be a truncated solution of  $\mathbf{x}$  whose sample paths are restricted to the compact set  $\overline{\mathcal{O}} + \varepsilon\mathbb{B}$ . Since no level set of  $\widehat{V}$  outside the set  $\mathcal{O}$  contains a weakly totally recurrent in probability set, this necessarily means that from Theorem 5.2 the sample paths  $\mathbf{x}_{\varepsilon}(\omega)$  are not complete. Then, it follows that for almost all  $\omega \in \Omega_c$ , since  $\mathbf{x}(\omega)$  is complete, the

solutions cannot stay in the set  $\overline{\mathcal{O}} + \varepsilon\mathbb{B}$ . Hence we have

$$\mathbb{P}\left(\text{graph}(\mathbf{x}(\omega)) \cap (\mathbb{R}^2 \times \mathbb{R}^n \setminus (\overline{\mathcal{O}} + \varepsilon\mathbb{B})) \neq \emptyset\right) \geq \rho_1. \quad (5.30)$$

We also have

$$\mathbb{P}\left(\text{graph}(\mathbf{x}(\omega)) \cap (\mathbb{R}^2 \times \mathbb{R}^n \setminus (\overline{\mathcal{O}} + \varepsilon\mathbb{B})) \neq \emptyset\right) \leq \rho_1/2. \quad (5.31)$$

This leads to a contradiction which establishes that  $\rho_1$  must be zero. ■

**Example 5.2** Consider the simple discrete-time system

$$x^+ = g(x, v) = \max\{0, x + v\} \quad (5.32)$$

where  $v$  takes values in the set  $\{-1, 1\}$  with equal probability and  $x \in D$  with  $D = \mathbb{Z}_{\geq 0}$ .

Consider the set  $\mathcal{O} = (-1, 1)$ . Let  $V(x) = |x|$ . Then for  $x \in D \setminus \mathcal{O}$ ,

$$\mathbb{E}[V(g(x, v))] = 0.5|x + 1| + 0.5|x - 1| = |x| = V(x) \quad (5.33)$$

Hence, we do not have strict decrease in expected value along solutions. It follows that for  $c \in \mathbb{Z}_{\geq 1}$  the set  $L_V(c) \cap (D \setminus \mathcal{O}) := \{c\}$  is non-empty. For every  $c \geq 1$ , it follows that solutions cannot stay in the set  $L_V(c) \cap (D \setminus \mathcal{O})$  almost surely since in one jump with probability 0.5, solutions reach the point  $c - 1$ . This establishes global recurrence of the set  $\mathcal{O}$ .

It can be observed from the statements of Theorems 5.8 and 5.9 that the weak sufficient conditions generated by Theorem 5.2 are sharper compared to the results from Theorem 5.1 due to the refined convergence results established using the Krasovskii-LaSalle functions.

### 5.7.2 Sufficient conditions based on strict decrease properties

In this section we present sufficient conditions for asymptotic stability in probability and recurrence that rely on Lyapunov functions satisfying strict decrease conditions on average. The results are a direct consequence of the Krasovskii- LaSalle function based weak sufficient conditions. We also refer the reader to [25, Thm 4.4, 4.5] for alternate proofs.

**Definition 5.9** *A continuously differentiable function  $V : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$  is a Lyapunov function relative to the compact set  $\mathcal{A} \subset \mathbb{R}^n$  for the system  $\mathcal{H}$  if  $V$  is radially unbounded,  $V \in \mathcal{PD}(\mathcal{A})$  and there exists a continuous function  $\rho \in \mathcal{PD}(\mathcal{A})$  such that*

$$\begin{aligned} \langle \nabla V(x), f \rangle &\leq -\rho(x), \forall x \in C, f \in F(x) \\ \int_{\mathbb{R}^m} \max_{g \in G(x,v)} V(g) \mu(dv) &\leq V(x) - \rho(x), \forall x \in D. \end{aligned}$$

**Theorem 5.10** *Let  $V$  be a Lyapunov function relative to the compact set  $\mathcal{A} \subset \mathbb{R}^n$  for the system  $\mathcal{H}$ . Then,  $\mathcal{A}$  is uniformly globally asymptotically stable in probability for  $\mathcal{H}$ .*

**Definition 5.10** *A continuously differentiable function  $V : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$  is a Lyapunov function relative to an open, bounded set  $\mathcal{O} \subset \mathbb{R}^n$  for the system  $\mathcal{H}$  if  $V$  is radially unbounded and and there exists a continuous function  $\rho : \mathbb{R}^n \rightarrow \mathbb{R}_{> 0}$  such that*

$$\begin{aligned} \langle \nabla V(x), f \rangle &\leq -\rho(x), \forall x \in C \setminus \mathcal{O}, f \in F(x) \\ \int_{\mathbb{R}^m} \max_{g \in G(x,v) \cap (\mathbb{R}^n \setminus \mathcal{O})} V(g) \mu(dv) &\leq V(x) - \rho(x), \forall x \in D \setminus \mathcal{O}. \end{aligned}$$

**Theorem 5.11** *Let  $V$  be a Lyapunov function relative to an open, bounded set  $\mathcal{O} \subset \mathbb{R}^n$  for the system  $\mathcal{H}$ . Then,  $\mathcal{O}$  is uniformly globally recurrent for  $\mathcal{H}$ .*

# Chapter 6

## Robust global recurrence in stochastic hybrid systems

### 6.1 Introduction

In this chapter, we focus on the global recurrence property for the class of stochastic hybrid systems in Chapter 5 and develop robustness results and a converse Lyapunov theorem. A converse theorem for a stronger version of recurrence called positive recurrence is in [22] for discrete-time stochastic systems and in [23, Thm 3.26] for switching diffusion processes. A converse theorem for the recurrence property in non-stochastic hybrid inclusions is in [31] and for stochastic difference inclusions in [55]. In this chapter we extend the results in Chapter 2 and Chapter 3 to a larger class of stochastic hybrid systems modeled by set-valued mappings. The results in this chapter are from [87].

## 6.2 Recurrence and Uniform recurrence

We briefly recall the stochastic hybrid system model in Chapter 5, the basic assumptions on the data of the model and the definitions of recurrence for open, bounded sets. Let the state  $x \in \mathbb{R}^n$  and the random input  $v \in \mathbb{R}^m$ . The stochastic hybrid system is written formally as

$$\dot{x} \in F(x), x \in C \tag{6.1a}$$

$$x^+ \in G(x, v^+), x \in D \tag{6.1b}$$

$$v \sim \mu(\cdot) \tag{6.1c}$$

We denote by  $\mathcal{S}_r(x)$ , the set of random solutions generated by  $\mathcal{H} := (C, F, D, G, \mu)$  from the initial condition  $x$ . The data  $(C, F, D, G, \mu)$  of the stochastic hybrid system  $\mathcal{H}$  are assumed to satisfy the conditions of Standing Assumption 5.1 which are restated below.

**Standing Assumption 6.1** *The data of the stochastic hybrid system  $\mathcal{H}$  satisfies the following conditions:*

1. *The sets  $C, D \subset \mathbb{R}^n$  are closed;*
2. *The mapping  $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$  is outer-semicontinuous, locally bounded with nonempty convex values on  $C$ ;*
3. *The mapping  $G : \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n$  is locally bounded and the mapping  $v \mapsto \text{graph}(G(\cdot, v)) := \{(x, y) \in \mathbb{R}^{2n} : y \in G(x, v)\}$  is measurable with closed values.*

**Definition 6.1** *An open, bounded set  $\mathcal{O} \subset \mathbb{R}^n$  is globally recurrent for  $\mathcal{H}$  if there are no*

finite escape times for (6.1a) and for each  $x \in \mathbb{R}^n$  and  $\mathbf{x} \in \mathcal{S}_r(x)$ ,

$$\lim_{\tau \rightarrow \infty} \mathbb{P} \left( \left( \text{graph}(\mathbf{x}) \subset (\Gamma_{<\tau} \times \mathbb{R}^n) \right) \vee \left( \text{graph}(\mathbf{x}) \cap (\Gamma_{\leq\tau} \times \mathcal{O}) \right) \right) = 1.$$

Loosely speaking, the above condition insists that almost surely the sample paths of the random solution  $\mathbf{x}$  are either not complete or hit the set  $\mathcal{O}$ .

**Example 6.1** Consider a stochastic hybrid system with a state  $x \in \mathbb{R}$  satisfying

$$\begin{aligned} \dot{x} &= f(x), x \in C \\ x^+ &= g(x, v), x \in D \end{aligned}$$

where  $f(x) = 1$ ,  $g(x, v) = vx$  with  $v \in \{0, 1\}$ ,  $\mu(0) = \mu(1) = 0.5$ ,  $C = (-\infty, 1]$  and  $D = [1, 2]$ . For this system, it can be observed that any set of the form  $\mathcal{O} = (-\epsilon, \epsilon)$  with  $0 < \epsilon < 1$  is globally recurrent. For any initial condition  $x_0 \in C$  such that  $x_0 \in (-\infty, \epsilon)$ , solutions hit the set  $\mathcal{O}$  due to the continuous-time dynamics. For initial conditions  $x_0 \in C$  such that  $x_0 \geq \epsilon$ , the solutions reach the set  $D$ . Then, for solutions from the set  $D$ , almost surely the random input  $v = 0$  appears in a sequence of random inputs  $\{v_i\}_{i=0}^{\infty}$ . Hence the solutions from the set  $D$  reaches the origin almost surely. This establishes global recurrence of the set  $\mathcal{O}$ . We can easily observe from this example that the set  $\mathcal{O}$  is not invariant in a probabilistic sense as the continuous-time dynamics ensures that solutions leave the set  $\mathcal{O}$  and reach  $D$ . Similarly, we can observe that the set  $\mathcal{O}$  does not have any stability-like property.

**Definition 6.2** An open, bounded set  $\mathcal{O} \subset \mathbb{R}^n$  is uniformly globally recurrent for  $\mathcal{H}$  if there are no finite escape times for (6.1a) and for each  $\rho > 0$  and compact set  $K$  there

exists  $\tau \geq 0$  such that for  $\xi \in K$  and  $\mathbf{x} \in \mathcal{S}_r(\xi)$ ,

$$\mathbb{P}\left(\left(\text{graph}(\mathbf{x}) \subset (\Gamma_{<\tau} \times \mathbb{R}^n)\right) \vee \left(\text{graph}(\mathbf{x}) \cap (\Gamma_{\leq\tau} \times \mathcal{O})\right)\right) \geq 1 - \rho.$$

**Example 6.2** Consider the stochastic hybrid system in Example 6.1. We are now going to establish that the set  $\mathcal{O}$  is uniformly globally recurrent. Let a compact set  $K$  and  $\rho > 0$  in the definition of uniform global recurrence be given. Choose  $\tau^* \in \mathbb{Z}_{\geq 0}$  such that  $1 - (0.5)^{\tau^*} \geq 1 - \rho$ . We first consider the case when the set  $K \subset D$ . In this case, we can choose  $\tau \geq \tau^*$ . If the compact set  $K \subset [\epsilon, 1]$ , the time  $\tau$  in the definition of uniform global recurrence is chosen such that  $\tau \geq (1 - \epsilon) + \tau^*$ . If the compact set  $K \subset (-\epsilon, \epsilon)$ , we can choose  $\tau = 0$ . Similarly, if the compact set  $K \subset (-\infty, -\epsilon]$ , the time  $\tau$  can be chosen such that  $\tau \geq \max_{x \in K} (|x| - \epsilon) + 1$ . The choice of  $\tau$  for any other compact set  $K$  can be derived from the above cases.

The following result establishes equivalence between uniform and non-uniform recurrence. We refer the reader to [85, Thm 6] for a proof.

**Proposition 6.1** *An open, bounded set  $\mathcal{O}$  is globally recurrent for  $\mathcal{H}$  if and only if it is uniformly globally recurrent for  $\mathcal{H}$ .*

### 6.3 Viability and reachability probabilities

It can be observed from the definition of global recurrence that the recurrence property needs to hold for every random solution generated from an initial condition. Hence, it is useful to work with worst case probabilities related to the recurrence property. As in [42] and [25] we characterize the recurrence property in terms of viability probabilities defined below.



For  $x \in \mathbb{R}^n$ ,  $\tau \geq 0$  and closed set  $S \subset \mathbb{R}^n$ , we define

$$m_{\mathcal{C}S}(\tau, x) := \sup_{\mathbf{x} \in \mathcal{S}_r(x)} \mathbb{P} \left( (\text{graph}(\mathbf{x}) \cap (\Gamma_{\geq \tau} \times \mathbb{R}^n) \neq \emptyset) \vee (\text{graph}(\mathbf{x}) \cap (\Gamma_{\leq \tau} \times \mathbb{R}^n) \subset \mathbb{R}^2 \times S) \right). \quad (6.2)$$

The viability probability  $m_{\mathcal{C}S}(\tau, x)$  is related to the largest probability that random solutions starting from  $x$  stay in the set  $S$  for hybrid time less than or equal to  $\tau$  and not stop before that time. This probability condition is complementary to the condition for recurrence when the set  $S = \mathbb{R}^n \setminus \mathcal{O}$  and when  $\tau \rightarrow \infty$ .

It is established in [25, Prop 10.2, Prop 9.1] that the supremum in the above definition is achieved for some random solution and the mapping  $(\tau, x) \mapsto m_{\mathcal{C}S}(\tau, x)$  is upper semicontinuous. We refer the reader to [25, Section 9] for more details. Define

$$\widehat{m}_{\mathcal{C}S}(x) := \lim_{\tau \rightarrow \infty} m_{\mathcal{C}S}(\tau, x). \quad (6.3)$$

The quantity  $\widehat{m}_{\mathcal{C}S}(x)$  is related to the largest infinite time viability probability. The limit is well defined due to the mapping  $\tau \mapsto m_{\mathcal{C}S}(\tau, x)$  being non-increasing for every  $x$ .

The following proposition proved in the appendix holds for any stochastic hybrid system satisfying Standing Assumption 6.1 and provides an equivalent characterization for global recurrence. Roughly, recurrence of a set  $\mathcal{O}$  implies that solutions keep returning to the set  $\mathcal{O}$  infinitely often with probability one. This implies that solutions cannot stay in the complement of the set  $\mathcal{O}$  for all time and hence, the set  $\mathbb{R}^n \setminus \mathcal{O}$  is not viable.

**Proposition 6.2** *Let  $\mathcal{O} \subset \mathbb{R}^n$  be an open, bounded set. The following statements are equivalent:*

1.  $\mathcal{O}$  is globally recurrent.

2.  $\widehat{m}_{\subset \mathbb{R}^n \setminus \mathcal{O}}(x) = 0$  for all  $x \in \mathbb{R}^n$ .

3. For every compact set  $K \subset \mathbb{R}^n$ , and  $\rho > 0$ , there exists  $\tau \geq 0$  such that

$$\sup_{x \in K} m_{\subset \mathbb{R}^n \setminus \mathcal{O}}(\tau, x) \leq \rho.$$

We also utilize reachability probabilities studied in [25, Section 8]. For  $x \in \mathbb{R}^n, \tau \geq 0$  and closed set  $S \subset \mathbb{R}^n$ , we define

$$m_{\cap S}(\tau, x) := \sup_{\mathbf{x} \in \mathcal{S}_r(x)} \mathbb{P}\left(\text{graph}(\mathbf{x}) \cap (\Gamma_{\leq \tau} \times S) \neq \emptyset\right). \quad (6.4)$$

The reachability probability  $m_{\cap S}(\tau, x)$  is related to the largest probability that random solutions starting from  $x$  reach the set  $S$  within hybrid time  $\tau$ . It can be established similar to [25, Prop 10.2] that the supremum in the above definition is achieved for some random solution.

## 6.4 Preliminary bounds on viability and reachability probabilities

In this section, we focus of stochastic hybrid systems  $\tilde{\mathcal{H}}$  that satisfies the following assumption. This assumption will be satisfied for the stochastic hybrid systems used in generating the robustness results and it also simplifies some of the proofs.

**Assumption 6.1** *The data of the stochastic hybrid system  $\tilde{\mathcal{H}}$  are such that, for every maximal random solution  $\mathbf{x}$  generated by  $\tilde{\mathcal{H}}$ , the sample paths  $\mathbf{x}(\omega)$  are almost surely complete.*

The probability bounds in this section are generated for the system  $\tilde{\mathcal{H}}$ . We now present a series of bounds related to viability and reachability probabilities in this section and the proofs are presented later in the appendix.

The first result establishes an equivalent characterization for the quantity  $\hat{m}_{CS}(x)$  defined in (6.3) for every closed set  $S \subset \mathbb{R}^n$  and  $x \in \mathbb{R}^n$ .

**Proposition 6.3** *Let Assumption 6.1 hold,  $S \subset \mathbb{R}^n$  be closed and  $x \in \mathbb{R}^n$ . Then, there exists a random solution  $\mathbf{x}^* \in \mathcal{S}_r(x)$  such that*

$$\hat{m}_{CS}(x) = \mathbb{P}(\text{graph}(\mathbf{x}^*) \subset (\mathbb{R}^2 \times S)) = \sup_{\mathbf{x} \in \mathcal{S}_r(x)} \mathbb{P}(\text{graph}(\mathbf{x}) \subset (\mathbb{R}^2 \times S)).$$

The following result when applied with the set  $S = \mathbb{R}^n \setminus \mathcal{O}$  gives an alternative characterization of recurrence of the set  $\mathcal{O}$  similar to [55, Lemma 3].

**Proposition 6.4** *Let Assumption 6.1 hold,  $S \subset \mathbb{R}^n$  be closed. If there exists  $\gamma < 1$  such that  $\sup_{x \in S} \hat{m}_{CS}(x) \leq \gamma$ , then  $\hat{m}_{CS}(x) = 0$  for all  $x \in S$ .*

The next result is motivated by the result in [42, Lemma 3] and is similar in nature to the semi-group property for non-stochastic systems.

**Proposition 6.5** *Let Assumption 6.1 hold. For closed sets  $S_0, S_1 \subset \mathbb{R}^n$  and  $(k_1, k_2, x) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \times \mathbb{R}^n$ ,*

$$m_{CS_0}(k_1 + k_2, x) \leq m_{CS_1}(k_1, x) + \sup_{\xi \in \mathbb{R}^n \setminus S_1} m_{CS_0}(k_2, \xi).$$

We now present a result that relates the viability and reachability probabilities. A similar result for discrete-time stochastic systems is in [42, Lemma 1].

**Proposition 6.6** *Let Assumption 6.1 hold. For closed sets  $S, S_1, S_2 \subset \mathbb{R}^n$  such that  $S \subset S_1 \cup S_2$  and for each  $x \in \mathbb{R}^n$  and  $\tau \geq 0$ ,*

$$m_{\subset S}(\tau, x) \leq m_{\subset S_1}(\tau, x) + m_{\subset S_2}(\tau, x).$$

The next result establishes that the reachability probabilities  $m_{\cap S}(\tau, x)$  can be made arbitrarily small for a fixed  $\tau \geq 0$  and initial conditions  $x$  in a compact set, when the set  $S = \mathbb{R}^n \setminus R\mathbb{B}^o$  by choosing  $R > 0$  sufficiently large. The proof is omitted as it follows along the same lines as [42, Lemma 4] using the fact that the reachable set from a compact set of initial conditions for finite time is bounded for  $x \in C, \dot{x} \in F(x)$  using [25, Lemma 6.16], the local boundedness  $G$  and the dynamic programming methods in [25, Section 8.1].

**Proposition 6.7** *For each  $k \in \mathbb{Z}_{\geq 0}$ ,  $\varepsilon > 0$  and  $r > 0$  there exists  $R > 0$  such that, with  $S = \mathbb{R}^n \setminus R\mathbb{B}^o$ ,  $m_{\cap S}(k, x) \leq \varepsilon$  for all  $x \in r\mathbb{B}$ .*

## 6.5 Robustness of recurrence

In this section we establish robustness of the recurrence property to various state dependent perturbations. We prove robustness of the recurrence property to three different types of perturbations. Firstly, we establish that recurrence of an open bounded set implies recurrence of a smaller open set within the original set. This type of result can be viewed as robustness to perturbations in the set. Secondly, we prove recurrence is preserved when the data of the stochastic hybrid system is modified to slow down recurrence. Slowing down the recurrence property loosely means that we make quantities related to the average worst case first hitting time to the recurrent set for solutions from every initial condition increase with the distance of the initial condition to the recurrent

set. Finally, we show that by perturbing the system data in a sufficiently small manner we preserve recurrence. This property establishes robustness of recurrence to measurement noise, additive disturbances and parameter uncertainty in system data. The importance of the results will become apparent in the next section which develops converse Lyapunov theorems.

In this section, we will work with stochastic hybrid systems for which the maximal random solutions have almost surely complete sample paths. This modification will preserve recurrence and will play an important role in developing converse Lyapunov theorems. If the open, bounded set  $\mathcal{O}$  is globally recurrent for  $\mathcal{H}$ , consider the inflated system

$$\widehat{\mathcal{H}} := (C, F, \mathbb{R}^n, \widehat{G}, \mu) \quad (6.5)$$

where  $\widehat{G}(x, v) = G_1(x, v) \cup G_2(x)$  with  $G_1(x, v) = G(x, v)$  for  $x \in D$ ,  $G_1(x, v) = \emptyset$  for  $x \notin D$ , and  $G_2(x) = x^*$  for some  $x^* \in \mathcal{O}$  and for all  $x \in \mathbb{R}^n$ . From the data of the hybrid system  $\widehat{\mathcal{H}}$ , recurrence of the set  $\mathcal{O}$  for  $\mathcal{H}$  and solutions of (6.1a) not exhibiting finite escape times it follows that for every random solution of  $\widehat{\mathcal{H}}$  that is maximal, the sample paths are almost surely complete. The proof of the next result follows directly using [42, Prop 2] and [84, Prop 14.11 b].

**Lemma 6.1** *The data of the SHS  $\widehat{\mathcal{H}}$  in (6.5) satisfies Standing Assumption 6.1 and Assumption 6.1. □*

The following result establishes that the recurrence property is preserved by the augmented system  $\widehat{\mathcal{H}}$  and the proof basically follows from the observation that maximal solutions of  $\widehat{\mathcal{H}}$  contains solutions of  $\mathcal{H}$  augmented with additional jumps to the recurrent set.

**Lemma 6.2** *If the open, bounded set  $\mathcal{O}$  is globally recurrent for  $\mathcal{H}$  then  $\mathcal{O}$  is globally recurrent for  $\widehat{\mathcal{H}}$ .*

*Proof:* Since the flow map for the hybrid system  $\widehat{\mathcal{H}}$  is the same as  $\mathcal{H}$ , the solutions generated by  $\widehat{\mathcal{H}}$  do not exhibit finite escape times. We will now establish global recurrence of  $\mathcal{O}$  for  $\widehat{\mathcal{H}}$  by contradiction. If  $\mathcal{O}$  is not globally recurrent for  $\widehat{\mathcal{H}}$ , then there exists  $\rho > 0$  and a random solution  $\mathbf{x}$  such that  $\mathbb{P}(\text{graph}(\mathbf{x}) \subset \mathbb{R}^2 \times (\mathbb{R}^n \setminus \mathcal{O})) \geq \rho$ . Without loss of generality we can assume that the solution  $\mathbf{x}$  is maximal and the sample paths are almost surely complete. We also observe that for  $\omega \in \Omega$  such that  $\text{graph}(\mathbf{x}(\omega)) \subset \mathbb{R}^2 \times (\mathbb{R}^n \setminus \mathcal{O})$ ,  $\mathbf{x}_\omega(t, j) \neq x^*$  for all  $(t, j) \in \text{dom}(\mathbf{x}(\omega))$ . We now define a solution  $\tilde{\mathbf{x}}$  for the system  $\mathcal{H}$  using  $\mathbf{x}$ . For the case when  $\omega$  is such that  $\mathbf{x}_\omega(t, j+1) = G_2(\mathbf{x}_\omega(t, j)) = x^*$  occurs for the first time  $(t, j) \in \text{dom}(\mathbf{x}(\omega))$ , we let  $\tilde{\mathbf{x}}_\omega(\bar{t}, \bar{j}) = \mathbf{x}_\omega(\bar{t}, \bar{j})$  for  $\bar{t} \leq t$ ,  $\bar{j} \leq j$  and the sample paths are stopped afterwards. Otherwise, we let  $\tilde{\mathbf{x}}(\omega) = \mathbf{x}(\omega)$ . It can be easily observed that  $\tilde{\mathbf{x}}$  is a truncation of the solution  $\mathbf{x}$ , truncated at first jump times where the mapping  $G_2$  is used in the sample paths. Then, we can establish that  $\tilde{\mathbf{x}}$  satisfies  $\mathcal{F}_i$  measurability of the mapping  $\omega \mapsto \text{graph}_{\leq i}(\tilde{\mathbf{x}}(\omega))$  from [25, Prop 2.1]. Let  $\Omega_i := \{\omega : \text{graph}(\tilde{\mathbf{x}}(\omega)) \cap (\Gamma_{\geq i} \times \mathbb{R}^n) \neq \emptyset\}$  for  $i \in \mathbb{Z}_{\geq 0}$ . Then,  $\Omega_i \in \mathcal{F}$  from [84, Thm 14.3(a), Prop 14.11(a)]. Since  $\Omega_i \in \mathcal{F}$  for all  $i \in \mathbb{Z}_{\geq 0}$ , it follows that  $\cap_i \Omega_i \in \mathcal{F}$ . Let  $\Omega_2 := \{\omega : \text{graph}(\tilde{\mathbf{x}}(\omega)) \subset \mathbb{R}^2 \times (\mathbb{R}^n \setminus \mathcal{O})\}$ . Then,  $\Omega_2 \in \mathcal{F}$  follows from [84, Thm 14.3(i)]. Then, necessarily we have  $\mathbb{P}(\Omega_c) \geq \rho$  where  $\Omega_c = \{\omega : \tilde{\mathbf{x}}(\omega) \text{ is complete and } \text{graph}(\tilde{\mathbf{x}}(\omega)) \subset \mathbb{R}^2 \times (\mathbb{R}^n \setminus \mathcal{O})\} = (\cap_i \Omega_i) \cap \Omega_2$  and  $\Omega_c \in \mathcal{F}$ . This contradicts global recurrence of  $\mathcal{O}$  for  $\mathcal{H}$  and establishes the result.  $\blacksquare$

We can also observe that since the solutions of  $\mathcal{H}$  are a subset of solutions of  $\widehat{\mathcal{H}}$ , if any set  $\widehat{\mathcal{O}}$  is globally recurrent for  $\widehat{\mathcal{H}}$ , then  $\widehat{\mathcal{O}}$  is also globally recurrent for  $\mathcal{H}$ .

**Example 6.3** Consider the stochastic hybrid system  $\mathcal{H}$  in Example 6.1. Now consider the inflated system  $\widehat{\mathcal{H}} = (C, f, \mathbb{R}^n, \widehat{G}, \mu)$ , where  $\widehat{G}(x, v) = g(x, v) \cup G_2(x)$  for  $x \in \mathbb{R}^n$  with  $G_2(x) = \{0\}$ . It follows that solutions generated by the augmented system  $\widehat{\mathcal{H}}$  that are

not solutions of the system  $\mathcal{H}$  jump to the origin through the mapping  $G_2$ . Hence, global recurrence of the set  $\mathcal{O} = (-\epsilon, \epsilon)$  where  $0 < \epsilon < 1$  is preserved for the inflated system  $\widehat{\mathcal{H}}$ .

### 6.5.1 Robustness to perturbations of the set

The probabilities used in this subsection are generated using the system  $\widehat{\mathcal{H}}$  for which the random solutions have almost surely complete sample paths. We define

$$\tilde{m}_{\subset S}(\ell, \xi) := \sup_{\mathbf{x} \in S_r(\xi)} \mathbb{P}(\text{graph}(\mathbf{x}) \cap (\Gamma_{< \ell} \times \mathbb{R}^n) \subset (\mathbb{R}^2 \times S)). \quad (6.6)$$

The motivation for defining the above quantity which is greater than or equal to  $m_{\subset S}(\ell, \xi)$  is that we can apply the sequential compactness results developed in [41] to prove the next result. We establish that that finite time viability probabilities related to a perturbation of a set  $S$  from a compact set of initial conditions can be made arbitrarily close to worst case probabilities related to the original set  $S$  provided the perturbation is small enough.

**Proposition 6.8** *Let  $S \subset \mathbb{R}^n$  be closed. For each  $(\ell, \rho) \in \mathbb{Z}_{\geq 0} \times \mathbb{R}_{> 0}$  and  $K \subset \mathbb{R}^n$  compact there exists a  $\varepsilon > 0$  such that, for every  $x \in K$ ,*

$$m_{\subset S + \varepsilon \mathbb{B}}(\ell, x) \leq \max_{\xi \in K} \tilde{m}_{\subset S}(\ell, \xi) + \rho.$$

We now state the first main result related to robustness of the recurrence property. The following theorem establishes that recurrence of an open, bounded set implies the existence of a smaller recurrent set within the original set. The proof is presented in the appendix.

**Theorem 6.1** *Let the open bounded set  $\mathcal{O} \subset \mathbb{R}^n$  be globally recurrent for  $\widehat{\mathcal{H}}$  in (6.5). Then, there exists an open bounded set  $\widehat{\mathcal{O}}$  and  $\varepsilon > 0$  such that  $\widehat{\mathcal{O}} + \varepsilon \mathbb{B}^o \subset \mathcal{O}$  and  $\widehat{\mathcal{O}}$  is globally recurrent for  $\widehat{\mathcal{H}}$ .*

**Example 6.4** Consider the SHS  $\mathcal{H}$  in Example 6.1. It follows from Example 6.3 that the set  $\mathcal{O} = (-\epsilon, \epsilon)$ , where  $0 < \epsilon < 1$  is globally recurrent for  $\widehat{\mathcal{H}}$ . It can be easily observed from the discrete-time dynamics that solutions exhibit jumps to the origin in an almost sure sense and hence the set  $\widehat{\mathcal{O}} := \{x : |x| < \epsilon/2\}$  is also globally recurrent for  $\widehat{\mathcal{H}}$  and satisfies  $\widehat{\mathcal{O}} + (\epsilon/3)\mathbb{B}^o \subset \mathcal{O}$ .

### 6.5.2 Robustness to slowing down recurrence

The next inflation of the data of  $\widehat{\mathcal{H}}$  results in preserving recurrence while making certain quantities related to the average value of worst case first hitting time for solutions to the set  $\mathcal{O}$  grow unbounded in the distance of the state to the set  $\mathcal{O}$ . The result is important in the context of developing converse Lyapunov theorems with radially unbounded Lyapunov functions.

For  $\nu \in \mathcal{K}_\infty$ , define the continuous set-valued mapping  $M_\nu(x) := \{x^*\} + \nu(|x - x^*|)\mathbb{B}$  for  $x^* \in \mathbb{R}^n$ . Consider the inflated mapping

$$\widehat{\mathcal{H}}_\nu := (C, F, \mathbb{R}^n, \widehat{G}_\nu, \mu) \quad (6.7)$$

where  $\widehat{G}(x) = G_1(x) \cup M_\nu(x)$ . The proof of the next result is very similar to Lemma 6.1.

**Lemma 6.3** *For every  $\nu \in \mathcal{K}_\infty$ , the data of the SHS  $\widehat{\mathcal{H}}_\nu$  in (6.7) satisfies Standing Assumption 6.1 and Assumption 6.1.*

The next theorem claims the existence of a  $\nu \in \mathcal{K}_\infty$  small enough to preserve recurrence of the set  $\mathcal{O}$  for the inflated system  $\widehat{\mathcal{H}}_\nu$  if  $\mathcal{O}$  is globally recurrent for  $\widehat{\mathcal{H}}$  and  $x^* \in \mathcal{O}$ . A similar result is established for stochastic difference inclusions in [55, Theorem 4] and the proof presented in the appendix differs only in the construction of the function  $\nu$ .



**Theorem 6.2** *Let the open, bounded set  $\mathcal{O} \subset \mathbb{R}^n$  be globally recurrent for  $\widehat{\mathcal{H}}$ . Then, for any  $x^* \in \mathcal{O}$ , there exists  $\nu \in \mathcal{K}_\infty$  such that  $\mathcal{O}$  is globally recurrent for  $\widehat{\mathcal{H}}_\nu$  in (6.7).  $\square$*

**Example 6.5** Consider the SHS  $\mathcal{H}$  in Example 6.1. Let the augmented system  $\widehat{\mathcal{H}}_\nu = (C, f, \mathbb{R}^n, \widehat{G}_\nu, \mu)$ , where  $\widehat{G}(x, v) = g(x, v) \cup M_\nu(x)$  for  $x \in \mathbb{R}^n$  with  $M_\nu(x) = \{x^*\} + \nu(|x^* - x|)\mathbb{B}$ . We choose  $\nu(s) = s/2$  and  $x^* = \{0\}$ . With this modification, it can be observed that the recurrence property is preserved for the set  $\mathcal{O}$  while making the worst first hitting time for solutions to the set  $\mathcal{O}$  increase with the size of the state. Compared to the solutions of  $\widehat{\mathcal{H}}$  in Example 3 where solutions from large initial conditions can reach the set  $\mathcal{O}$  in one jump, the solutions of  $\widehat{\mathcal{H}}_\nu$  have a worst first hitting time to the set  $\mathcal{O}$  that is proportional to the size of the initial conditions. Hence, the recurrence property is slowed in  $\widehat{\mathcal{H}}_\nu$  by preventing some solutions from jumping to the origin in one step from large initial conditions.

### 6.5.3 Robustness to perturbations of system data

Finally, we analyze the robustness of recurrence to sufficiently small state dependent perturbations. For a continuous, positive-valued function  $\delta : \mathbb{R}^n \rightarrow \mathbb{R}_{>0}$ , we denote the perturbed version of  $\widehat{\mathcal{H}}$  by

$$\widehat{\mathcal{H}}_\delta := (C_\delta, F_\delta, \mathbb{R}^n, \widehat{G}_\delta, \mu) \quad (6.8)$$

with the data defined as

$$\begin{aligned} C_\delta &:= \{x \in \mathbb{R}^n : (x + \delta(x)\mathbb{B}) \cap C \neq \emptyset\} \\ F_\delta(x) &:= \overline{\text{con}}F(x + \delta(x)\mathbb{B}) \cap C + \delta(x)\mathbb{B} \\ \widehat{G}_\delta(x, v) &:= \{w \in \mathbb{R}^n : w \in g + \delta(g)\mathbb{B}, g \in \widehat{G}(x + \delta(x)\mathbb{B}, v)\} \end{aligned}$$

where  $\overline{\text{con}}$  refers to the closed convex hull. The next result follows from [14, Proposition 6.28] and [42, Prop 8].

**Lemma 6.4** *For every continuous  $\delta : \mathbb{R}^n \rightarrow \mathbb{R}_{>0}$ , the data of the hybrid system  $\widehat{\mathcal{H}}_\delta$  in (6.8) satisfies Standing Assumption 6.1 and Assumption 6.1.*

The next result establishes closeness of probabilities between the perturbed and unperturbed SHS. For constant perturbations we use  $\delta(x) \equiv \delta$  for all  $x \in \mathbb{R}^n$ . In this subsection we denote the probabilities generated by the system  $\widehat{\mathcal{H}}_\delta$  with the subscript  $\delta$ .

**Proposition 6.9** *Let  $S \subset \mathbb{R}^n$  be closed. For each  $(\ell, \rho) \in \mathbb{Z}_{\geq 0} \times \mathbb{R}_{>0}$  and  $K \subset \mathbb{R}^n$  compact there exists a  $\delta > 0$  such that, for every  $x \in K$ ,*

$$m_{\delta, \subset S}(\ell, x) \leq \max_{\xi \in K} \tilde{m}_{\subset S}(\ell, \xi) + \rho.$$

The next result establishes that recurrence of the set open, bounded  $\mathcal{O}$  set can be preserved when the state dependent perturbations are sufficiently small. The proof presented in the appendix follows along the same lines as [55, Thm 5].

**Theorem 6.3** *Let the open bounded set  $\mathcal{O} \subset \mathbb{R}^n$  be globally recurrent for  $\widehat{\mathcal{H}}$ . Then, there exists a continuous function  $\delta : \mathbb{R}^n \rightarrow \mathbb{R}_{>0}$  such that  $\mathcal{O}$  is globally recurrent for  $\widehat{\mathcal{H}}_\delta$  in (6.8).*

## 6.6 Necessary and sufficient condition for global recurrence

In this section we present a Lyapunov function based characterization of the recurrence property. A smooth function  $V : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$  is a Lyapunov function with respect

to the set  $\mathcal{O}$  for  $\mathcal{H}$  if it is radially unbounded and there exists a continuous function  $\rho : \mathbb{R}^n \rightarrow \mathbb{R}_{>0}$  and  $\mu > 0$  such that

$$\langle \nabla V(x), f \rangle \leq -\rho(x) + \mu \mathbb{1}_{\mathcal{O}}(x), \quad \forall x \in C, f \in F(x) \quad (6.9)$$

$$\int_{\mathbb{R}^m} \max_{g \in G(x,v)} V(g) \mu(dv) \leq V(x) - \rho(x) + \mu \mathbb{1}_{\mathcal{O}}(x), \quad \forall x \in D. \quad (6.10)$$

The conditions (6.9) and (6.10) imply that the Lyapunov function  $V$  decreases strictly during flows outside the set  $\mathcal{O}$  and decreases strictly on average along jumps outside the set  $\mathcal{O}$ . It can be noted that the Lyapunov function  $V$  can increase along solutions in the set  $\mathcal{O}$ .

We note that weak sufficient conditions for characterizing global recurrence that do not rely on a Lyapunov-like function satisfying strict decrease conditions on average are established in [25] in terms of Matorosov functions and in Chapter 4 using the invariance principle. The following theorem establishes necessary and sufficient conditions for global recurrence of open, bounded sets in terms of the Lyapunov conditions (6.9) and (6.10).

**Theorem 6.4** *An open, bounded set  $\mathcal{O} \subset \mathbb{R}^n$  is globally recurrent for  $\mathcal{H}$  if and only if there exists a smooth Lyapunov function  $V$  with respect to the set  $\mathcal{O}$  for  $\mathcal{H}$ .*

The proof of Theorem 6.4 involves two parts. The sufficiency of the Lyapunov condition is already established in [25, Thm 4.3]. The necessity of the existence of a Lyapunov function satisfying (2.8) and (2.9) is proved in detail in the subsections below. The outline of the proof for the construction of the Lyapunov function is stated below.

A preliminary Lyapunov function  $V_0$  is constructed for a perturbed version of the nominal system with respect to a set contained within the set  $\mathcal{O}$  such that it is radially unbounded, satisfies strict decrease conditions during flows and strict decrease on average along jumps. The preliminary Lyapunov function is not necessarily smooth. Hence, the

final step involves constructing a smooth Lyapunov function  $V$  with respect to the set  $\mathcal{O}$  for the nominal system from  $V_0$  in a manner that preserves the main decrease properties of the function  $V_0$ .

### 6.6.1 Preliminary Lyapunov function

Since the set  $\mathcal{O}$  is globally recurrent for the SHS  $\mathcal{H}$ , it follows from Lemma 6.2 that  $\mathcal{O}$  is globally recurrent for  $\widehat{\mathcal{H}}$ . Then, from Theorem 6.1 we have that there exists  $\varepsilon > 0$  such that  $\widehat{\mathcal{O}}_2 + \varepsilon\mathbb{B}^o \subset \mathcal{O}$  and  $\widehat{\mathcal{O}}_2$  is globally recurrent for  $\widehat{\mathcal{H}}$ . It now follows from Theorem 6.2 that there exists  $\nu \in \mathcal{K}_\infty$  such that  $\widehat{\mathcal{O}}_2$  is globally recurrent for the system  $\widehat{\mathcal{H}}_\nu$ . Finally, it follows from Theorem 6.3 that there exists a continuous function  $\delta : \mathbb{R}^n \rightarrow \mathbb{R}_{>0}$  such that  $\widehat{\mathcal{O}}_2$  is globally recurrent for the system  $\widehat{\mathcal{H}}_{\nu,\delta}$  where the SHS  $\widehat{\mathcal{H}}_{\nu,\delta}$  refers to a  $\delta$  perturbation of  $\widehat{\mathcal{H}}_\nu$ . We now construct a preliminary Lyapunov function to certify recurrence of the set  $\widehat{\mathcal{O}}_2$  for the system  $\widehat{\mathcal{H}}_{\nu,\delta}$ . The probability functions used in this section are generated from the system  $\widehat{\mathcal{H}}_{\nu,\delta}$ .

**Proposition 6.10** *There exist a locally absolutely continuous function  $\kappa \in \mathcal{K}_\infty$  such that  $W(x) := \int_0^\infty \kappa'(\tau) m_{\mathbb{C}\mathbb{R}^n \setminus \widehat{\mathcal{O}}_2}(\tau, x) d\tau$  is well defined, locally bounded and upper semicontinuous.*

*Proof:* Since the set  $\widehat{\mathcal{O}}_2$  is uniformly globally recurrent, for any compact set  $K$  we can bound  $m_{\mathbb{C}\mathbb{R}^n \setminus \widehat{\mathcal{O}}_2}(\tau, x)$  for all  $(\tau, x) \in \mathbb{R}_{\geq 0} \times K$  by a function  $\sigma_K \in \mathcal{L}$  such that  $m_{\mathbb{C}\mathbb{R}^n \setminus \widehat{\mathcal{O}}_2}(\tau, x) \leq \sigma_K(\tau)$  for all  $\tau \in \mathbb{R}_{\geq 0}$ . Without loss of generality we can assume that  $\sigma_{2^i\mathbb{B}}(\tau) \leq \sigma_{2^{i+1}\mathbb{B}}(\tau), \forall (i, \tau) \in \mathbb{Z}_{\geq 0} \times \mathbb{R}_{\geq 0}$ . Let  $\ell \in \mathcal{K}_\infty$  satisfy  $\sigma_{2^i\mathbb{B}}(\ell(i)) \leq 2^{-i}$  for all  $i \in \mathbb{Z}_{\geq 1}$ . Without loss of generality we can assume  $\ell(i+1) \geq \ell(i) + 1$  for all  $i \in \mathbb{Z}_{\geq 0}$ . The function  $\ell$  can be linearly interpolated between the points  $i, i+1$  for every  $i \in \mathbb{Z}_{\geq 0}$ . Hence for every  $i \in \mathbb{Z}_{\geq 0}$  and  $s \in (i, i+1)$ , we have  $\ell'(s) \geq 1$ . Define  $\kappa(s) := \ell^{-1}(s)$  for all  $s \geq 0$ . We observe that  $\kappa \in \mathcal{K}_\infty$ . Also, for any interval  $[a, b]$ , since the set

$E = \{\tau \in [a, b] : \ell'(\tau) = 0\}$  is of measure zero, it follows from [88, Exercise 3.21] that  $\kappa$  is absolutely continuous on  $[a, b]$ . Given  $x \in \mathbb{R}^n$ , let  $k \in \mathbb{Z}_{\geq 1}$  be such that  $x \in 2^k \mathbb{B}$ . Then, it follows that

$$\begin{aligned}
\int_0^\infty \kappa'(\tau) m_{\mathbb{C}\mathbb{R}^n \setminus \widehat{\mathcal{O}}_2}(\tau, x) d\tau &= \sum_{j=0}^\infty \int_{\ell(j)}^{\ell(j+1)} \kappa'(\tau) m_{\mathbb{C}\mathbb{R}^n \setminus \widehat{\mathcal{O}}_2}(\tau, x) d\tau \\
&\leq \sum_{j=0}^\infty \int_{\ell(j)}^{\ell(j+1)} \kappa'(\tau) \sigma_{2^k \mathbb{B}}(\tau) d\tau \\
&\leq \sum_{j=0}^\infty \sigma_{2^k \mathbb{B}}(\ell(j)) \int_{\ell(j)}^{\ell(j+1)} \kappa'(\tau) d\tau \\
&= \sum_{j=0}^\infty \sigma_{2^k \mathbb{B}}(\ell(j)) [\kappa(\ell(j+1)) - \kappa(\ell(j))] = \sum_{j=0}^\infty \sigma_{2^k \mathbb{B}}(\ell(j)) \\
&\leq \sum_{j=0}^{k-1} \sigma_{2^k \mathbb{B}}(\ell(j)) + \sum_{j=k}^\infty \sigma_{2^j \mathbb{B}}(\ell(j)) \\
&\leq k \sigma_{2^k \mathbb{B}}(\ell(0)) + \sum_{j=k}^\infty 2^{-j}.
\end{aligned}$$

Since  $\sum_{j=0}^\infty 2^{-j} < \infty$ , it follows that  $W$  is well defined and locally bounded on compact sets. Next, we establish that  $W$  is upper semicontinuous. It follows from reorganizing the calculations above that for every  $x \in \mathbb{R}^n$ ,  $\delta > 0$  and  $\gamma > 0$  there exists  $\tau^* \in \mathbb{Z}_{\geq 1}$  such that  $W(z) \leq \int_0^{\tau^*} \kappa'(\tau) m_{\mathbb{C}\mathbb{R}^n \setminus \widehat{\mathcal{O}}_2}(\tau, z) d\tau + \gamma$  for all  $z \in \{x\} + \delta \mathbb{B}$ . Let  $\{x_i\}_{i=0}^\infty$  be a sequence of points that converges to  $x$ . Since  $\kappa$  is locally absolutely continuous and the viability probabilities are upper bounded by one, it follows that  $\int_0^{\tau^*} \kappa'(\tau) m_{\mathbb{C}\mathbb{R}^n \setminus \widehat{\mathcal{O}}_2}(\tau, x_i) d\tau \leq \int_0^{\tau^*} \kappa'(\tau) d\tau = \kappa(\tau^*)$ . Then, from Fatou's lemma we have

that

$$\begin{aligned}
\limsup_{i \rightarrow \infty} W(x_i) &= \limsup_{i \rightarrow \infty} \int_0^\infty \kappa'(\tau) m_{\mathbb{C}\mathbb{R}^n \setminus \widehat{\mathcal{O}}_2}(\tau, x_i) d\tau \\
&\leq \limsup_{i \rightarrow \infty} \int_0^{\tau^*} \kappa'(\tau) m_{\mathbb{C}\mathbb{R}^n \setminus \widehat{\mathcal{O}}_2}(\tau, x_i) d\tau + \gamma \\
&\leq \int_0^{\tau^*} \kappa'(\tau) \limsup_{i \rightarrow \infty} m_{\mathbb{C}\mathbb{R}^n \setminus \widehat{\mathcal{O}}_2}(\tau, x_i) d\tau + \gamma \\
&\leq \int_0^{\tau^*} \kappa'(\tau) m_{\mathbb{C}\mathbb{R}^n \setminus \widehat{\mathcal{O}}_2}(\tau, x) d\tau + \gamma \\
&\leq W(x) + \gamma.
\end{aligned}$$

Since  $\gamma > 0$  is arbitrary, the upper semicontinuity of  $W$  follows. ■

**Proposition 6.11** *The function  $W$  is radially unbounded.*

*Proof:* We first establish that for every  $k \in \mathbb{Z}_{>0}$ , there exists  $R > 0$  such that for all  $x \in \mathbb{R}^n \setminus (\widehat{\mathcal{O}}_2 + R\mathbb{B})$ , we have  $m_{\mathbb{C}\mathbb{R}^n \setminus \widehat{\mathcal{O}}_2}(\tau, x) = 1$  for all  $\tau \leq k$ . Let  $k \in \mathbb{Z}_{>0}$  be given. Let  $\hat{R} > 0$  be such that  $\widehat{\mathcal{O}}_2 \subset \{x^*\} + \hat{R}\mathbb{B}$ . We now pick  $\tilde{R} > 0$  such that  $\nu^k(\tilde{R}) \geq 2\hat{R}$ , where  $\nu^k$  is the composition of the function  $\nu$  for  $k$  times. Now pick  $R > 0$  such that  $\{x^*\} + \tilde{R}\mathbb{B} \subset \widehat{\mathcal{O}}_2 + R\mathbb{B}$ . We now consider a random process  $\mathbf{x}$  generated by the system  $x^+ \in \{x^*\} + \nu(|x - x^*|)\mathbb{B}$  with initial condition  $x \in \mathbb{R}^n \setminus (\widehat{\mathcal{O}}_2 + R\mathbb{B})$ . In particular, any process satisfying  $\mathbf{x}_\omega(0, j+1) \in \partial(\{x^*\} + \nu(|\mathbf{x}_\omega(0, j) - x^*|)\mathbb{B})$ , where  $\partial S$  represents the boundary of a set  $S \subset \mathbb{R}^n$ . Then, it follows that  $\mathbf{x}_\omega(0, j) \in \mathbb{R}^n \setminus \widehat{\mathcal{O}}_2$  for every  $j \in \{0, \dots, k\}$  and  $\omega \in \Omega$  when  $x \in \mathbb{R}^n \setminus (\widehat{\mathcal{O}}_2 + R\mathbb{B})$ . Then, we have  $m_{\mathbb{C}\mathbb{R}^n \setminus \widehat{\mathcal{O}}_2}(\tau, x) = 1$  for  $\tau \leq k$ .

Now, we prove radial unboundedness. Let  $k \in \mathbb{Z}_{>0}$ . Then, there exists  $R > 0$  such that for all  $x \in \mathbb{R}^n \setminus (\widehat{\mathcal{O}}_2 + R\mathbb{B})$ , we have  $m_{\mathbb{C}\mathbb{R}^n \setminus \widehat{\mathcal{O}}_2}(\tau, x) = 1$  for all  $\tau \leq k$ . It now follows

that

$$\begin{aligned}
W(x) &= \int_0^\infty \kappa'(\tau) m_{\mathbb{C}\mathbb{R}^n \setminus \widehat{\mathcal{O}}_2}(\tau, x) d\tau \\
&\geq \int_0^k \kappa'(\tau) m_{\mathbb{C}\mathbb{R}^n \setminus \widehat{\mathcal{O}}_2}(\tau, x) d\tau \\
&\geq \int_0^k \kappa'(\tau) d\tau = \kappa(k).
\end{aligned}$$

Since  $\kappa \in \mathcal{K}_\infty$  and  $k > 0$  is arbitrary it follows that  $W$  is radially unbounded.  $\blacksquare$

The preliminary Lyapunov function  $V_0$  that we will consider is given by  $V_0(x) = \int_0^\infty \kappa'(\tau) m_{\mathbb{C}\mathbb{R}^n \setminus \widehat{\mathcal{O}}_2}(\tau, x) d\tau + \alpha \mathbb{I}_{\mathbb{R}^n \setminus \widehat{\mathcal{O}}_2}(x) + \beta$  for some  $\alpha, \beta > 0$ . We now explain the motivation for the structure of  $V_0$ . For recurrence of open, bounded sets in non-stochastic systems, it is established in [31] and [22, Thm 11.2.1] that the (worst-case) first hitting time for solutions to the recurrent set is a Lyapunov function candidate. A natural extension to the case of stochastic systems would be to consider the average value of the (worst-case) first hitting time for solutions to the recurrent set as a Lyapunov function candidate. In general, the average value of the (worst-case) first hitting time for solutions to the recurrent set need not be finite. In fact, it is finite and well defined only if a stronger property like positive recurrence of the set is assumed. The function  $V_0$  is closely related to the average value of the average value of the (worst-case) first hitting time for solutions to the recurrent set and the role of the function  $\kappa$  is to make the function  $V_0$  well defined.

It follows that  $V_0$  is upper semicontinuous and radially unbounded. We first claim that for every  $s > 0$ ,  $\text{essinf}_{\tau \in [0, s]} \kappa'(\tau) > 0$ . Since the mapping  $\ell$  in Proposition 6.10 has bounded derivatives almost everywhere it follows that for every  $s > 0$ , there exists  $\gamma_s > 0$  such that  $1 \leq \ell'(\tau) \leq \gamma_s$  for almost every  $\tau \in [0, \kappa(s)]$ . Then,  $\text{essinf}_{\tau \in [0, s]} \kappa'(\tau) = \text{essinf}_{\tau \in [0, s]} 1/\ell'(\kappa(\tau)) \geq 1/\gamma_s$ . Now, we establish the decrease properties along solutions

for the function  $V_0$ . In particular, we prove that the function  $V_0$  decreases strictly along solutions during flows outside  $\widehat{\mathcal{O}}_2$  and decreases strictly on average during jumps outside the set  $\widehat{\mathcal{O}}_2$ .

**Proposition 6.12** *For each compact set  $K$  there exists  $\gamma > 0$  such that for each  $x \in (C_\delta \setminus \widehat{\mathcal{O}}_2) \cap K$  from which there exists a solution  $\phi$  to  $(C_\delta \setminus \widehat{\mathcal{O}}_2, F_\delta)$  with  $t > 0$  and  $t \in \text{dom}(\phi)$  we have*

$$V_0(\phi(t)) \leq V_0(x) - \gamma t. \quad (6.11)$$

*Proof:* Let the initial condition  $x \in C_\delta \setminus \widehat{\mathcal{O}}_2$  and  $\phi$  be a solution of  $\dot{x} \in F_\delta(x)$ ,  $x \in C_\delta \setminus \widehat{\mathcal{O}}_2$  with  $t \in \text{dom}(\phi)$ ,  $t > 0$ . We first observe that  $m_{\mathbb{C}\mathbb{R}^n \setminus \widehat{\mathcal{O}}_2}(\tau, \phi(t)) \leq m_{\mathbb{C}\mathbb{R}^n \setminus \widehat{\mathcal{O}}_2}(\tau + t, x)$ . This inequality is a direct consequence of the definition of the viability probabilities and the properties of the solution  $\phi$ . Hence,

$$\begin{aligned} V_0(\phi(t)) &= \int_0^\infty \kappa'(\tau) m_{\mathbb{C}\mathbb{R}^n \setminus \widehat{\mathcal{O}}_2}(\tau, \phi(t)) d\tau + \alpha + \beta \\ &\leq \int_0^\infty \kappa'(\tau) m_{\mathbb{C}\mathbb{R}^n \setminus \widehat{\mathcal{O}}_2}(\tau + t, x) d\tau + \alpha + \beta \\ &\leq \int_0^\infty \kappa'(\tau) m_{\mathbb{C}\mathbb{R}^n \setminus \widehat{\mathcal{O}}_2}(\tau, x) d\tau + \alpha + \beta - \tilde{\rho}_c(t, x) \\ &= V_0(x) - \tilde{\rho}_c(t, x) \end{aligned}$$

where  $\tilde{\rho}_c(t, x) = \int_0^\infty \kappa'(\tau) [m_{\mathbb{C}\mathbb{R}^n \setminus \widehat{\mathcal{O}}_2}(\tau, x) - m_{\mathbb{C}\mathbb{R}^n \setminus \widehat{\mathcal{O}}_2}(\tau + t, x)] d\tau$ .

We now establish that  $\tilde{\rho}_c$  is positive on compact sets  $(C_\delta \setminus \widehat{\mathcal{O}}_2) \cap K$ . From uniform global recurrence of  $\widehat{\mathcal{O}}_2$ , it follows that  $m_{\mathbb{C}\mathbb{R}^n \setminus \widehat{\mathcal{O}}_2}(\tau, x) \leq \sigma_K(\tau)$  for all  $\tau \geq 0$ ,  $x \in K$  and some  $\sigma_K \in \mathcal{L}$ . Let  $\ell > 0$  be such that  $\sigma_K(\ell) \leq 0.5$ . From the construction of  $\kappa$ , it follows that there exists  $\gamma_0 > 0$  such that  $\text{essinf}_{\tau \in [0, \ell]} \kappa'(\tau) = \gamma_0$ . Let  $x \in K$  be such that there exists a solution  $\phi$  to  $(C_\delta \setminus \widehat{\mathcal{O}}_2, F_\delta)$  from  $x$  with  $t \in \text{dom}(\phi)$  and  $t > 0$ . We observe that



$m_{\mathbb{C}\mathbb{R}^n \setminus \widehat{\mathcal{O}}_2}(\tau, x) = 1$  for  $0 \leq \tau \leq t$  due to the existence of a solution  $\phi$  that remains in the set  $\mathbb{R}^n \setminus \widehat{\mathcal{O}}_2$  till time  $t$ . Then,

$$\begin{aligned}
\tilde{\rho}_c(t, x) &\geq \int_0^\ell \kappa'(\tau) [m_{\mathbb{C}\mathbb{R}^n \setminus \widehat{\mathcal{O}}_2}(\tau, x) - m_{\mathbb{C}\mathbb{R}^n \setminus \widehat{\mathcal{O}}_2}(\tau + t, x)] d\tau \\
&\geq \operatorname{ess\,inf}_{\tau \in [0, \ell]} \kappa'(\tau) \int_0^\ell [m_{\mathbb{C}\mathbb{R}^n \setminus \widehat{\mathcal{O}}_2}(\tau, x) - m_{\mathbb{C}\mathbb{R}^n \setminus \widehat{\mathcal{O}}_2}(\tau + t, x)] d\tau \\
&= \gamma_0 \int_0^\ell [m_{\mathbb{C}\mathbb{R}^n \setminus \widehat{\mathcal{O}}_2}(\tau, x) - m_{\mathbb{C}\mathbb{R}^n \setminus \widehat{\mathcal{O}}_2}(\tau + t, x)] d\tau \\
&= \gamma_0 \left[ \int_0^t m_{\mathbb{C}\mathbb{R}^n \setminus \widehat{\mathcal{O}}_2}(\tau, x) d\tau - \int_\ell^{\ell+t} m_{\mathbb{C}\mathbb{R}^n \setminus \widehat{\mathcal{O}}_2}(\tau, x) d\tau \right] \\
&\geq \gamma_0 [t - t/2] = \gamma t
\end{aligned}$$

where  $\gamma = \gamma_0/2$ . The result now follows and since  $t > 0$ , the bound (6.11) establishes that the function  $V_0$  decreases strictly along solutions outside the set  $\widehat{\mathcal{O}}_2$ .  $\blacksquare$

**Proposition 6.13** *There exists  $\tilde{\rho}_d : \mathbb{R}^n \setminus \widehat{\mathcal{O}}_2 \rightarrow \mathbb{R}_{>0}$  such that for every compact set  $K \subset \mathbb{R}^n \setminus \widehat{\mathcal{O}}_2$ ,  $\inf_{x \in K} \tilde{\rho}_d(x) > 0$  and*

$$\int_{\mathbb{R}^m} \max_{g \in \widehat{G}_{\nu, \delta}(x, v)} V_0(g) \mu(dv) \leq V_0(x) - \tilde{\rho}_d(x), \forall x \in \mathbb{R}^n \setminus \widehat{\mathcal{O}}_2. \quad (6.12)$$

*Proof:* Let  $x \in \mathbb{R}^n \setminus \widehat{\mathcal{O}}_2$ , then

$$\begin{aligned}
\int_{\mathbb{R}^m} \max_{g \in \widehat{G}_{\nu, \delta}(x, v)} V_0(g) \mu(dv) &= \int_{\mathbb{R}^m} \max_{g \in \widehat{G}_{\nu, \delta}(x, v)} \left[ \int_0^\infty \kappa'(\tau) m_{\mathbb{C}\mathbb{R}^n \setminus \widehat{\mathcal{O}}_2}(\tau, g) d\tau \right. \\
&\quad \left. + \alpha \mathbb{I}_{\mathbb{R}^n \setminus \widehat{\mathcal{O}}_2}(g) \right] \mu(dv) + \beta \\
&\leq \int_{\mathbb{R}^m} \max_{g \in \widehat{G}_{\nu, \delta}(x, v)} \int_0^\infty \kappa'(\tau) m_{\mathbb{C}\mathbb{R}^n \setminus \widehat{\mathcal{O}}_2}(\tau, g) d\tau \mu(dv) \\
&\quad + \alpha \int_{\mathbb{R}^m} \max_{g \in \widehat{G}_{\nu, \delta}(x, v)} \mathbb{I}_{\mathbb{R}^n \setminus \widehat{\mathcal{O}}_2}(g) \mu(dv) + \beta \\
&\leq \int_{\mathbb{R}^m} \int_0^\infty \max_{g \in \widehat{G}_{\nu, \delta}(x, v)} \kappa'(\tau) m_{\mathbb{C}\mathbb{R}^n \setminus \widehat{\mathcal{O}}_2}(\tau, g) d\tau \mu(dv) \\
&\quad + \alpha \int_{\mathbb{R}^m} \max_{g \in \widehat{G}_{\nu, \delta}(x, v)} \mathbb{I}_{\mathbb{R}^n \setminus \widehat{\mathcal{O}}_2}(g) \mu(dv) + \beta \\
&= \int_0^\infty \kappa'(\tau) \int_{\mathbb{R}^m} \max_{g \in \widehat{G}_{\nu, \delta}(x, v)} m_{\mathbb{C}\mathbb{R}^n \setminus \widehat{\mathcal{O}}_2}(\tau, g) \mu(dv) d\tau \\
&\quad + \alpha \int_{\mathbb{R}^m} \max_{g \in \widehat{G}_{\nu, \delta}(x, v)} \mathbb{I}_{\mathbb{R}^n \setminus \widehat{\mathcal{O}}_2}(g) \mu(dv) + \beta \\
&\leq \int_0^\infty \kappa'(\tau) m_{\mathbb{C}\mathbb{R}^n \setminus \widehat{\mathcal{O}}_2}(\tau + 1, x) d\tau \\
&\quad + \alpha \int_{\mathbb{R}^m} \max_{g \in \widehat{G}_{\nu, \delta}(x, v)} \mathbb{I}_{\mathbb{R}^n \setminus \widehat{\mathcal{O}}_2}(g) \mu(dv) + \beta \\
&= V_0(x) - \tilde{\rho}_d(x)
\end{aligned}$$

where

$$\begin{aligned}
\tilde{\rho}_d(x) &:= \int_0^\infty \kappa'(\tau) [m_{\mathbb{C}\mathbb{R}^n \setminus \widehat{\mathcal{O}}_2}(\tau, x) - m_{\mathbb{C}\mathbb{R}^n \setminus \widehat{\mathcal{O}}_2}(\tau + 1, x)] d\tau \\
&\quad + \alpha [1 - \int_{\mathbb{R}^m} \max_{g \in \widehat{G}_{\nu, \delta}(x, v)} \mathbb{I}_{\mathbb{R}^n \setminus \widehat{\mathcal{O}}_2}(g) \mu(dv)].
\end{aligned}$$

We now establish  $\tilde{\rho}_d$  is bounded away from zero on compact subsets  $K \subset \mathbb{R}^n \setminus \widehat{\mathcal{O}}_2$ . From uniform global recurrence of  $\widehat{\mathcal{O}}_2$ , it follows that  $m_{\mathbb{C}\mathbb{R}^n \setminus \widehat{\mathcal{O}}_2}(\tau, x) \leq \sigma_K(\tau)$  for  $\tau \geq 0, x \in K$  and some  $\sigma_K \in \mathcal{L}$ . Let  $\ell > 0$  be such that  $\sigma_K(\ell) \leq 0.25$ . We also have

$\text{essinf}_{\tau \in [0, \ell]} \kappa'(\tau) = \gamma > 0$ . For  $x \in K$ , let  $\zeta(x) = \int_{\mathbb{R}^m} \max_{g \in \widehat{G}_{\nu, \delta}(x, v)} \mathbb{I}_{\mathbb{R}^n \setminus \widehat{\mathcal{O}}_2}(g) \mu(dv)$ . Let  $K_1 := \{x \in K : \zeta(x) \leq 0.5\}$  and  $K_2 := \{x \in K : \zeta(x) \geq 0.5\}$ . We observe that  $K_1 \cup K_2 = K$ . Then, we have

$$\begin{aligned} \tilde{\rho}_d(x) &\geq \int_0^\ell \kappa'(\tau) [m_{\mathbb{C}\mathbb{R}^n \setminus \widehat{\mathcal{O}}_2}(\tau, x) - m_{\mathbb{C}\mathbb{R}^n \setminus \widehat{\mathcal{O}}_2}(\tau + 1, x)] d\tau \\ &\quad + \alpha \left[ 1 - \int_{\mathbb{R}^m} \max_{g \in \widehat{G}_{\nu, \delta}(x, v)} \mathbb{I}_{\mathbb{R}^n \setminus \widehat{\mathcal{O}}_2}(g) \mu(dv) \right] \\ &\geq \gamma \left[ \int_0^1 m_{\mathbb{C}\mathbb{R}^n \setminus \widehat{\mathcal{O}}_2}(\tau, x) d\tau - \int_\ell^{\ell+1} m_{\mathbb{C}\mathbb{R}^n \setminus \widehat{\mathcal{O}}_2}(\tau, x) d\tau \right] \\ &\quad + \alpha \left[ 1 - \int_{\mathbb{R}^m} \max_{g \in \widehat{G}_{\nu, \delta}(x, v)} \mathbb{I}_{\mathbb{R}^n \setminus \widehat{\mathcal{O}}_2}(g) \mu(dv) \right]. \end{aligned}$$

It follows that for  $x \in K_1$ ,  $\tilde{\rho}_d(x) \geq \alpha/2$  and for  $x \in K_2$ ,  $\tilde{\rho}_d(x) \geq \gamma/4$  and hence for  $x \in K$ ,  $\tilde{\rho}_d(x) \geq \min\{\gamma/4, \alpha/2\}$ . Hence  $\tilde{\rho}_d(x)$  is bounded away from zero on compact sets outside the set  $\widehat{\mathcal{O}}_2$ . The bound (6.12) establishes that the function  $V_0$  decreases strictly on average along jumps outside the set  $\widehat{\mathcal{O}}_2$ . ■

### 6.6.2 Smoothing the preliminary Lyapunov function $V_0$

The bounds (6.11) and (6.12) establish that the preliminary Lyapunov function  $V_0$  satisfies strict decrease conditions on average outside the set  $\widehat{\mathcal{O}}_2$ . The next step in the development of the converse Lyapunov theorem is the smoothing process where the preliminary non-smooth Lyapunov function is used to derive a smooth Lyapunov function by exploiting the robustness of the recurrence property. We consider the discrete-time and continuous-time Lyapunov conditions separately.

### Discrete-time condition

Define  $V_d(x) := \int_{\mathbb{R}^n} V_0(x + \sigma_d(x)\xi)\Psi(\xi)d\xi$  where  $\sigma_d : \mathbb{R}^n \rightarrow \mathbb{R}_{>0}$  is a continuous positive function for all  $x \in \mathbb{R}^n$  and  $\Psi : \mathbb{R}^n \rightarrow [0, 1]$  is any infinitely differentiable function such that  $\Psi(x) = 0$  for  $x \notin \mathbb{B}$  and  $\int \Psi(x)dx = 1$ . The next results involves smoothing the discrete-time condition. The structure of the function  $V_d$  is motivated by similar constructions in [45], [7], [31] and [55]. We will establish in the next proposition that the function  $V_d$ , through appropriate choice of  $\sigma_d$ , satisfies a condition related to the bound (2.9). We note that conditions related to radial unboundedness and smoothness of  $V_d$  will be explained in the subsequent sections.

**Proposition 6.14** *There exists a concave function  $\Gamma \in \mathcal{K}_\infty$ , continuous positive functions  $\sigma_d, \rho_d : \mathbb{R}^n \rightarrow \mathbb{R}_{>0}$ , and  $\mu_d > 0$  such that*

$$\int_{\mathbb{R}^m} \max_{g \in \widehat{G}(x,v)} \Gamma(V_d(g))\mu(dv) \leq \Gamma(V_d(x)) - \rho_d(x) + \mu_d \mathbb{I}_{\mathcal{O}}(x), x \in D. \quad (6.13)$$

*Proof:* We note that the proof of the proposition follows along the same lines as the proof of [55, Theorem 2]. Let  $\sigma^* > 0$  be such that  $\widehat{\mathcal{O}}_2 + \sigma^*\mathbb{B}^o \subset \mathcal{O}$ . We pick  $\sigma_d$  to satisfy  $\sigma_d(x) \leq \sigma^*/2$ . We also choose  $\sigma_d$  sufficiently small so that

$$\sigma_d(x) \leq 0.5\delta(x) \leq \delta(x + \sigma_d(x)\xi) \quad \forall (x, \xi) \in \mathbb{R}^n \times \mathbb{B}. \quad (6.14)$$

It follows from (6.14) that

$$x \in \{x + \sigma_d(x)\xi\} + \delta(x + \sigma_d(x)\mathbb{B}) \quad \forall (x, \xi) \in \mathbb{R}^n \times \mathbb{B}. \quad (6.15)$$

We also note that  $\widehat{G}(x, v) \subset \widehat{G}_\nu(x, v)$  for all  $(x, v) \in \mathbb{R}^n \times \mathbb{R}^m$  and

$$\begin{aligned} \widehat{G}_{\nu, \delta}(x + \sigma_d(x)\xi, v) = \\ \{w : w = \{g\} + \delta(g)\mathbb{B}, g \in \widehat{G}_\nu(x + \sigma_d(x)\xi + \delta(x + \sigma_d(x)\xi)\mathbb{B}, v)\}. \end{aligned}$$

It follows from (6.14), (6.15) that

$$g \in \widehat{G}(x, v), \tilde{g} = g + \sigma_d(g)\xi, \xi \in \mathbb{B} \Big\} \Rightarrow \tilde{g} \in \widehat{G}_{\nu, \delta}(x + \sigma_d(x)\xi, v).$$

Since  $\sigma_d(x) \leq \sigma^*/2$ , it follows that for  $x \in \mathbb{R}^n \setminus \mathcal{O}$ ,  $x + \sigma_d(x)\xi \in \mathbb{R}^n \setminus \widehat{\mathcal{O}}_2$  for every  $\xi \in \mathbb{B}$ .

Then, from the above conditions and (6.12) it follows that for all  $x \in \mathbb{R}^n \setminus \mathcal{O}$ ,

$$\begin{aligned} \int_{\mathbb{R}^m} \max_{g \in \widehat{G}(x, v)} V_d(g) \mu(dv) &= \int_{\mathbb{R}^m} \max_{g \in \widehat{G}(x, v)} \left( \int_{\mathbb{R}^n} V_0(g + \sigma_d(g)\xi) \Psi(\xi) d\xi \right) \mu(dv) \\ &\leq \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^m} \max_{g \in \widehat{G}(x, v)} V_0(g + \sigma_d(g)\xi) \Psi(\xi) d\xi \right) \mu(dv) \\ &= \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^m} \max_{g \in \widehat{G}(x, v)} V_0(g + \sigma_d(g)\xi) \mu(dv) \right) \Psi(\xi) d\xi \\ &\leq \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^m} \max_{g \in \widehat{G}_{\nu, \delta}(x + \sigma_d(x)\xi, v)} V_0(g) \mu(dv) \right) \Psi(\xi) d\xi \\ &\leq \int_{\mathbb{R}^n} \left( V_0(x + \sigma_d(x)\xi) - \tilde{\rho}_d(x + \sigma_d(x)\xi) \right) \Psi(\xi) d\xi \\ &= V_d(x) - \varrho_d(x) \end{aligned}$$

where  $\varrho_d : \mathbb{R}^n \setminus \mathcal{O} \rightarrow \mathbb{R}_{>0}$  is given by  $\varrho_d(x) := \int_{\mathbb{R}^n} \tilde{\rho}_d(x + \sigma_d(x)\xi) \Psi(\xi) d\xi$ . We now establish that  $\varrho_d$  is bounded away from zero on compact subsets of  $\mathbb{R}^n \setminus \mathcal{O}$ . Let  $K \subset \mathbb{R}^n \setminus \mathcal{O}$  be compact. Then,  $K_1 := \overline{\bigcup_{x \in K} (x + \sigma_d(x)\mathbb{B})}$  is a compact set which is a subset of  $\mathbb{R}^n \setminus \widehat{\mathcal{O}}_2$ .

Hence, from (6.12) we have

$$\begin{aligned} \varrho_d(x) &= \int_{\mathbb{R}^n} \tilde{\rho}_d(x + \sigma_d(x)\xi) \Psi(\xi) d\xi \\ &\geq \int_{\mathbb{R}^n} \inf_{z \in K_1} \tilde{\rho}_d(z) \Psi(\xi) d\xi = \inf_{z \in K_1} \tilde{\rho}_d(z) > 0. \end{aligned}$$

We now analyze the quantity  $\sup_{x \in \mathcal{O}} \int_{\mathbb{R}^m} \max_{g \in \hat{G}(x,v)} V_d(g) \mu(dv)$ . As, illustrated in [55, Example 1], it is not necessary for this quantity to be finite even though the function  $V_d$  satisfies strict decrease conditions on average outside the set  $\mathcal{O}$ . Hence, we adopt the solution proposed in [55, Prop 1] which involves constructing a concave function  $\Gamma \in \mathcal{K}_\infty$  such that  $\sup_{x \in \mathcal{O}} \int_{\mathbb{R}^m} \max_{g \in \hat{G}(x,v)} \Gamma(V_d(g)) \mu(dv) < \infty$ . The concavity assumption is needed to preserve the strict decrease on average property of  $V_d$  outside the set  $\mathcal{O}$ .

From [55, Prop 1] we have that there exists  $\Gamma \in \mathcal{K}_\infty$ , concave such that

$$\int_{\mathbb{R}^m} \max_{g \in \hat{G}(x,v)} \Gamma(V_d(g)) \mu(dv) \leq \mu_d/2, \forall x \in \mathcal{O}$$

for some  $\mu_d > 0$ . Since  $\Gamma$  is concave, it follows from Jensen's inequality that for  $x \in \mathbb{R}^n \setminus \mathcal{O}$ ,

$$\begin{aligned} \int_{\mathbb{R}^m} \max_{g \in \hat{G}(x,v)} \Gamma(V_d(g)) \mu(dv) &\leq \int_{\mathbb{R}^m} \Gamma\left(\max_{g \in \hat{G}(x,v)} V_d(g)\right) \mu(dv) \\ &\leq \Gamma\left(\int_{\mathbb{R}^m} \max_{g \in \hat{G}(x,v)} V_d(g) \mu(dv)\right) \\ &\leq \Gamma(V_d(x) - \varrho_d(x)). \end{aligned}$$

Since  $\Gamma \in \mathcal{K}_\infty$  and  $\varrho_d$  is bounded away from zero on compact sets contained in  $\mathbb{R}^n \setminus \mathcal{O}$ , it follows that  $\Gamma(V_d(x) - \varrho_d(x)) < \Gamma(V_d(x))$  for  $x \in \mathbb{R}^n \setminus \mathcal{O}$ . Let the continuous function  $\tilde{\varrho}_d : \mathbb{R}^n \setminus \mathcal{O} \rightarrow \mathbb{R}_{>0}$  be defined such that  $\Gamma(V_d(x) - \varrho_d(x)) \leq \Gamma(V_d(x)) - \tilde{\varrho}_d(x)$  for all  $x \in \mathbb{R}^n \setminus \mathcal{O}$ . Now choose a function  $\hat{\varrho}_d : \mathbb{R}^n \rightarrow \mathbb{R}_{>0}$  such that  $\hat{\varrho}_d(x) \leq \min\{\mu_d/2, \tilde{\varrho}_d(x)\}$  where by convention  $\tilde{\varrho}_d(x) = \infty$  for  $x \notin \mathbb{R}^n \setminus \mathcal{O}$ . We now construct a continuous function

$\rho_d(x) := \inf_{\xi \in \mathbb{R}^n} (\widehat{\varrho}_d(\xi) + |\xi - x|)$  for all  $x \in \mathbb{R}^n$ . Since  $\widehat{\varrho}_d$  is bounded away from zero on compact sets, it follows that  $\rho_d$  inherits the same property. Also, from the construction we have that  $\rho_d(x) \leq \widehat{\varrho}_d(x)$  for all  $x \in \mathbb{R}^n$ . Hence,

$$\int_{\mathbb{R}^m} \max_{g \in \widehat{G}(x,v)} \Gamma(V_d(g)) \mu(dv) \leq \Gamma(V_d(x)) - \rho_d(x) + \mu_d \mathbb{I}_{\mathcal{O}}(x), \forall x \in \mathbb{R}^n.$$

It follows from the above bound and using  $G(x, v) \subset \widehat{G}(x, v), \forall (x, v) \in D \times \mathbb{R}^m$  that

$$\int_{\mathbb{R}^m} \max_{g \in G(x,v)} \Gamma(V_d(g)) \mu(dv) \leq \Gamma(V_d(x)) - \rho_d(x) + \mu_d \mathbb{I}_{\mathcal{O}}(x), \forall x \in D.$$

The result of the proposition is thus established. ■

### Continuous-time condition

Define  $V_c(x) := \int_{\mathbb{R}^n} V_0(x + \sigma_c(x)\xi) \Psi(\xi) d\xi$  where  $\sigma_c : \mathbb{R}^n \rightarrow \mathbb{R}_{>0}$  is a continuous positive function for all  $x \in \mathbb{R}^n$  and  $\Psi : \mathbb{R}^n \rightarrow [0, 1]$  is any infinitely differentiable function such that  $\Psi(x) = 0$  for  $x \notin \mathbb{B}$  and  $\int \Psi(x) dx = 1$ . The next result involves smoothing the continuous-time condition. We will establish in the next proposition that the function  $V_c$ , through appropriate choice of  $\sigma_c$ , satisfies a condition related to the bound (2.8).

**Proposition 6.15** *There exist continuous positive functions  $\sigma_c, \rho_c : \mathbb{R}^n \rightarrow \mathbb{R}_{>0}$  and  $\mu_c > 0$  such that the function  $V_c$  satisfies*

$$\langle \nabla V_c(x), f \rangle \leq -\rho_c(x) + \mu_c \mathbb{I}_{\mathcal{O}}(x), \forall x \in C, f \in F(x). \quad (6.16)$$

*Proof:* Let  $\sigma^* > 0$  be such that  $\widehat{\mathcal{O}}_2 + \sigma^* \mathbb{B}^o \subset \mathcal{O}$ . Choose the open set  $\widehat{\mathcal{O}}_1$  such that  $\widehat{\mathcal{O}}_2 + (\sigma^*/3) \mathbb{B}^o \subset \widehat{\mathcal{O}}_1$  and  $\widehat{\mathcal{O}}_1 + (\sigma^*/3) \mathbb{B}^o \subset \mathcal{O}$ . Define  $\varrho_c(x) := \min\{\delta(x), \sigma^*/4\}$ . Let  $\sigma_c$  come from [14, Lemma 7.37] using  $\varrho_c$ . Then, the function  $\sigma_c$  is continuous and positive

on bounded sets. We can also conclude that if  $x \in \mathbb{R}^n \setminus \widehat{\mathcal{O}}_1$ , then  $x + \sigma_c(x)\mathbb{B} \subset \mathbb{R}^n \setminus \widehat{\mathcal{O}}_2$ . The smoothness of  $V_c$  on  $\mathbb{R}^n$  follows from the results in [14, Section 7.36].

From [14, Lemma 7.37], for every solution  $\phi$  generated by  $(C_{\sigma_c}, F_{\sigma_c})$  and  $\eta \in \mathbb{B}$ , there exists a solution  $\psi_\eta$  generated by  $(C_{\varrho_c}, F_{\varrho_c})$  such that  $\text{dom}(\phi) = \text{dom}(\psi)$ ,  $\psi_\eta(0) = \phi(0) + \sigma_c(\phi(0))\eta$  and  $\psi_\eta(t) = \phi(t) + \sigma_c(\phi(t))\eta$ . Let  $i \in \mathbb{Z}_{\geq 1}$  and  $K_i = 2^i\mathbb{B}$  be a sequence of compact sets. For every  $i \in \mathbb{Z}_{\geq 1}$ , let  $\gamma_i > 0$  come from Proposition 6.12 using the sets  $K_i + \sigma^*/4\mathbb{B}$ . Then, for solutions  $\phi$  of  $(C_{\sigma_c}, F_{\sigma_c})$  from  $x \in K_i$  satisfying  $\text{graph}(\phi) \cap ([0, t] \times \mathbb{R}^n) \subset \mathbb{R} \times \mathbb{R}^n \setminus \widehat{\mathcal{O}}_1$  for some  $(t) \in \text{dom}(\phi)$  with  $t > 0$  we have

$$\begin{aligned} V_c(\phi(t)) &= \int_{\mathbb{R}^n} V_0(\phi(t) + \sigma_c(\phi(t))\eta)\Psi(\eta)d\eta \\ &= \int_{\mathbb{R}^n} V_0(\psi_\eta(t))\Psi(\eta)d\eta \\ &\leq \int_{\mathbb{R}^n} V_0(\psi_\eta(0))\Psi(\eta)d\eta - \gamma_i t \\ &= V_c(x) - \gamma_i t. \end{aligned}$$

We can now conclude that for every  $i \in \mathbb{Z}_{\geq 1}$ , there exists  $\gamma_i > 0$  such that

$$\langle \nabla V_c(x), f \rangle \leq -\gamma_i, \forall x \in (C \setminus \mathcal{O}) \cap K_i, f \in F(x) \quad (6.17)$$

For every  $x \in C \setminus \mathcal{O}$ , let  $i(x) = \min_{j \in \mathbb{Z}_{\geq 1}} \{j : x \in (C \setminus \mathcal{O}) \cap K_j\}$ . Define  $\widehat{\varrho}_c(x) := \inf_{\xi \in C \setminus \mathcal{O}} (\gamma_{i(x)} + |\xi - x|)$ . Since  $\mathcal{O}$  is bounded,  $V_c$  is smooth and  $F$  is locally bounded, there exists  $\mu_c > 0$  such that  $\sup_{f \in F(x)} \langle \nabla V_c(x), f \rangle \leq \mu_c/2$  for all  $x \in C \cap \mathcal{O}$ . Define the function  $\widehat{\rho}_c(x) = \min\{\widehat{\varrho}_c(x), \mu_c/2\}$  for all  $x \in \mathbb{R}^n$  where  $\widehat{\varrho}_c(x) = \infty$  for  $x \notin C \setminus \mathcal{O}$ . Now define the function  $\rho_c(x) = \inf_{z \in \mathbb{R}^n} (\widehat{\rho}_c(z) + |x - z|)$ . Then  $\rho_c$  is continuous, positive-valued and bounded away from zero on compact sets. Then, it follows that

$$\langle \nabla V_c(x), f \rangle \leq -\rho_c(x) + \mu_c \mathbb{I}_{\mathcal{O}}(x), \forall x \in C.$$



The result of the proposition is thus established. ■

### 6.6.3 Smooth Lyapunov function for recurrence

We now combine the results from Proposition 6.15 and 6.14 to establish a smooth Lyapunov function with respect to the set  $\mathcal{O}$  for the system  $\mathcal{H}$ . Define  $V_s(x) := \int_{\mathbb{R}^n} V_0(x + \sigma(x)\xi)\Psi(\xi)d\xi$  where  $\sigma : \mathbb{R}^n \rightarrow \mathbb{R}_{>0}$  is a continuous positive function for all  $x \in \mathbb{R}^n$  and  $\Psi : \mathbb{R}^n \rightarrow [0, 1]$  is any infinitely differentiable function such that  $\Psi(x) = 0$  for  $x \notin \mathbb{B}$  and  $\int \Psi(x)dx = 1$ . The next results completes the proof of Theorem 6.4.

**Proposition 6.16** *There exist continuous functions  $\sigma, \rho : \mathbb{R}^n \rightarrow \mathbb{R}_{>0}$ ,  $\mu > 0$  and a concave function  $\Gamma \in \mathcal{K}_\infty$  that is smooth on  $\mathbb{R}_{>0}$  with  $\Gamma'(s) > 0$  for  $s > 0$  such that the function  $\Gamma(V_s)$  is smooth, radially unbounded and satisfies*

$$\begin{aligned} \langle \nabla \Gamma(V_s(x)), f \rangle &\leq -\rho(x) + \mu \mathbb{I}_{\mathcal{O}}(x), \quad \forall x \in C, f \in F(x) \\ \int_{\mathbb{R}^m} \max_{g \in G(x,v)} \Gamma(V_s(g)) \mu(dv) &\leq \Gamma(V_s(x)) - \rho(x) + \mu \mathbb{I}_{\mathcal{O}}(x), \quad \forall x \in D. \end{aligned}$$

*Proof:* It follows from the proof of [55, Prop 1] that without loss of generality, the function  $\Gamma$  used in Proposition 6.14 can be taken to be smooth on  $\mathbb{R}_{>0}$  with  $\Gamma'(s) > 0$  for  $s > 0$ . Let  $\sigma_c, \rho_c, \mu_c$  come from Proposition 6.15 and  $\Gamma, \sigma_d, \rho_d, \mu_d$  come from Proposition 6.14. Choose the continuous function  $\sigma$  such that  $\sigma(x) = \min\{\sigma_c(x), \sigma_d(x)\}$ , choose  $\mu = \max\{\sup_{x \in \mathcal{O}}(\mu_c \Gamma'(V_s(x))), \mu_d\}$  and the continuous function  $\rho$  such that  $\rho(x) = \min\{\Gamma'(V_s(x))\rho_c(x), \rho_d(x)\}$ . Since  $V_s(x) \geq \beta > 0$ , it follows that  $\mu$  is well defined and  $\Gamma(V_s)$  is smooth. Since  $V_0$  is radially unbounded and  $\Gamma \in \mathcal{K}_\infty$ , it follows that  $\Gamma(V_s)$  is radially unbounded. Finally, it follows from the results in Propositions 6.15 and 6.14

that the function  $\Gamma(V_s)$  satisfies

$$\begin{aligned} \langle \nabla \Gamma(V_s(x)), f \rangle &\leq -\rho(x) + \mu \mathbb{I}_{\mathcal{O}}(x), \quad \forall x \in C, f \in F(x) \\ \int_{\mathbb{R}^m} \max_{g \in G(x,v)} \Gamma(V_s(g)) \mu(dv) &\leq \Gamma(V_s(x)) - \rho(x) + \mu \mathbb{I}_{\mathcal{O}}(x), \quad \forall x \in D. \end{aligned}$$

The proof of Theorem 6.4 is now complete. A summary of the results on global recurrence of open, bounded sets for (6.1) is in Figure 6.1.

■

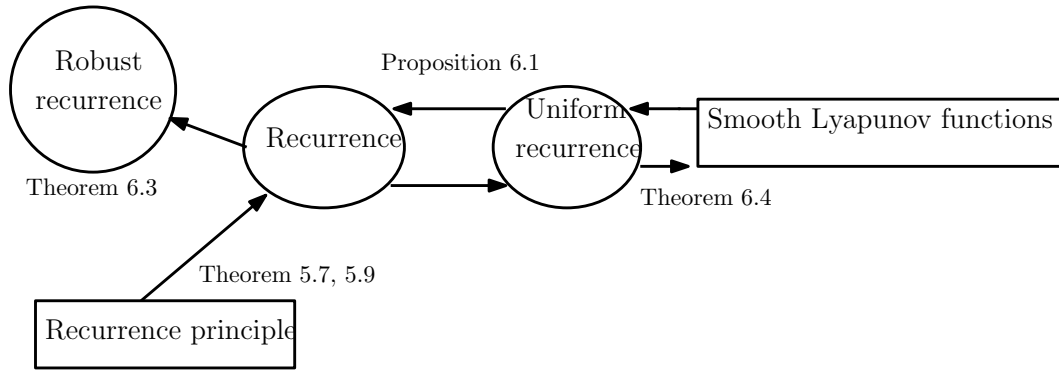


Figure 6.1: Summary of results for recurrence in (6.1)

# Chapter 7

## Conclusions

In this chapter, we summarize the main contributions of the dissertation and point out future research directions.

### 7.1 Summary

In Chapter 2, we studied hybrid systems modeled by set-valued mappings and presented a Lyapunov function characterization for a property called recurrence. In particular, under mild regularity properties for the system we establish that the existence of a smooth Lyapunov function that decreases strictly along solutions outside an open, bounded set is a necessary and a sufficient condition for recurrence of that set. Robustness of the recurrence property to various state dependent perturbations is a key result that aids the development of the converse theorem.

In Chapter 3, we introduced a class of systems called stochastic difference inclusions and extend the results of Chapter 2 to stochastic systems. We present a solution concept for stochastic difference inclusions, establish Lyapunov function based sufficient conditions, weak sufficient conditions, converse Lyapunov theorems and robust stability

conditions for recurrence of an open, bounded set. Similar results for asymptotic stability in probability are also discussed.

In Chapter 4, constrained discrete-time stochastic systems stabilized by discontinuous feedback laws are studied. In particular, robustness of asymptotic stability in probability for the closed loop stochastic system is analyzed. Since robustness results from Chapter 3 are not necessarily applicable in this scenario, we establish a Lyapunov function based approach to verify robustness as opposed to asserting robustness from system regularity properties.

In Chapters 5, we study a class of stochastic hybrid systems modeled by set-valued mappings where the randomness is restricted only to the discrete-time dynamics. We introduce the concept of weakly totally recurrent in probability sets and establish convergence of bounded random solutions to such sets. An extension of the result under the existence of a non-increasing on average Lyapunov-like function is also presented and convergence of sample paths of the random solution to weakly totally recurrent in probability sets inside level sets of the Lyapunov-like function is established. Application of the results to establishing weak sufficient conditions for asymptotic stability in probability and recurrence are also discussed.

Chapter 6 extends the results of Chapters 2-3 to a larger class of stochastic hybrid systems studied in Chapter 5. In particular, we study the recurrence property in detail and establish robustness results and a converse Lyapunov theorem.

## 7.2 Future directions

We now present possible research directions that expand upon the work in this dissertation.

- A closely related property to recurrence is called positive recurrence. For non-

stochastic systems, the recurrence property in Chapter 2 is equivalent to positive recurrence but for stochastic systems they are not equivalent. It is a stronger property than recurrence since positive recurrence also requires the expected value of the time for solutions to hit the set be finite. Robustness of positive recurrence and equivalence of positive recurrence to the existence of smooth Lyapunov functions are problems that need to be explored further. Results on robustness of global asymptotic stability in probability and an associated converse Lyapunov theorem for the class of systems studied in Chapters 5-6 also need to be established. Similarly, analysis tools related to other asymptotic stability notions such as mean square asymptotic stability and mean square exponential stability need to be established to develop a more complete stability theory for set-valued stochastic systems.

- In this dissertation we have not investigated robustness of stability properties with respect to uncertainties in the probability distribution. For example, consider the system  $x^+ = \max\{0, x+v\}$  with  $v \in \{-1, 1\}$ ,  $x \in \mathbb{Z}_{\geq 0}$  and  $\mu(\{-1\}) = \mu(\{1\}) = 0.5$ . It can be observed that  $V(x) = |x|$  is a weak-Lyapunov function for the set  $(-1, 1)$  and using the invariance principle from Chapter 2, it can be concluded that the set  $(-1, 1)$  is globally recurrent. However, even for arbitrarily small  $\delta \in (0, 0.5)$ , if the distribution function of  $v$  is modified to  $\mu(\{-1\}) = 0.5 - \delta$ ,  $\mu(\{1\}) = 0.5 + \delta$ , the set  $(-1, 1)$  is no longer recurrent. Results related to robustness of stability with respect to uncertainties in the probability distribution would be useful in the analysis of networked control systems where exact statistical information might not be available.

In [89], [90] results regarding robustness of positive recurrence in Markov chains to perturbations in the transition probabilities are established. For a particular class of stochastic hybrid systems like Markov jump linear systems, results on robustness of

stability to perturbations in the transition rate matrices or transition probability matrices are available in [91], [92], [93], [94] and [95]. However, the proofs rely on a Lyapunov function assumption for the nominal system. A general robustness result with respect to uncertainties in probability distributions for stochastic hybrid systems without relying on Lyapunov function based assumption for the nominal system needs to be explored further.

- Finally, an extension of the results in Chapters 3-6 to the general class of stochastic hybrid systems studied in [82] and modeled by

$$dx \in F(x)dt + B(x)dW, x \in C$$

$$x^+ \in G(x, v), x \in D$$

is a natural step towards understanding stability theory for complex systems. Results related to sequential compactness for stochastic hybrid systems in [82] will likely be crucial to establishing equivalence between uniform and non uniform versions of stability, robustness of stability, invariance principle and converse Lyapunov theorems.

# Appendix A

## Mathematical review

- Let  $I \subset \mathbb{R}$  be an interval. A function  $\phi : I \mapsto \mathbb{R}$  is absolutely continuous on  $I$ , if for every  $\varepsilon > 0$ , there exists a  $\delta > 0$  such that  $\sum_{k=1}^l |\phi(b_k) - \phi(a_k)| \leq \varepsilon$  for every finite number of non-overlapping intervals  $(a_k, b_k)$ ,  $k \in \{1, \dots, l\}$  with  $[a_k, b_k] \subset I$  and  $\sum_{k=1}^l |b_k - a_k| \leq \delta$ . The function  $\phi$  is locally absolutely continuous if it is absolutely continuous on every interval  $[a, b] \subset I$ .
- A set-valued mapping  $M : \mathbb{R}^p \rightrightarrows \mathbb{R}^n$  is *outer semicontinuous* if, for each  $(x_i, y_i) \rightarrow (x, y) \in \mathbb{R}^p \times \mathbb{R}^n$  satisfying  $y_i \in M(x_i)$  for all  $i \in \mathbb{Z}_{\geq 0}$ ,  $y \in M(x)$ .
- A mapping  $M$  is *locally bounded* if, for each bounded set  $K \subset \mathbb{R}^p$ ,  $M(K) := \bigcup_{x \in K} M(x)$  is bounded.
- Let  $\mathcal{T}$  be a topological space. A function  $\Psi : \mathcal{T} \rightarrow \mathbb{R}_{\geq 0}$  is *upper semicontinuous* if for every sequence  $\{t_i\}_{i=0}^{\infty}$  such that  $t_i \rightarrow t$ , we have  $\limsup_{i \rightarrow \infty} \Psi(t_i) \leq \Psi(t)$ . A function  $\kappa : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$  is *lower semicontinuous* if for every converging sequence  $\{x_i\} \rightarrow x$ ,  $\liminf_{i \rightarrow \infty} \kappa(x_i) \geq \kappa(x)$ .
- For a measurable space  $(\Omega, \mathcal{F})$ , a mapping  $M : \Omega \rightrightarrows \mathbb{R}^n$  is *measurable* [84, Def. 14.1], if for each open set  $\mathcal{O} \subset \mathbb{R}^n$ , the set  $M^{-1}(\mathcal{O}) := \{\omega \in \Omega : M(\omega) \cap \mathcal{O} \neq \emptyset\} \in \mathcal{F}$ .

$\mathcal{F}$ . A measurable function  $\mathbf{T} : \Omega \rightarrow \mathbb{Z}_{\geq 0} \cup \{\infty\}$  is a *stopping time* [84, § 11.3, Def. 5] with respect to the filtration  $\{\mathcal{F}_n\}_{n \in \mathbb{Z}_{\geq 0}}$  if the event  $\{\mathbf{T} \leq k\}$  (or equivalently the event  $\{\mathbf{T} = k\} \in \mathcal{F}_k$  for every  $k \in \mathbb{Z}_{\geq 0}$ ).

- A set  $F \subset \mathbb{R}^m$  is measurable if  $F \in \mathbf{B}(\mathbb{R}^m)$
- A function  $\alpha : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  is of class- $\mathcal{K}$  if it is continuous, strictly increasing and  $\alpha(0) = 0$ .
- A function  $\alpha : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  is of class- $\mathcal{K}_{\infty}$  if it is of class- $\mathcal{K}$  and unbounded.
- A function  $\psi : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{R}_{> 0}$  is of class  $\mathcal{L}$  if it is non-increasing and  $\lim_{\ell \rightarrow \infty} \psi(\ell) = 0$ .
- Fatou's lemma: Let  $\{f_i\}_{i=1}^{\infty}$  be a sequence of measurable functions defined on a measure space  $(S, \Sigma, \mu)$ . If there exists an integrable function  $g$  such that  $f_i \leq g$  for all  $i \in \mathbb{Z}_{\geq 1}$ , then

$$\limsup_{i \rightarrow \infty} \int_S f_i d\mu \leq \int_S \limsup_{i \rightarrow \infty} f_i d\mu.$$

- Monotone convergence theorem: Let  $\{f_i\}_{i=1}^{\infty}$  be a sequence of pointwise non-decreasing measurable functions defined on a measure space  $(S, \Sigma, \mu)$ . Then,

$$\lim_{i \rightarrow \infty} \int_S f_i d\mu = \int_S \lim_{i \rightarrow \infty} f_i d\mu.$$

- Dominated convergence theorem: Let  $\{f_i\}_{i=1}^{\infty}$  be a sequence of measurable functions defined on a measure space  $(S, \Sigma, \mu)$ . If there exists an integrable function  $g$  such that  $f_i \leq g$  for all  $i \in \mathbb{Z}_{\geq 1}$ , then

$$\lim_{i \rightarrow \infty} \int_S f_i d\mu = \int_S \lim_{i \rightarrow \infty} f_i d\mu.$$



- Jensen's inequality: If  $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$  is a convex function and  $\mathbf{x} : \Omega \rightarrow \mathbb{R}^n$  is a random variable, then

$$\phi(\mathbb{E}[\mathbf{x}]) \leq \mathbb{E}[\phi(\mathbf{x})].$$

If  $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$  is a concave function, then

$$\mathbb{E}[\phi(\mathbf{x})] \leq \phi(\mathbb{E}[\mathbf{x}]).$$

- Inf convolution: Let  $\rho : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$  be bounded away from zero on compact subsets of  $\mathbb{R}^n$ . Then, define the function

$$\tilde{\rho}(x) := \inf_{z \in \mathbb{R}^n} (\rho(z) + |x - z|), x \in \mathbb{R}^n$$

The function  $\tilde{\rho} : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$  is bounded away from zero on compact subsets of  $\mathbb{R}^n$  and Lipschitz on  $\mathbb{R}^n$ .

- Set convergence: Let  $\{S_i\}_{i=1}^{\infty}$  be a sequence of sets in  $\mathbb{R}^n$ .
  1. The inner limit of the sequence  $\{S_i\}_{i=1}^{\infty}$  is the set of all points  $x \in \mathbb{R}^n$  such that, there exists points  $x_i \in S_i, i = 1, 2, \dots$ , such that  $\lim_{i \rightarrow \infty} x_i = x$ .
  2. The outer limit of the sequence  $\{S_i\}_{i=1}^{\infty}$  is the set of all points  $x \in \mathbb{R}^n$  for which there exists a subsequence  $\{S_{i_k}\}_{k=1}^{\infty}$  of  $\{S_i\}_{i=1}^{\infty}$  and points  $x_k \in S_{i_k}, k = 1, 2, \dots$ , such that  $\lim_{i \rightarrow \infty} x_k = x$ .

When the inner limit and the outer limit of the sequence  $\{S_i\}_{i=1}^{\infty}$  are equal, the sequence  $\{S_i\}_{i=1}^{\infty}$  is convergent, and its limit is given by

$$\liminf_{i \rightarrow \infty} S_i = \limsup_{i \rightarrow \infty} S_i = \lim_{i \rightarrow \infty} S_i.$$

# Appendix B

## Sequential compactness: Hybrid systems and stochastic hybrid systems

In this section, we state results on sequential compactness for hybrid systems studied in Chapter 2 and stochastic hybrid systems studied in Chapter 5-6. A metric space is sequentially compact if every sequence has a converging subsequence which converges to a point in the metric space. In this section, we state results related to sequential compactness for the solution space of hybrid systems and stochastic hybrid systems.

### B.1 Hybrid systems

We briefly recall the framework for modeling hybrid systems and the Standing Assumptions imposed in Chapter 2. A hybrid system with a state  $x \in \mathbb{R}^n$  is written formally

as

$$\dot{x} \in F(x), \quad x \in C \tag{B.1a}$$

$$x^+ \in G(x), \quad x \in D \tag{B.1b}$$

In this section we will assume that the conditions of Standing Assumption 2.1 are satisfied by the data of the hybrid system. The conditions of Standing Assumption 2.1 are stated below.

**Assumption B.1** *The data of the hybrid system (B.1) satisfies the following conditions:*

1. *The sets  $C, D \subset \mathbb{R}^n$  are closed.*
2. *The mapping  $F$  is outer semicontinuous, locally bounded, convex valued and non-empty on  $C$ .*
3. *The mapping  $G$  is outer semicontinuous, locally bounded and non-empty on  $D$ .*

A sequence of hybrid arcs  $\{\phi_i\}_{i=1}^\infty$  converges if the sequence of sets  $\{\text{graph}(\phi_i)\}_{i=1}^\infty$  converges in the sense of set convergence. A sequence of solutions  $\{\phi_i\}_{i=1}^\infty$  for the hybrid system (B.1) is said to be locally eventually bounded if for every  $\tau \geq 0$ , there exists  $i^*$  and  $M > 0$  such that for all  $i \geq i^*$  and all  $(t, j) \in \text{dom}(\phi_i)$  with  $t + j \leq \tau$ ,  $\phi_i(t, j) \in M\mathbb{B}$ .

We now state the main result related to sequential compactness for (B.1). The following result is from [14, Theorem 6.8].

**Theorem B.1** *Let the hybrid system (B.1) satisfy the conditions of Assumption B.1. For every locally eventually bounded sequence  $\{\phi_i\}_{i=1}^\infty$  of hybrid arcs generated by (B.1), there exists a subsequence  $\{\phi_{i_k}\}_{k=1}^\infty$  which converges graphically to a hybrid arc  $\phi$  generated by (B.1).*

## B.2 Stochastic hybrid systems

We briefly recall the framework for modeling stochastic hybrid system and the Standing Assumptions imposed in Chapter 5. A stochastic hybrid system with a state  $x \in \mathbb{R}^n$  and random input  $v \in \mathbb{R}^m$  is written formally as

$$\dot{x} \in F(x), x \in C \tag{B.2a}$$

$$x^+ \in G(x, v^+), x \in D \tag{B.2b}$$

$$v \sim \mu(\cdot) \tag{B.2c}$$

In this section we will assume that the conditions of Standing Assumption 5.1 are satisfied by the data of the stochastic hybrid system. The conditions of Standing Assumption 5.1 are stated below.

**Assumption B.2** *The data of the stochastic hybrid system  $\mathcal{H}$  satisfies the following conditions:*

1. *The sets  $C, D \subset \mathbb{R}^n$  are closed;*
2. *The mapping  $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$  is outer-semicontinuous, locally bounded with nonempty convex values on  $C$ ;*
3. *The mapping  $G : \mathbb{R}^n \times \mathbb{R}^m \rightrightarrows \mathbb{R}^n$  is locally bounded and the mapping  $v \mapsto \text{graph}(G(\cdot, v)) := \{(x, y) \in \mathbb{R}^{2n} : y \in G(x, v)\}$  is measurable with closed values.*

We recall that the set of hybrid arcs with closed graphs can be thought of as a subset in the space of not-identically empty-valued outer semicontinuous set-valued mappings from  $\mathbb{R}^2$  to  $\mathbb{R}^n$ . It follows from [84, Theorem 5.50], equipped with the metric of graph distance, this space is a separable, locally compact, complete (and  $\sigma$ -compact) metric space, which we denote  $(\mathcal{X}, \mathbf{d})$ . For each  $j \in \mathbb{Z}_{\geq 0}$  we define  $E_j := \mathbb{R}_{\geq 0} \times \{0, \dots, j\}$  and

for each hybrid arc  $\phi$ , we define  $\phi|_{E_j}$  to be the hybrid arc with domain  $\text{dom}(\phi) \cap E_j$  such that  $\phi|_{E_j}(t, j) = \phi(t, j)$  for  $(t, j) \in \text{dom}(\phi|_{E_j})$ . For each  $S \in \mathcal{X}$ , we use  $S|_{E_j}$  to refer to the outer semicontinuous set-valued mapping from  $\mathbb{R}^2$  to  $\mathbb{R}^n$  with domain  $\text{dom}(S) \cap E_j$  such that  $S|_{E_j}(t, j) = S(t, j)$  for  $(t, j) \in \text{dom}(S|_{E_j})$ .

A sequence of random solutions  $\{\mathbf{x}_i\}_{i=0}^\infty$  for the stochastic hybrid system (B.2) is said to be almost surely locally eventually bounded if the sequence  $\mathbf{x}_i(\omega)$  is locally eventually bounded for almost every  $\omega \in \Omega$ . As noted in [85], in order for sequential compactness results for stochastic hybrid system to be useful in the context of developing a robust stability theory we also need to characterize and relate the statistical properties of the sequence  $\mathbf{x}_i$  and the limiting solution. Hence, we impose the following assumption related to the functions that characterize the behavior of the random solutions.

**Assumption B.3** *The functions  $\varphi, \varphi_i : \mathcal{X} \rightarrow \mathbb{R}_{\geq 0}, i \in \mathbb{Z}_{\geq 0}$  are upper semicontinuous, bounded and for each  $\varepsilon > 0$ , there exists  $i^*, j^* \in \mathbb{Z}_{\geq 0}$  such that for each  $i \geq i^*$  and each hybrid arc  $\phi$ ,*

$$\varphi_i(\phi) \leq \varphi_i(\phi|_{E_j}) + \varepsilon, \forall j \geq j^* \quad (\text{B.3})$$

and, for each unbounded  $\mathcal{N} \subset \mathbb{Z}_{\geq 0}$  and each sequence of locally eventually bounded hybrid arcs  $\{S_i\}_{i=1}^\infty$  and each  $j \geq j^*$ ,

$$\lim_{i \rightarrow \infty, i \in \mathcal{N}} S_i = S \implies \limsup_{i \rightarrow \infty, i \in \mathcal{N}} \varphi_i(S_i|_{E_j}) \leq \varphi(S|_{E_j}). \quad (\text{B.4})$$

We now state the main result related to sequential compactness for (B.2). The following result is from [85, Theorem 1].

**Theorem B.2** *Let the SHS (B.2) satisfy the conditions of Assumption B.2 and let Assumption B.3 hold. Let  $\{\mathbf{x}_i\}_{i=1}^\infty$  generated by (B.2) be an almost surely locally eventually*

bounded sequence of solutions, let  $\{\Delta_i\}_{i=0}^{\infty}$  be a sequence of non-negative real numbers such that  $\mathbb{E}[\varphi_i(\mathbf{x}_i)] \geq \Delta_i$  for all  $i \in \mathbb{Z}_{\geq 0}$ . Then, there exists a solution  $\mathbf{x}$  generated by (B.2) in the pointwise outerlimit of the sequence  $\mathbf{x}_i$  such that  $\mathbb{E}[\varphi(\mathbf{x})] \geq \limsup_{i \rightarrow \infty} \Delta_i$ .

# Appendix C

## Stochastic stability properties

In this section, we will give definitions of stochastic stability properties that are frequently studied in the literature. We state the definitions for the simpler class of stochastic difference inclusions

$$x^+ \in G(x, v), x \in \mathbb{R}^n \quad (\text{C.1})$$

studied in Chapter 3. We refer the reader to [27] for equivalent definitions in the case of stochastic hybrid systems. For completeness we also include the definitions of recurrence and global asymptotic stability in probability.

**Definition C.1** *An open, bounded set  $\mathcal{O} \subset \mathbb{R}^n$  is said to be globally recurrent for (C.1) if for every  $x \in \mathbb{R}^n$  and  $\mathbf{x} \in \mathcal{S}_r(x)$ ,  $\mathbb{E} \left[ \prod_{i \in \mathbb{Z}_{\geq 1}} \mathbb{I}_{\mathbb{R}^n \setminus \mathcal{O}}(\mathbf{x}_i) \right] = 0$ .*

**Definition C.2** *An open, bounded set  $\mathcal{O} \subset \mathbb{R}^n$  is said to be uniformly globally recurrent for (C.1) if for every compact set  $K \subset \mathbb{R}^n$  and  $\rho > 0$  there exists  $J \in \mathbb{Z}_{\geq 1}$  such that  $\mathbb{E} \left[ \prod_{i=1}^J \mathbb{I}_{\mathbb{R}^n \setminus \mathcal{O}}(\mathbf{x}_i) \right] \leq \rho$  for every  $\mathbf{x} \in \mathcal{S}_r(K)$ .*

**Definition C.3** *An open, bounded set  $\mathcal{O} \subset \mathbb{R}^n$  is globally positively recurrent for (C.1) if for every  $\mathbf{x} \in \mathcal{S}_r(\mathbb{R}^n)$ ,  $\mathbb{E}[\inf\{k \in \mathbb{Z}_{\geq 0}, \mathbf{x}_k \in \mathcal{O}\}] < \infty$ .*

**Definition C.4** An open, bounded set  $\mathcal{O} \subset \mathbb{R}^n$  is uniformly globally positively recurrent for (C.1) if for every compact set  $K \subset \mathbb{R}^n$ , there exists  $M > 0$  such that for all  $\mathbf{x} \in \mathcal{S}_r(K)$ ,  $\mathbb{E}[\inf\{k \in \mathbb{Z}_{\geq 0}, \mathbf{x}_k \in \mathcal{O}\}] \leq M$ .

**Definition C.5** A compact set  $\mathcal{A} \subset \mathbb{R}^n$  is globally exponentially stable in the  $p$ th mean for (C.1) if there exists  $\lambda \in [0, 1)$  and  $\gamma > 0$  such that for every  $\mathbf{x} \in \mathcal{S}_r(x)$   $\mathbb{E}[|\mathbf{x}_k|_{\mathcal{A}}^p] \leq \gamma \lambda^k |x|_{\mathcal{A}}^p$ .

**Definition C.6** A compact set  $\mathcal{A} \subset \mathbb{R}^n$  is globally asymptotically stable in the  $p$ th mean for (C.1) if

1.  $\lim_{i \rightarrow \infty} \sup_{k \in \mathbb{Z}_{\geq 0}} \mathbb{E}[|\mathbf{x}_k|_{\mathcal{A}}^p] = 0$  for each sequence  $\mathbf{x}_i \in \mathcal{S}_r(x_i)$  and each bounded sequence  $x_i$  satisfying  $\lim_{i \rightarrow \infty} |x_i|_{\mathcal{A}} = 0$ .
2.  $\sup_{k \in \mathbb{Z}_{\geq 0}} \mathbb{E}[|\mathbf{x}_k|_{\mathcal{A}}^p] < \infty$  for each  $\mathbf{x} \in \mathcal{S}_r(\mathbb{R}^n)$ .
3.  $\lim_{k \rightarrow \infty} \mathbb{E}[|\mathbf{x}_k|_{\mathcal{A}}^p] = 0$  for each  $\mathbf{x} \in \mathcal{S}_r(\mathbb{R}^n)$ .

**Definition C.7** A compact set  $\mathcal{A} \subset \mathbb{R}^n$  is uniformly globally asymptotically stable in the  $p$ th mean for (C.1) if

1. For every  $\varepsilon > 0$  there exists a  $\delta > 0$  such that for all  $\mathbf{x} \in \mathcal{S}_r(\mathcal{A} + \delta\mathbb{B})$ ,  $\mathbb{E}[|\mathbf{x}_k|_{\mathcal{A}}^p] \leq \delta$ ,  $\forall k \in \mathbb{Z}_{\geq 0}$ .
2. For every  $\delta > 0$  there exists a  $\varepsilon > 0$  such that for all  $\mathbf{x} \in \mathcal{S}_r(\mathcal{A} + \delta\mathbb{B})$ ,  $\mathbb{E}[|\mathbf{x}_k|_{\mathcal{A}}^p] \leq \delta$ ,  $\forall k \in \mathbb{Z}_{\geq 0}$ .
3. For every  $\Delta > 0$ ,  $\delta > 0$  there exists  $J \in \mathbb{Z}_{\geq 0}$  such that for every  $\mathbf{x} \in \mathcal{S}_r(\mathcal{A} + \Delta\mathbb{B})$ ,  $\mathbb{E}[|\mathbf{x}_k|_{\mathcal{A}}^p] \leq \delta$ ,  $\forall k \in \mathbb{Z}_{\geq J}$ .

**Definition C.8** A compact set  $\mathcal{A} \subset \mathbb{R}^n$  is globally asymptotically stable in probability for (C.1) if



1. For every  $\varepsilon > 0$  and  $\rho > 0$  there exists a  $\delta > 0$  such that for all  $\mathbf{x} \in \mathcal{S}_r(\mathcal{A} + \delta\mathbb{B})$ ,

$$\mathbb{P}(\text{graph}(\mathbf{x}) \subset (\mathbb{Z}_{\geq 0} \times (\mathcal{A} + \varepsilon\mathbb{B}^o))) \geq 1 - \rho.$$

2. For every  $\mathbf{x} \in \mathcal{S}_r(\mathbb{R}^n)$ ,  $\lim_{i \rightarrow \infty} |\mathbf{x}_i(\omega)|_{\mathcal{A}} = 0$  for almost every  $\omega \in \Omega$ .

**Definition C.9** A compact set  $\mathcal{A} \subset \mathbb{R}^n$  is uniformly globally asymptotically stable in probability for (C.1) if

1. For every  $\varepsilon > 0$  and  $\rho > 0$  there exists a  $\delta > 0$  such that for all  $\mathbf{x} \in \mathcal{S}_r(\mathcal{A} + \delta\mathbb{B})$ ,

$$\mathbb{P}(\text{graph}(\mathbf{x}) \subset (\mathbb{Z}_{\geq 0} \times (\mathcal{A} + \varepsilon\mathbb{B}^o))) \geq 1 - \rho.$$

2. For every  $\delta > 0$  and  $\rho > 0$  there exists a  $\varepsilon > 0$  such that for all  $\mathbf{x} \in \mathcal{S}_r(\mathcal{A} + \delta\mathbb{B})$ ,

$$\mathbb{P}(\text{graph}(\mathbf{x}) \subset (\mathbb{Z}_{\geq 0} \times (\mathcal{A} + \varepsilon\mathbb{B}^o))) \geq 1 - \rho.$$

3. For every  $\Delta > 0$ ,  $\delta > 0$  and  $\rho > 0$ , there exists  $J \in \mathbb{Z}_{\geq 0}$  such that for every  $\mathbf{x} \in \mathcal{S}_r(\mathcal{A} + \Delta\mathbb{B})$ ,

$$\mathbb{P}(\text{graph}(\mathbf{x}) \cap (\mathbb{Z}_{\geq J} \times \mathbb{R}^n) \subset (\mathbb{Z}_{\geq 0} \times (\mathcal{A} + \delta\mathbb{B}))) \geq 1 - \rho.$$

# Appendix D

## Proofs

### D.1 Proof of Proposition 2.4

Since  $\mathcal{O}$  is recurrent for  $\mathcal{H}$  in (2.2), it follows from Lemma 2.2 that  $\mathcal{O}$  is recurrent for the modified system  $\widehat{\mathcal{H}}$  in (2.5). From Proposition 2.2 it follows that solutions of  $\widehat{\mathcal{H}}$  are ultimately bounded with ultimate bound  $M$ . Let  $S = (M+1)\mathbb{B}$ . Then, there exists  $T > 0$  such that for  $\phi \in \mathcal{S}_{\widehat{\mathcal{H}}}(S)$  if  $(t, j) \in \text{dom } \phi$  satisfies  $t + j \geq T$ , then  $\phi(t, j) \in M\mathbb{B}$ . Then, it follows that  $\Omega(S) \subset M\mathbb{B} \subset \text{int}(S)$ . Since the maximal solutions of  $\widehat{\mathcal{H}}$  are complete  $\Omega(S)$  is non-empty. Then, from [14, Corollary 7.7], it follows that  $\Omega(S)$  is compact and asymptotically stable with basin of attraction  $S$ . Since every solution eventually enters the set  $S$ , it follows that  $\Omega(S)$  is uniformly globally asymptotically stable for  $\widehat{\mathcal{H}}$ . Since the solutions of  $\mathcal{H}$  are also solutions of  $\widehat{\mathcal{H}}$ , it follows that  $\mathcal{A} := \Omega(S)$  is UGAS for  $\mathcal{H}$ .

### D.2 Proof of Theorem 3.1

We denote the weak viability probabilities for (3.1) as  $m_{G,CS}(i, x)$  for  $(i, x) \in \mathbb{Z}_{\geq 0} \times \mathbb{R}^n$ .

**Proposition D.1** *Let  $\mathcal{O} \subset \mathbb{R}^n$  be an open, bounded set. If there exists a sufficient recurrence-Lyapunov function for (3.1) relative to  $\mathcal{O} \subset \mathbb{R}^n$  then, for every  $R > 0$  such that  $\mathcal{O} \subset R\mathbb{B}$  and for all  $x \in R\mathbb{B} \setminus \mathcal{O}$ ,  $\widehat{m}_{G, \subset R\mathbb{B} \setminus \mathcal{O}}(x) = 0$ .*

*Proof:* Define  $S_1 := R\mathbb{B} \setminus \mathcal{O}$ . Let  $V : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$  be a sufficient recurrence-Lyapunov function for (3.1) relative to  $\mathcal{O}$  and  $\varrho : \mathbb{R}^n \rightarrow \mathbb{R}_{> 0}$  continuous, satisfy (3.7). We begin by proving that for every  $\sigma^* > 0$  there exists a function  $c \in \mathcal{L}$  such that for all  $(i, x) \in \mathbb{Z}_{\geq 0} \times R\mathbb{B} \setminus \mathcal{O}$ ,

$$m_{G, \subset S_1}(i, x) \leq \sigma^* V(x) + c(i). \quad (\text{D.1})$$

Let  $c(0) \geq 1$ . Then, the bounds holds when  $i = 0$  for all  $x \in R\mathbb{B} \setminus \mathcal{O}$  from the definition. Now assume that the bound holds for some  $i \in \mathbb{Z}_{\geq 0}$  and for all  $x \in R\mathbb{B} \setminus \mathcal{O}$ . Now pick  $c$  such that for all  $i \in \mathbb{Z}_{\geq 0}$ ,

$$c(i) = \max\{0, c(0) - i\sigma^* \inf_{x \in R\mathbb{B} \setminus \mathcal{O}} \varrho(x)\}.$$

This choice gives a function that belongs to class- $\mathcal{L}$  as  $\inf_{x \in R\mathbb{B} \setminus \mathcal{O}} \varrho(x) > 0$  due to the compactness of  $R\mathbb{B} \setminus \mathcal{O}$  and continuity of  $\varrho$ . Since  $x \in R\mathbb{B} \setminus \mathcal{O}$ , it follows from (D.1) that

$$\begin{aligned} m_{G, \subset S_1}(i+1, x) &= \int_{\mathbb{R}^m} \max_{g \in G(x, v)} \mathbb{I}_{S_1}(g) m_{G, \subset S_1}(i, g) \mu(dv) \\ &\leq \sigma^* \left( \int_{\mathbb{R}^m} \max_{g \in G(x, v)} V(g) \mu(dv) \right) + c(i) \\ &\leq \sigma^* V(x) - \sigma^* \varrho(x) + c(i) \\ &\leq \sigma^* V(x) + c(i+1). \end{aligned}$$

Then (D.1) holds by induction. Now for  $x \in R\mathbb{B} \setminus \mathcal{O}$  if  $\widehat{m}_{G, \subset S_1}(x) = \varepsilon > 0$ , choose  $\sigma^*$  such that  $\sigma^* V(x) \leq \frac{1}{2}\varepsilon$ . Since  $c \in \mathcal{L}$  it follows from (D.1) that  $\varepsilon \leq \sigma^* V(x) \leq \frac{1}{2}\varepsilon$ . This

is a contradiction which implies  $\varepsilon = 0$ . Then, it follows that  $\widehat{m}_{G, \subset R\mathbb{B} \setminus \mathcal{O}}(x) = 0$  for all  $x \in R\mathbb{B} \setminus \mathcal{O}$ . ■

We now consider the set-valued mapping  $\widehat{G}$  defined as

$$\widehat{G}(x, v) := \begin{cases} G(x, v) & , (x, v) \in \mathbb{R}^n \setminus \mathcal{O} \times \mathbb{R}^m \\ \emptyset & , (x, v) \in \mathcal{O} \times \mathbb{R}^m. \end{cases} \quad (\text{D.2})$$

It can be noted from the definition that  $\widehat{G}(x, v)$  satisfies the conditions of Standing Assumption 3.1. Since  $\widehat{G}(x, v) \subseteq G(x, v)$  for all  $(x, v) \in \mathbb{R}^n \times \mathbb{R}^m$  it follows that if  $V$  is a sufficient recurrence-Lyapunov function for (3.1) relative to  $\mathcal{O} \subset \mathbb{R}^n$  then, for all  $x \in \mathbb{R}^n$ ,

$$\int_{\mathbb{R}^m} \max_{g \in \widehat{G}(x, v)} V(g) \mu(dv) \leq V(x). \quad (\text{D.3})$$

For the system  $x^+ \in \widehat{G}(x, v)$ , we denote the weak reachability probabilities for a closed set  $S \subset \mathbb{R}^n$  as  $m_{\widehat{G}, \cap S}(i, x)$  for  $(i, x) \in \mathbb{Z}_{\geq 0} \times \mathbb{R}^n$ . The proof of the following result is similar to [61, Thm. 1].

**Proposition D.2** *If there exists a function that satisfies (D.3) for  $x^+ \in \widehat{G}(x, v)$  then, for every  $x \in \mathbb{R}^n$  and  $\gamma \in (0, 1)$  there exists a  $R > 0$  such that  $\lim_{i \rightarrow \infty} m_{\widehat{G}, \cap \mathbb{R}^n \setminus R\mathbb{B}^o}(i, x) \leq \gamma$ .*

*Proof:* Let  $V$  satisfy (D.3). Since  $V$  is radially unbounded, there exists  $\alpha_1 \in \mathcal{K}_\infty, c_1 > 0$  such that  $\alpha_1(|x|) \leq V(x) + c_1$ . Define  $S_2 := \mathbb{R}^n \setminus R\mathbb{B}^o$ . To prove the above statement we first establish that for all  $x \in \mathbb{R}^n$  and  $i \in \mathbb{Z}_{\geq 0}$ ,

$$\alpha_1(R) m_{\widehat{G}, \cap S_2}(i, x) \leq V(x) + c_1.$$

The bound holds for  $i = 0$  by definition for all  $x \in \mathbb{R}^n$ . Now assume that the bound holds for some  $i \in \mathbb{Z}_{\geq 0}$  and all  $x \in \mathbb{R}^n$ . Then, from the bound  $\alpha_1(|x|) \leq V(x) + c_1$  it

follows that

$$\begin{aligned}
 \alpha_1(R)m_{\widehat{G},\cap S_2}(i+1,x) &= \int_{\mathbb{R}^m} \max_{g \in \widehat{G}(x,v)} \max \{ \alpha_1(R)\mathbb{I}_{S_2}(g), \alpha_1(R)m_{\widehat{G},\cap S_2}(i,g) \} \mu(dv) \\
 &\leq \int_{\mathbb{R}^m} \max_{g \in \widehat{G}(x,v)} \max \{ V(g) + c_1, V(g) + c_1 \} \mu(dv) \\
 &= \int_{\mathbb{R}^m} \max_{g \in \widehat{G}(x,v)} V(g) \mu(dv) + c_1 \leq V(x) + c_1.
 \end{aligned}$$

Then, the result follows by induction. This bound implies that for all  $(i,x) \in \mathbb{Z}_{\geq 0} \times \mathbb{R}^n$ ,

$$m_{\widehat{G},\cap S_2}(i,x) \leq \frac{1}{\alpha_1(R)}(V(x) + c_1).$$

So given  $x \in \mathbb{R}^n$ , we can choose  $R > 0$  such that  $\frac{1}{\alpha_1(R)}(V(x) + c_1) \leq \gamma$ . Then  $\lim_{i \rightarrow \infty} m_{\widehat{G},\cap \mathbb{R}^n \setminus R\mathbb{B}^\circ}(i,x) \leq \gamma$ . ■

Let  $\gamma \in (0,1)$ . Now given  $x \in \mathbb{R}^n \setminus \mathcal{O}$ , it follows from D.2 that there exists a  $R > 0$ , such that  $\mathcal{O} \subset R\mathbb{B}$ ,  $x \in R\mathbb{B}$  and  $\lim_{i \rightarrow \infty} m_{\widehat{G},\cap \mathbb{R}^n \setminus R\mathbb{B}^\circ}(i,x) \leq \gamma$ . Then, from 3.1 we have that for all  $i \in \mathbb{Z}_{\geq 0}$ ,

$$m_{\widehat{G},\subset \mathbb{R}^n \setminus \mathcal{O}}(i,x) \leq m_{\widehat{G},\subset R\mathbb{B} \setminus \mathcal{O}}(i,x) + m_{\widehat{G},\cap \mathbb{R}^n \setminus R\mathbb{B}^\circ}(i,x).$$

Since  $x \in \mathbb{R}^n \setminus \mathcal{O}$ , using 3.2 it follows that for all  $i \in \mathbb{Z}_{\geq 0}$ ,  $m_{\widehat{G},\subset \mathbb{R}^n \setminus \mathcal{O}}(i,x) = m_{G,\subset \mathbb{R}^n \setminus \mathcal{O}}(i,x)$  and  $m_{\widehat{G},\subset R\mathbb{B} \setminus \mathcal{O}}(i,x) = m_{G,\subset R\mathbb{B} \setminus \mathcal{O}}(i,x)$ . Then

$$m_{G,\subset \mathbb{R}^n \setminus \mathcal{O}}(i,x) \leq m_{G,\subset R\mathbb{B} \setminus \mathcal{O}}(i,x) + m_{\widehat{G},\cap \mathbb{R}^n \setminus R\mathbb{B}^\circ}(i,x).$$

Then, from Propositions D.1 and D.2 it follows that

$$\widehat{m}_{G,\subset \mathbb{R}^n \setminus \mathcal{O}}(x) \leq \widehat{m}_{G,\subset R\mathbb{B} \setminus \mathcal{O}}(x) + \lim_{i \rightarrow \infty} m_{\widehat{G},\cap \mathbb{R}^n \setminus R\mathbb{B}^\circ}(i,x) \leq \gamma.$$

Therefore, for every  $x \in \mathbb{R}^n \setminus \mathcal{O}$  we have  $\widehat{m}_{G, \mathbb{C}\mathbb{R}^n \setminus \mathcal{O}}(x) \leq \gamma$ . Then, from the definition of  $m_{G, \mathbb{C}\mathbb{R}^n \setminus \mathcal{O}}(i, x)$  for  $(i, x) \in \mathbb{Z}_{\geq 0} \times \mathbb{R}^n$ , we have that  $\widehat{m}_{G, \mathbb{C}\mathbb{R}^n \setminus \mathcal{O}}(x) \leq \gamma$  for all  $x \in \mathbb{R}^n$ . Finally, from Proposition 3.2 it follows that for all  $x \in \mathbb{R}^n$ ,  $\widehat{m}_{G, \mathbb{C}\mathbb{R}^n \setminus \mathcal{O}}(x) = 0$  and consequently globally recurrent of  $\mathcal{O}$  for (3.1) follows.

### D.3 Proof of Proposition 3.5

We first establish conditions under which the two notions of sufficient recurrence-Lyapunov functions and Lyapunov functions coincide. If  $V$  is a sufficient recurrence-Lyapunov function and

$$\sup_{x \in \mathcal{O}} \int_{\mathbb{R}^m} \max_{g \in G(x, v)} V(g) \mu(dv) \leq \lambda$$

for some  $\lambda > 0$ , then  $V/\lambda$  is both a sufficient recurrence-Lyapunov function and a Lyapunov function. The existence of such a  $\lambda$  is also guaranteed if  $\mu(\cdot)$  has compact support, due to the local boundedness of  $G$ .

If the conditions listed above are not satisfied, then the following proof explains the construction of a concave,  $\mathcal{K}_\infty$  function  $\kappa$  such that if  $V$  is a sufficient recurrence-Lyapunov function then,  $\kappa(V)$  is a Lyapunov function.

Since  $V$  is upper semicontinuous, it follows that it is locally bounded. Also from Standing Assumption 3.1, we have that  $G$  is locally bounded. Then, there exists  $\gamma \in \mathcal{K}_\infty$  and  $\gamma_0 \in \mathbb{R}_{\geq 0}$  such that, for all  $x \in \mathcal{O}$ ,

$$\max_{g \in G(x, v)} V(g) \leq \gamma(|v|) + \gamma_0.$$

Let  $\mathbb{B}$  be the closed unit ball in  $\mathbb{R}^m$ . Define  $F_i := 2^i \mathbb{B}$ ,  $F_{-1} := \emptyset$  and  $m(i) := \mu(F_i \setminus F_{i-1})$

for all  $i \in \mathbb{Z}_{\geq 0}$ . We now prove that there exists  $\bar{\tau} \in \mathcal{K}_\infty$  such that

$$\sum_{i=0}^{\infty} \bar{\tau}(i+1)m(i) \leq 1.$$

Since  $\mu$  is a measure it follows that for  $j \in \mathbb{Z}_{\geq 0}$ ,

$$\sum_{i=j}^{\infty} m(i) = \sum_{i=j}^{\infty} \mu(F_i \setminus F_{i-1}) = \mu\left(\bigcup_{i=j}^{\infty} (F_i \setminus F_{i-1})\right) = \mu(\mathbb{R}^m \setminus F_{j-1}).$$

Let  $\sigma \in \mathcal{L}$  be such that  $\mu(\mathbb{R}^m \setminus F_{j-1}) \leq \sigma(j)$  for all  $j \in \mathbb{Z}_{\geq 0}$ . Then  $\sum_{i=j}^{\infty} m(i) \leq \sigma(j)$ . Let  $\ell : \mathbb{Z}_{\geq 0} \rightarrow \mathbb{Z}_{\geq 0}$  be a strictly increasing unbounded mapping satisfying  $\ell(0) = 0$ ,  $\ell(1) > 1$  and  $\sigma(\ell(j)) \leq 2^{-j}$  for all  $j \in \mathbb{Z}_{\geq 0}$ . Let  $\tilde{\ell} \in \mathcal{K}_\infty$  satisfy  $\tilde{\ell}(i) = \ell(i)$  for each  $i \in \mathbb{Z}_{\geq 0}$ . Define  $\tau(s) := \tilde{\ell}^{-1}(s)$  for all  $s \geq 0$ . Then

$$\begin{aligned} \sum_{i=0}^{\infty} \tau(i+1)m(i) &= \sum_{j=0}^{\infty} \left( \sum_{i=\ell(j)}^{\ell(j+1)-1} \tau(i+1)m(i) \right) \leq \sum_{j=0}^{\infty} \left( \sum_{i=\ell(j)}^{\ell(j+1)-1} (j+1)m(i) \right) \\ &\leq \sum_{j=0}^{\infty} (j+1) \left( \sum_{i=\ell(j)}^{\infty} m(i) \right) \\ &\leq \sum_{j=0}^{\infty} (j+1) \sigma(\ell(j)) \leq \sum_{j=0}^{\infty} (j+1) 2^{-j} \leq 4. \end{aligned}$$

Now define  $\bar{\tau}(s) := \tau(s)/4$  for all  $s \geq 0$ . Then  $\sum_{i=0}^{\infty} \bar{\tau}(i+1)m(i) \leq 1$ . Let  $\hat{\kappa} \in \mathcal{K}_\infty$  be such that  $\hat{\kappa}(2\gamma(2^i)) \leq \bar{\tau}(i+1)$  for all  $i \in \mathbb{Z}_{\geq 0}$ . Then, it follows that

$$\begin{aligned} \int_{\mathbb{R}^m} \hat{\kappa}(2\gamma(|v|))\mu(dv) &= \sum_{i=0}^{\infty} \int_{F_i \setminus F_{i-1}} \hat{\kappa}(2\gamma(|v|))\mu(dv) \leq \sum_{i=0}^{\infty} \hat{\kappa}(2\gamma(2^i)) \int_{F_i \setminus F_{i-1}} \mu(dv) \\ &= \sum_{i=0}^{\infty} \hat{\kappa}(2\gamma(2^i))\mu(F_i \setminus F_{i-1}) \\ &\leq \sum_{i=0}^{\infty} \bar{\tau}(i+1)m(i) \leq 1. \end{aligned}$$

Define  $c := 2\gamma(1)$ . We now prove that there exists a function  $\bar{\kappa} \in \mathcal{K}_\infty$  that is concave and satisfies  $\bar{\kappa}(s) \leq \widehat{\kappa}(s)$  for all  $s \geq c$ . To prove the existence of  $\bar{\kappa}$ , we first prove the existence of a convex,  $\mathcal{K}_\infty$  function  $\alpha$  such that  $\widehat{\kappa}^{-1}(r) \leq \alpha(r)$  for all  $r \geq \tilde{c} := \widehat{\kappa}(c)$ . Now choose  $\beta \in \mathcal{K}_\infty$  such that  $\beta(s) \geq (2/\tilde{c})\widehat{\kappa}^{-1}(2s)$  and define  $\alpha(s) := \int_0^s \beta(t)dt$  for all  $s \geq 0$ . Then, by construction  $\alpha \in \mathcal{K}_\infty$  is convex and for  $r \geq \tilde{c}$ ,

$$\alpha(r) \geq \int_{r/2}^r \beta(t)dt \geq (r/2)\beta(r/2) \geq (\tilde{c}/2)\beta(r/2) \geq \widehat{\kappa}^{-1}(r). \quad (\text{D.4})$$

Define  $\bar{\kappa}(s) := \alpha^{-1}(s)$  for all  $s \geq 0$ . Then  $\bar{\kappa} \in \mathcal{K}_\infty$  is concave since  $\alpha$  is convex and strictly increasing. From the construction of  $\bar{\kappa}$  it follows that  $\bar{\kappa}(s) \leq \widehat{\kappa}(s)$  for all  $s \geq c$ ,

$$\begin{aligned} \int_{\mathbb{R}^m} \bar{\kappa}(2\gamma(|v|))\mu(dv) &\leq \int_{|v|<1} \bar{\kappa}(2\gamma(|v|))\mu(dv) + \int_{|v|\geq 1} \bar{\kappa}(2\gamma(|v|))\mu(dv) \\ &\leq \bar{\kappa}(2\gamma(1)) + \int_{|v|\geq 1} \widehat{\kappa}(2\gamma(|v|))\mu(dv) \leq \bar{\kappa}(2\gamma(1)) + 1. \end{aligned}$$

We now define  $\overline{W}(x) := \bar{\kappa}(V(x))$  for all  $x \in \mathbb{R}^n$ . Then, it follows from Jensen's inequality [96, Sec. 3.1.8] and the construction of  $\bar{\kappa}$  that, for  $x \in \mathbb{R}^n \setminus \mathcal{O}$ ,

$$\begin{aligned} \int_{\mathbb{R}^m} \max_{g \in G(x,v)} \overline{W}(g)\mu(dv) &= \int_{\mathbb{R}^m} \max_{g \in G(x,v)} \bar{\kappa}(V(g))\mu(dv) \leq \int_{\mathbb{R}^m} \bar{\kappa}\left(\max_{g \in G(x,v)} V(g)\right)\mu(dv) \\ &\leq \bar{\kappa}\left(\int_{\mathbb{R}^m} \max_{g \in G(x,v)} V(g)\mu(dv)\right) \\ &\leq \bar{\kappa}(V(x) - \varrho(x)). \end{aligned}$$

Since  $\bar{\kappa} \in \mathcal{K}_\infty$ , and  $\varrho$  is bounded away from zero on compact sets, it follows that  $\bar{\kappa}(V(x) - \varrho(x)) < \bar{\kappa}(V(x)) = \overline{W}(x)$ . Hence, there exists a function  $\tilde{\rho} : \mathbb{R}^n \rightarrow \mathbb{R}_{>0}$  that is bounded away from zero on compact sets such that  $\bar{\kappa}(V(x) - \varrho(x)) \leq \overline{W}(x) - \tilde{\rho}(x)$  for all  $x \in \mathbb{R}^n \setminus \mathcal{O}$ .

We now construct a continuous function  $\rho(x) := \inf_{\xi \in \mathbb{R}^n} (\tilde{\rho}(\xi) + |\xi - x|)$  for all  $x \in \mathbb{R}^n$ .

Since  $\tilde{\rho}$  is bounded away from zero on compact sets, it follows that  $\rho$  inherits the same



property. Also from the construction we have that  $\rho(x) \leq \tilde{\rho}(x)$  for all  $x \in \mathbb{R}^n$ . Therefore

$$\int_{\mathbb{R}^m} \max_{g \in G(x,v)} \overline{W}(g) \mu(dv) \leq \overline{W}(x) - \rho(x) \quad \forall x \in \mathbb{R}^n \setminus \mathcal{O}. \quad (\text{D.5})$$

Finally, for  $x \in \mathcal{O}$ ,

$$\begin{aligned} \int_{\mathbb{R}^m} \max_{g \in G(x,v)} \overline{W}(g) \mu(dv) &= \int_{\mathbb{R}^m} \max_{g \in G(x,v)} \overline{\kappa}(V(g)) \mu(dv) \\ &\leq \int_{\mathbb{R}^m} \overline{\kappa}(\gamma(|v|) + \gamma_0) \mu(dv) \\ &\leq \int_{\mathbb{R}^m} (\overline{\kappa}(2\gamma(|v|)) + \overline{\kappa}(2\gamma_0)) \mu(dv) \\ &\leq 1 + \overline{\kappa}(2\gamma_0) + \overline{\kappa}(2\gamma(1)). \end{aligned} \quad (\text{D.6})$$

Now define  $\hat{c} := 1 + \overline{\kappa}(2\gamma_0) + \overline{\kappa}(2\gamma(1)) + \sup_{x \in \mathcal{O}} \rho(x)$ ,  $\kappa(s) := \frac{1}{\hat{c}} \overline{\kappa}(s)$  for all  $s \geq 0$ ,  $\hat{\varrho}(x) := \frac{1}{\hat{c}} \rho(x)$  and  $W(x) := \kappa(V(x))$  for all  $x \in \mathbb{R}^n$ . Then, it follows from (D.5), (D.6) that

$$\int_{\mathbb{R}^m} \max_{g \in G(x,v)} W(g) \mu(dv) \leq W(x) - \hat{\varrho}(x) + \mathbb{I}_{\mathcal{O}}(x) \quad \forall x \in \mathbb{R}^n.$$

Hence  $W$  is a Lyapunov function relative to  $\mathcal{O}$  for (3.1).

## D.4 Proof of Theorem 3.2

We assume that  $\mathcal{O} \subset \mathbb{R}^n$  is globally recurrent for (3.1) and prove that there exists  $\varepsilon > 0$  and an open, bounded set  $\hat{\mathcal{O}}$  such that  $\hat{\mathcal{O}} + \varepsilon \mathbb{B} \subset \mathcal{O}$  and  $\hat{\mathcal{O}}$  is strongly globally recurrent for (3.1). In order to prove the theorem, we begin with some initial results. The proof of the following result is similar to that used in [42, Prop. 15].

**Claim D.1** *Let  $\{x_i\}_{i=1}^{\infty}$  be a sequence of points that converges to  $x \in \mathbb{R}^n$  and  $\phi_i : \mathbb{R}^n \rightarrow$*

$\mathbb{R}_{\geq 0}$  be a sequence of upper semicontinuous, bounded, non-increasing functions (with respect to  $i$ ). Then, for every  $v \in \mathbb{R}^m$ ,

$$\limsup_{i \rightarrow \infty} \max_{g \in G(x_i, v)} \phi_i(g) \leq \max_{g \in G(x, v)} \limsup_{i \rightarrow \infty} \phi_i(g).$$

*Proof:* Let  $g_i$  be such that for all  $i \in \mathbb{Z}_{\geq 1}$ ,  $\phi_i(g_i) = \max_{g \in G(x_i, v)} \phi_i(g)$ . Due to local boundedness of  $G$  we can assume without loss of generality that there exists at least a subsequence converging to  $g^*$ . Then, by outer semicontinuity of  $G$  for a fixed  $v$ , it follows that  $g^* \in G(x, v)$ . Also for every  $\varepsilon > 0$ , there exists  $i^* \in \mathbb{Z}_{\geq 0}$ ,

$$\phi_{i^*}(g^*) \leq \lim_{i \rightarrow \infty} \phi_i(g^*) + \varepsilon.$$

Then, it follows that

$$\begin{aligned} \limsup_{i \rightarrow \infty} \max_{g \in G(x_i, v)} \phi_i(g) &= \limsup_{i \rightarrow \infty} \phi_i(g_i) \leq \limsup_{i \rightarrow \infty} \phi_{i^*}(g_i) \\ &\leq \phi_{i^*}(g^*) \leq \lim_{i \rightarrow \infty} \phi_i(g^*) + \varepsilon \\ &\leq \max_{g \in G(x, v)} \limsup_{i \rightarrow \infty} \phi_i(g) + \varepsilon. \end{aligned}$$

Since  $\varepsilon > 0$  is arbitrary the result of the claim follows. ■

**Claim D.2** Let  $\{x_i\}_{i=1}^{\infty}$  be a sequence of points converging to  $x \in \mathbb{R}^n$  and  $S \subset \mathbb{R}^n$  be closed. Then, for all  $k \in \mathbb{Z}_{\geq 0}$ ,

$$\limsup_{i \rightarrow \infty} \max_{g \in G(x_i, v)} \mathbb{I}_{S+\frac{1}{i}\mathbb{B}}(g) m_{CS+\frac{1}{i}\mathbb{B}}(k, g) \leq \max_{g \in G(x, v)} \mathbb{I}_S(g) m_{CS}(k, g). \quad (\text{D.7})$$

*Proof:* Define  $\phi_i(x) := \mathbb{I}_{S+\frac{1}{i}\mathbb{B}}(x) m_{CS+\frac{1}{i}\mathbb{B}}(k, x)$  for  $x \in \mathbb{R}^n$ . Then, from the upper semicontinuity of  $\mathbb{I}_{S+\frac{1}{i}\mathbb{B}}(\cdot)$  and  $m_{CS+\frac{1}{i}\mathbb{B}}(k, \cdot)$  for each  $i \in \mathbb{Z}_{\geq 1}$  and  $k \in \mathbb{Z}_{\geq 0}$  it follows that

$\phi_i$  is upper semicontinuous for each  $i$ . From the monotonicity of the viability probabilities and the indicator function with respect to  $i$ , it follows that  $\phi_i$  is monotonically nonincreasing. Also  $\phi_i(x) \leq m_{\mathbb{C}S+\frac{1}{i}\mathbb{B}}(k, x) \leq 1$ . Then, from Claim D.1 it follows that

$$\begin{aligned} \limsup_{i \rightarrow \infty} \max_{g \in G(x, v)} \mathbb{I}_{S+\frac{1}{i}\mathbb{B}}(g) m_{\mathbb{C}S+\frac{1}{i}\mathbb{B}}(k, g) &\leq \max_{g \in G(x, v)} \limsup_{i \rightarrow \infty} \mathbb{I}_{S+\frac{1}{i}\mathbb{B}}(g) m_{\mathbb{C}S+\frac{1}{i}\mathbb{B}}(k, g) \\ &\leq \max_{g \in G(x, v)} \mathbb{I}_S(g) \limsup_{i \rightarrow \infty} m_{\mathbb{C}S+\frac{1}{i}\mathbb{B}}(k, g). \end{aligned} \quad (\text{D.8})$$

We now prove that

$$\limsup_{i \rightarrow \infty} m_{\mathbb{C}S+\frac{1}{i}\mathbb{B}}(k, x) \leq m_{\mathbb{C}S}(k, x) \quad \forall (k, x) \in \mathbb{Z}_{\geq 0} \times \mathbb{R}^n. \quad (\text{D.9})$$

The bound (D.9) holds for  $k = 0$  and for all  $x \in \mathbb{R}^n$  by definition. Now assume that (D.9) holds for some  $k \in \mathbb{Z}_{\geq 0}$  and for all  $x \in \mathbb{R}^n$ . Then, by Fatou's Lemma and (D.8) it follows that

$$\begin{aligned} \limsup_{i \rightarrow \infty} m_{\mathbb{C}S+\frac{1}{i}\mathbb{B}}(k+1, x) &= \limsup_{i \rightarrow \infty} \int_{\mathbb{R}^m} \max_{g \in G(x, v)} \mathbb{I}_{S+\frac{1}{i}\mathbb{B}}(g) m_{\mathbb{C}S+\frac{1}{i}\mathbb{B}}(k, g) \mu(dv) \\ &\leq \int_{\mathbb{R}^m} \limsup_{i \rightarrow \infty} \max_{g \in G(x, v)} \mathbb{I}_{S+\frac{1}{i}\mathbb{B}}(g) m_{\mathbb{C}S+\frac{1}{i}\mathbb{B}}(k, g) \mu(dv) \\ &\leq \int_{\mathbb{R}^m} \max_{g \in G(x, v)} \mathbb{I}_S(g) \limsup_{i \rightarrow \infty} m_{\mathbb{C}S+\frac{1}{i}\mathbb{B}}(k, g) \mu(dv) \\ &\leq \int_{\mathbb{R}^m} \max_{g \in G(x, v)} \mathbb{I}_S(g) m_{\mathbb{C}S}(k, g) \mu(dv) \leq m_{\mathbb{C}S}(k+1, x). \end{aligned}$$

Then, the bound (D.9) holds by induction. The result of the claim then follows from (D.8) and (D.9). ■

We then use the above results to prove the the following Lemma which illustrates the effect of perturbations of the set on the weak viability probabilities.

**Lemma D.1** *For each  $(\ell, \rho) \in \mathbb{Z}_{\geq 0} \times \mathbb{R}_{>0}$  and  $K \subset \mathbb{R}^n$  compact there exists a  $\varepsilon > 0$*

such that, for every  $x \in K$  compact,

$$m_{\subset S+\varepsilon\mathbb{B}}(\ell, x) \leq \max_{\xi \in K} m_{\subset S}(\ell, \xi) + \rho.$$

*Proof:* Suppose the lemma is false, then there exists  $\ell \in \mathbb{Z}_{\geq 0}$ ,  $\rho > 0$  and  $K$  compact such that, for each  $i \in \mathbb{Z}_{\geq 1}$  there exists  $x_i \in K$  satisfying

$$m_{\subset S+\frac{1}{i}\mathbb{B}}(\ell, x_i) > \max_{\xi \in K} m_{\subset S}(\ell, \xi) + \rho.$$

Without loss of generality we assume that  $x_i$  converges to some  $x \in K$ . Then, it follows from Fatou's Lemma and (D.7) that

$$\begin{aligned} \limsup_{i \rightarrow \infty} m_{\subset S+\frac{1}{i}\mathbb{B}}(\ell, x_i) &= \limsup_{i \rightarrow \infty} \int_{\mathbb{R}^m} \max_{g \in G(x_i, v)} \mathbb{I}_{S+\frac{1}{i}\mathbb{B}}(g) m_{\subset S+\frac{1}{i}\mathbb{B}}(\ell - 1, g) \mu(dv) \\ &\leq \int_{\mathbb{R}^m} \limsup_{i \rightarrow \infty} \max_{g \in G(x_i, v)} \mathbb{I}_{S+\frac{1}{i}\mathbb{B}}(g) m_{\subset S+\frac{1}{i}\mathbb{B}}(\ell - 1, g) \mu(dv) \\ &\leq \int_{\mathbb{R}^m} \max_{g \in G(x, v)} \mathbb{I}_S(g) m_{\subset S}(\ell - 1, g) \mu(dv) \\ &= m_{\subset S}(\ell, x). \end{aligned}$$

The bound contradicts the initial assumption for  $i$  large and thus establishes the lemma. ■

Let  $\ell_0 \in \mathbb{Z}_{> 0}$  be such that  $m_{\subset \mathbb{R}^n \setminus \mathcal{O}}(\ell_0, x) \leq 0.25/2$  for all  $x \in \overline{\mathcal{O} + \mathbb{B}^\circ}$ . This bound follows from the uniform global recurrence of the set  $\mathcal{O}$ . We now use the result of Lemma D.1 with  $K := \overline{\mathcal{O} + \mathbb{B}^\circ}$ . Then, there exists  $\tilde{\varepsilon} \in (0, 1)$  such that, for every  $x \in K$ ,

$$\begin{aligned} m_{\subset (\mathbb{R}^n \setminus \mathcal{O})+\tilde{\varepsilon}\mathbb{B}}(\ell_0, x) &\leq \max_{\xi \in \overline{\mathcal{O} + \mathbb{B}^\circ}} m_{\subset \mathbb{R}^n \setminus \mathcal{O}}(\ell_0, \xi) + 0.25/2 \\ &\leq 0.25. \end{aligned}$$

Let the open, bounded set  $\widehat{\mathcal{O}}$  be such that  $\widehat{\mathcal{O}} := \mathbb{R}^n \setminus ((\mathbb{R}^n \setminus \mathcal{O}) + \tilde{\varepsilon}\mathbb{B})$ . Then  $|\xi|_{\overline{\mathcal{O}}} = \tilde{\varepsilon}$  for all  $\xi \in \partial\widehat{\mathcal{O}}$ . Hence, it follows that for  $\varepsilon = \tilde{\varepsilon}/2 > 0$ ,  $\widehat{\mathcal{O}} + \varepsilon\mathbb{B} \subset \mathcal{O}$ . Since  $\tilde{\varepsilon} < 1$ ,  $\mathcal{O} \subset \widehat{\mathcal{O}} + \mathbb{B}^o$ . Then, for all  $x \in \widehat{\mathcal{O}} + \mathbb{B}^o$ ,

$$m_{\mathbb{C}\mathbb{R}^n \setminus \widehat{\mathcal{O}}}(\ell_0, x) \leq m_{\mathbb{C}\mathbb{R}^n \setminus \mathcal{O} + \varepsilon\mathbb{B}}(\ell_0, x) \leq 0.25. \quad (\text{D.10})$$

Now let  $S_i := \mathbb{R}^n \setminus (\widehat{\mathcal{O}} + i\mathbb{B}^o)$  be a sequence of closed sets for  $i \in \mathbb{Z}_{\geq 0}$ . Given  $x \in \mathbb{R}^n$ , there exists  $j \in \mathbb{Z}_{\geq 1}$  such that  $x \in \overline{\mathbb{R}^n \setminus S_{j+1}}$ . Since  $\mathcal{O} \subset \widehat{\mathcal{O}} + \mathbb{B}^o$ , it follows from uniform global recurrence of the set  $\mathcal{O}$  that for all  $\xi \in \overline{\mathbb{R}^n \setminus S_{j+1}}$  there exists  $\ell_j \in \mathbb{Z}_{>0}$  such that

$$m_{\mathbb{C}S_1}(\ell_j, \xi) \leq 0.25. \quad (\text{D.11})$$

Then, from Lemma 3.3, (D.10) and (D.11) it follows that

$$\begin{aligned} m_{\mathbb{C}S_0}(\ell_j + \ell_0, x) &\leq m_{\mathbb{C}S_1}(\ell_j, x) + \sup_{\xi \in \mathbb{R}^n \setminus S_1} m_{\mathbb{C}S_0}(\ell_0, \xi) \\ &\leq 0.5. \end{aligned}$$

Then, from the monotonicity of the viability probabilities we have that for every  $x \in \mathbb{R}^n$ ,  $\widehat{m}_{\mathbb{C}\mathbb{R}^n \setminus \widehat{\mathcal{O}}}(x) \leq 0.5$ . Hence, it follows from Proposition 3.2 that for all  $x \in \mathbb{R}^n$ ,  $\widehat{m}_{\mathbb{C}\mathbb{R}^n \setminus \widehat{\mathcal{O}}}(x) = 0$ . This equality implies that the set  $\widehat{\mathcal{O}}$  is globally recurrent for (3.1).

## D.5 Proof of Theorem 3.3

We begin with a result similar to [42, Lemma 4] that establishes that the reachability probabilities  $m_{\cap S}(k, x)$  can be made arbitrarily small for a fixed  $k$ , for  $x$  in a compact set, when  $S = \mathbb{R}^n \setminus R\mathbb{B}^o$  by using the local boundedness of  $G$  and choosing  $R > 0$  sufficiently

large.

**Lemma D.2** *For each  $k \in \mathbb{Z}_{\geq 0}$ ,  $\varepsilon > 0$  and  $r > 0$  there exists  $R > 0$  such that, with  $S = \mathbb{R}^n \setminus R\mathbb{B}^o$ ,  $m_{\cap S}(k, x) \leq \varepsilon$  for all  $x \in r\mathbb{B}$ .*

We denote the probabilities of the system (3.9) with the subscript  $\nu$ . Let  $\widehat{\mathcal{O}}$  be chosen according to Theorem 3.2. Now let  $S_i = \mathbb{R}^n \setminus (\widehat{\mathcal{O}} + i\mathbb{B}^o)$  be a sequence of closed sets and  $\varepsilon_i \leq (\frac{1}{2})^{i+2}$  for all  $i \in \mathbb{Z}_{\geq 0}$ . Then, for every  $i \in \mathbb{Z}_{\geq 0}$ , choose  $\ell_i$  such that

$$m_{\subset S_i}(\ell_i, x) \leq \frac{1}{2}\varepsilon_i \quad \forall x \in \overline{\mathbb{R}^n \setminus S_{i+1}}. \quad (\text{D.12})$$

This bound follows from the uniform global recurrence of the set  $\widehat{\mathcal{O}}$ . Let  $\beta_i \in \mathbb{Z}_{\geq 0}$ . Then, choose  $\beta_i \geq i + 1$  such that, with  $\nu(s) = s$  for all  $s \geq 0$ ,

$$m_{\nu, \cap S_{\beta_i}}(\ell_i, x) \leq \frac{1}{2}\varepsilon_i \quad \forall x \in \overline{\mathbb{R}^n \setminus S_{i+1}}. \quad (\text{D.13})$$

The values  $\beta_i$  exists according to Lemma D.2. Without loss of generality we can assume the function  $i \mapsto \beta_i$  is strictly increasing and unbounded. Define  $K_i := (\overline{\widehat{\mathcal{O}} + \beta_i\mathbb{B}^o}) \setminus (\widehat{\mathcal{O}} + i\mathbb{B}^o)$ ,  $\gamma_i := \sup_{x \in \partial(\widehat{\mathcal{O}} + \beta_i\mathbb{B}^o)} |x - x_0|$  and  $r_i := \inf_{y \in \partial(\widehat{\mathcal{O}} + i\mathbb{B}^o)} |y - x_0|$ . Then, it follows that the functions  $i \mapsto \gamma_i, r_i$  are strictly increasing and unbounded. Let  $\nu \in \mathcal{K}_\infty$  with  $\nu(s) < s$  for all  $s > 0$  and satisfy  $\nu(\gamma_i) < r_i/2$  for all  $i \in \mathbb{Z}_{\geq 0}$ . Then, we have that  $\{x_0\} + \nu(\gamma_i)\mathbb{B} \subset \{x_0\} + \frac{1}{2}r_i\mathbb{B} \subset \mathbb{R}^n \setminus K_i$  and hence

$$\max_{g \in \{x_0\} + \nu(\gamma_i)\mathbb{B}} \mathbb{I}_{K_i}(g) = 0. \quad (\text{D.14})$$

Next we show that for all  $x \in \overline{\mathbb{R}^n \setminus S_{\beta_i}}$  and all  $k \in \mathbb{Z}_{\geq 0}$ ,

$$m_{\nu, \subset K_i}(k, x) = m_{\subset K_i}(k, x). \quad (\text{D.15})$$

The equality (D.15) holds when  $k = 0$  for all  $x \in \mathbb{R}^n$ . Assume it holds for some  $k \in \mathbb{Z}_{\geq 0}$  and for all  $x \in \overline{\mathbb{R}^n \setminus S_{\beta_i}}$ . Using (D.14),  $G(x, v) \subset G_\nu(x, v)$  for all  $(x, v) \in \mathbb{R}^n \times \mathbb{R}^m$ , and  $K_i \subset \widehat{\mathcal{O}} + \beta_i \mathbb{B}^o$ , it follows that if  $x \in \overline{\mathbb{R}^n \setminus S_{\beta_i}}$  then

$$\begin{aligned} \max_{g \in G(x, v)} \mathbb{I}_{K_i}(g) m_{\subset K_i}(k, g) &\leq \max_{g \in G_\nu(x, v)} \mathbb{I}_{K_i}(g) m_{\nu, \subset K_i}(k, g) \\ &\leq \max_{g \in G(x, v)} \mathbb{I}_{K_i}(g) m_{\nu, \subset K_i}(k, g) + \max_{g \in \{x_0\} + \nu(\gamma_i) \mathbb{B}} \mathbb{I}_{K_i}(g) m_{\nu, \subset K_i}(k, g) \\ &= \max_{g \in G(x, v)} \mathbb{I}_{K_i}(g) m_{\nu, \subset K_i}(k, g) \\ &= \max_{g \in G(x, v)} \mathbb{I}_{K_i}(g) m_{\subset K_i}(k, g). \end{aligned}$$

Hence for all  $x \in \overline{\mathbb{R}^n \setminus S_{\beta_i}}$

$$\max_{g \in G_\nu(x, v)} \mathbb{I}_{K_i}(g) m_{\nu, \subset K_i}(k, g) = \max_{g \in G(x, v)} \mathbb{I}_{K_i}(g) m_{\subset K_i}(k, g)$$

Therefore for all  $x \in \overline{\mathbb{R}^n \setminus S_{\beta_i}}$

$$\begin{aligned} m_{\nu, \subset K_i}(k+1, x) &= \int_{\mathbb{R}^m} \max_{g \in G_\nu(x, v)} \mathbb{I}_{K_i}(g) m_{\nu, \subset K_i}(k, g) \mu(dv) \\ &= \int_{\mathbb{R}^m} \max_{g \in G(x, v)} \mathbb{I}_{K_i}(g) m_{\subset K_i}(k, g) \mu(dv) \\ &= m_{\subset K_i}(k+1, x). \end{aligned}$$

Then (D.15) holds for all  $k \in \mathbb{Z}_{\geq 0}$  and all  $x \in \overline{\mathbb{R}^n \setminus S_{\beta_i}}$  by induction. Now using Lemma 3.1, (D.12), (D.13), (D.15) we have that for all  $i \in \mathbb{Z}_{\geq 0}$  and every  $x \in \overline{\mathbb{R}^n \setminus S_{i+1}}$ ,

$$\begin{aligned} m_{\nu, \subset S_i}(\ell_i, x) &\leq m_{\nu, \cap S_{\beta_i}}(\ell_i, x) + m_{\nu, \subset K_i}(\ell_i, x) \\ &\leq \frac{1}{2} \varepsilon_i + m_{\subset K_i}(\ell_i, x) \leq \frac{1}{2} \varepsilon_i + m_{\subset S_i}(\ell_i, x) \\ &\leq \varepsilon_i. \end{aligned} \tag{D.16}$$

Given  $x \in \mathbb{R}^n$ , let  $i \in \mathbb{Z}_{\geq 1}$  be such that  $x \in \overline{\mathbb{R}^n \setminus S_{i+1}}$ . Then, from (D.16) it follows that  $m_{\nu, \subset S_i}(\ell_i, x) \leq \varepsilon_i$ , and similarly we have that for every  $k \in \{0, \dots, i-1\}$ ,  $\sup_{\xi \in \mathbb{R}^n \setminus S_{k+1}} m_{\nu, \subset S_k}(\ell_k, \xi) \leq \varepsilon_k$ . Then, from Lemma 3.3, (D.16) it follows that

$$\begin{aligned} m_{\nu, \subset S_0} \left( \sum_{j=0}^i \ell_j, x \right) &\leq m_{\nu, \subset S_1} \left( \sum_{j=1}^i \ell_j, x \right) + \sup_{\xi \in \mathbb{R}^n \setminus S_1} m_{\nu, \subset S_0}(\ell_0, \xi) \\ &\leq m_{\nu, \subset S_i}(\ell_i, x) + \sum_{k=0}^{i-1} \sup_{\xi \in \mathbb{R}^n \setminus S_{k+1}} m_{\nu, \subset S_k}(\ell_k, \xi) \\ &\leq \sum_{k=0}^i \varepsilon_k \leq 0.5. \end{aligned}$$

Hence, from the monotonicity of the viability probabilities we can conclude that  $\widehat{m}_{\nu, \subset S_0}(x) \leq 0.5$  for all  $x \in \mathbb{R}^n$ . Then, from Proposition 3.2 it follows that  $\widehat{m}_{\nu, \subset \mathbb{R}^n \setminus \widehat{\mathcal{O}}}(x) = 0$  for every  $x \in \mathbb{R}^n$  and global recurrence follows.

## D.6 Proof of Theorem 3.4

In order to prove this theorem we use the following result from [42, Corollary 3] which illustrates the effect of perturbations of the system on viability probabilities. Let  $\rho > 0$ . We denote the probabilities of the perturbed system  $x^+ \in G_\rho(x, v) := \{w \in \mathbb{R}^n : w \in g + \rho\mathbb{B}, g \in G(x + \rho\mathbb{B}, v)\}$  with a subscript  $\rho$ .

**Lemma D.3** *For each closed set  $S \subset \mathbb{R}^n$ , compact set  $K \subset \mathbb{R}^n$ ,  $\ell \in \mathbb{Z}_{\geq 0}$  and  $\varepsilon > 0$  there exists  $\rho > 0$  such that, for all  $k \in \{0, \dots, \ell\}$*

$$m_{\rho, \subset S}(k, \xi) \leq \max_{\zeta \in (\xi + \varepsilon\mathbb{B}) \cap K} m_{\subset S}(k, \zeta) + \varepsilon \quad \forall \xi \in K + \rho\mathbb{B}.$$

Let  $\widehat{\mathcal{O}}$  be chosen according to Theorem 3.2 and  $\nu \in \mathcal{K}_\infty$  be chosen according to Theorem 3.3. Now define  $S_i := \mathbb{R}^n \setminus (\widehat{\mathcal{O}} + i\mathbb{B}^\circ)$ .



**Claim D.3** *If the set  $\widehat{\mathcal{O}}$  is globally recurrent for (3.9), then for every  $\varepsilon_i \leq (\frac{1}{2})^{i+2}$  where  $i \in \mathbb{Z}_{\geq 0}$ , there exists  $\ell_i \in \mathbb{Z}_{>0}$  and  $\rho_i > 0$ , such that  $\max_{x \in \overline{\mathbb{R}^n \setminus S_{i+1}}} m_{\rho_i, C S_i}(\ell_i, x) \leq \varepsilon_i$ .*

*Proof:* Let  $\ell_i$  be chosen such that, for (3.9) we have that  $\max_{\xi \in \overline{\mathbb{R}^n \setminus S_{i+1}}} m_{C S_i}(\ell_i, \xi) \leq \frac{1}{2} \varepsilon_i$ . This bound follows from the uniform strong global recurrence of the set  $\widehat{\mathcal{O}}$ . Since  $x$  belongs to the compact set  $\overline{\mathbb{R}^n \setminus S_{i+1}}$ , it follows from Lemma D.2 that there exists  $\rho_i > 0$  such that, for every  $x \in \overline{\mathbb{R}^n \setminus S_{i+1}}$ ,

$$m_{\rho_i, C S_i}(\ell_i, x) \leq \max_{\xi \in \overline{\mathbb{R}^n \setminus S_{i+1}}} m_{C S_i}(\ell_i, \xi) + \frac{1}{2} \varepsilon_i \leq \varepsilon_i.$$

■

Given  $x \in \mathbb{R}^n$ , define  $i(x) := \min_{j \geq 1} \{j : x \in \overline{\mathbb{R}^n \setminus S_{j+1}}\}$ . Then, from D.3 it follows that there exists  $\rho_i, \ell_i > 0$  such that  $m_{\rho_i, C S_i}(\ell_i, x) \leq \varepsilon_i$ . Similarly we have that for every  $k \in \{0, \dots, i-1\}$  there exists  $\rho_k, \ell_k > 0$  such that  $\sup_{\xi \in \overline{\mathbb{R}^n \setminus S_{k+1}}} m_{\rho_k, C S_k}(\ell_k, \xi) \leq \varepsilon_k$ . Then, define a continuous state dependent perturbation  $\rho : \mathbb{R}^n \rightarrow \mathbb{R}_{>0}$  as follows,

$$\begin{aligned} \widehat{\rho}(x) &:= \min_{k \in \{0, 1, \dots, i(x)\}} \rho_k \\ \rho(x) &:= \inf_{\xi \in \mathbb{R}^n} (\widehat{\rho}(\xi) + |\xi - x|). \end{aligned}$$

Since  $\widehat{\rho}(x)$  is bounded away from zero on compact sets, it ensures that  $\rho(x)$  is positive for all  $x \in \mathbb{R}^n$ . The choice of  $\rho$  implies that  $\rho(x) \leq \widehat{\rho}(x)$ . Then, it follows from Lemma

3.3 that

$$\begin{aligned}
 m_{\rho, \subset S_0} \left( \sum_{j=0}^{i(x)} \ell_j, x \right) &\leq m_{\rho, \subset S_1} \left( \sum_{j=1}^{i(x)} \ell_j, x \right) + \sup_{\xi \in \mathbb{R}^n \setminus S_1} m_{\rho, \subset S_0}(\ell_0, \xi) \\
 &\leq m_{\rho, \subset S_{i(x)}}(\ell_{i(x)}, x) + \sum_{k=0}^{i(x)-1} \sup_{\xi \in \mathbb{R}^n \setminus S_{k+1}} m_{\rho, \subset S_k}(\ell_k, \xi) \\
 &\leq \sum_{k=0}^{i(x)} \varepsilon_k \leq 0.5.
 \end{aligned}$$

Hence, from monotonicity of the viability probabilities we can conclude that  $\widehat{m}_{\rho, \subset S_0}(x) \leq 0.5$ , for all  $x \in \mathbb{R}^n$ . Then, it follows from Proposition 3.2 that for all  $x \in \mathbb{R}^n$ ,  $\widehat{m}_{\rho, \subset \mathbb{R}^n \setminus \mathcal{O}}(x) = 0$ , which proves that global recurrence is robust to sufficiently small state dependent perturbations.

## D.7 Proof of Theorem 3.7

We first begin by proving that for the modified system  $\widehat{G}(x, v) = G(x, v) \cap \mathbb{R}^n \setminus \mathcal{O}$  we have boundedness in reachability probabilities. It follows from the construction that,  $\widehat{G}$  satisfies the conditions of the Standing assumption 3.1. The modified system  $\widehat{G}$  ensures that solutions cannot grow arbitrarily large with probability one. Since the function  $V$  is radially unbounded and locally bounded for all  $x \in \mathbb{R}^n \setminus \mathcal{O}$ , there exists  $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$  and  $c_1, c_2 > 0$  such that,

$$\begin{aligned}
 \alpha_1(|x|) &\leq V(x) + c_1 \\
 V(x) &\leq \alpha_2(|x|) + c_2.
 \end{aligned}$$

**Proposition D.3** *Under condition 1 of Theorem 3.7, for every  $x \in \mathbb{R}^n \setminus \mathcal{O}$  and  $\gamma > 0$ , there exists a  $R > 0$  such that  $\lim_{i \rightarrow \infty} m_{\widehat{G}, \mathbb{R}^n \setminus (R\mathbb{B}^0), \cup}(i, x) \leq \gamma$ .*

*Proof:* Let  $S = \mathbb{R}^n \setminus R\mathbb{B}^o$ . To prove the above statement we first establish that for all  $x \in \mathbb{R}^n \setminus \mathcal{O}$  and  $i \in \mathbb{Z}_{\geq 0}$ ,

$$\alpha_1(R)m_{\widehat{G},S,\cup}(i, x) \leq V(x) + c_1.$$

The bound holds for  $i = 0$  by definition for all  $x \in \mathbb{R}^n \setminus \mathcal{O}$ . Now assume that the bound holds for some  $i \in \mathbb{Z}_{\geq 0}$  and all  $x \in \mathbb{R}^n \setminus \mathcal{O}$ . Then from the bound  $\alpha_1(|x|) \leq V(x) + c_1$  it follows that,

$$\begin{aligned} \alpha_1(R)m_{\widehat{G},S,\cup}(i+1, x) &= \int_{\mathbb{R}^m} \max_{g \in \widehat{G}(x,v)} \max\{\alpha_1(R)\mathbb{I}_S(g), \alpha_1(R)m_{\widehat{G},S,\cup}(i, g)\} \mu(dv) \\ &\leq \int_{\mathbb{R}^m} \max_{g \in \widehat{G}(x,v)} \max\{V(g) + c_1, V(g) + c_1\} \mu(dv) \\ &= \int_{\mathbb{R}^m} \max_{g \in \widehat{G}(x,v)} V(g) \mu(dv) + c_1 \leq V(x) + c_1. \end{aligned}$$

Then the result follows by induction. This implies that for all  $(i, x) \in \mathbb{Z}_{\geq 0} \times \mathbb{R}^n \setminus \mathcal{O}$ ,

$$m_{\widehat{G},S,\cup}(i, x) \leq \frac{1}{\alpha_1(R)}(V(x) + c_1) \leq \frac{1}{\alpha_1(R)}(\alpha_2(|x|) + c_2 + c_1).$$

So given  $x \in \mathbb{R}^n \setminus \mathcal{O}$ , we can choose  $R > 0$  such that  $\frac{1}{\alpha_1(R)}(\alpha_2(|x|) + c_2 + c_1) \leq \gamma$ . This implies that  $\lim_{i \rightarrow \infty} m_{\widehat{G},\mathbb{R}^n \setminus R\mathbb{B}^o,\cup}(i, x) \leq \gamma$ . ■

Next using the nested matrosov property, we establish that on compact sets bounded away from the set  $\mathcal{O}$ , the viability probabilities can be made arbitrarily small.

**Proposition D.4** *Under condition 2 of Theorem 3.7, for every  $R > 0$  and  $\gamma > 0$ , there exists a  $J > 0$  such that  $m_{G,(\mathbb{R}^n \setminus \mathcal{O}) \cap R\mathbb{B}, \cap}(J, x) \leq \gamma$  for all  $x \in (\mathbb{R}^n \setminus \mathcal{O}) \cap R\mathbb{B}$ .*

*Proof:* From the result in [97] it can be shown that condition 2 in Theorem 3.7 implies the existence of positive real numbers  $\{K_i\}_{i=1}^N$  and  $\rho > 0$  such that,

$$U(x) = \sum_{i=1}^N K_i Y_i(x) \leq -\rho \quad \forall x \in (\mathbb{R}^n \setminus \mathcal{O}) \cap R\mathbb{B}.$$

Now define,

$$\bar{V}(x) = \sum_{i=1}^N K_i W_i(x).$$

This implies that for all  $x \in (\mathbb{R}^n \setminus \mathcal{O}) \cap R\mathbb{B}$ ,

$$\begin{aligned} \int_{\mathbb{R}^m} \max_{g \in G(x,v) \cap (\mathbb{R}^n \setminus \mathcal{O}) \cap R\mathbb{B}} \bar{V}(g) \mu(dv) &= \int_{\mathbb{R}^m} \max_{g \in G(x,v) \cap (\mathbb{R}^n \setminus \mathcal{O}) \cap R\mathbb{B}} \left( \sum_{i=1}^N K_i W_i(g) \right) \mu(dv) \\ &\leq \sum_{i=1}^N K_i \int_{\mathbb{R}^m} \max_{g \in G(x,v) \cap (\mathbb{R}^n \setminus \mathcal{O}) \cap R\mathbb{B}} W_i(g) \mu(dv) \\ &\leq \sum_{i=1}^N K_i (W_i(x) + Y_i(x)) \\ &= \bar{V}(x) + U(x) \leq \bar{V}(x) - \rho. \end{aligned}$$

Let  $S = (\mathbb{R}^n \setminus \mathcal{O}) \cap R\mathbb{B}$ . We now claim that  $\rho j m_{G,S,\cap}(j, x) \leq \bar{V}(x) \quad \forall (j, x) \in \mathbb{Z}_{\geq 0} \times (\mathbb{R}^n \setminus \mathcal{O}) \cap R\mathbb{B}$ . This bound holds for  $j = 0$  since  $0 \leq \bar{V}(x)$  for all  $x \in (\mathbb{R}^n \setminus \mathcal{O}) \cap R\mathbb{B}$ . Assume that the bound holds for some  $j \in \mathbb{Z}_{\geq 0}$  and all  $x \in (\mathbb{R}^n \setminus \mathcal{O}) \cap R\mathbb{B}$ . Then,

$$\begin{aligned} \rho(j+1) m_{G,S,\cap}(j+1, x) &= (\rho + \rho j) \int_{\mathbb{R}^m} \max_{g \in G(x,v)} \mathbb{I}_{(\mathbb{R}^n \setminus \mathcal{O}) \cap R\mathbb{B}}(g) m_{G,S,\cap}(j, g) \mu(dv) \\ &\leq \rho + \int_{\mathbb{R}^m} \max_{g \in G(x,v) \cap (\mathbb{R}^n \setminus \mathcal{O}) \cap R\mathbb{B}} \bar{V}(g) \mu(dv) \leq \bar{V}(x). \end{aligned}$$

Then the bound holds by induction for all  $(x, j) \in (\mathbb{R}^n \setminus \mathcal{O}) \cap R\mathbb{B} \times \mathbb{Z}_{\geq 0}$ . Now pick  $J \in \mathbb{Z}_{\geq 0}$  large enough so that,  $\bar{V}(x) \leq \rho J \gamma$ . Then it follows that for all  $x \in (\mathbb{R}^n \setminus \mathcal{O}) \cap R\mathbb{B}$ ,

$$m_{G,(\mathbb{R}^n \setminus \mathcal{O}) \cap R\mathbb{B}, \cap}(J, x) \leq \gamma. \quad \blacksquare$$

Next we prove a result that relates the viability measures of the modified system to that of the original system.

**Lemma D.4** *Let  $S \subset \mathbb{R}^n$  be a closed set such that  $S \cap \mathcal{O} = \emptyset$ . Then  $m_{\widehat{G}, S, \cap}(i, x) = m_{G, S, \cap}(i, x)$  for all  $(i, x) \in \mathbb{Z}_{\geq 0} \times S$ .*

*Proof:* The equality holds by definition when  $i = 0$  for all  $x \in S$ . Now assume that the equality holds for some  $i \in \mathbb{Z}_{\geq 0}$  and all  $x \in S$ . Then it follows that for  $(x, v) \in S \times \mathbb{R}^m$ ,

$$\max_{g \in G(x, v) \cap \mathbb{R}^n \setminus \mathcal{O}} \mathbb{I}_S(g) m_{\widehat{G}, S, \cap}(i, g) = \max_{g \in G(x, v)} \mathbb{I}_S(g) m_{G, S, \cap}(i, g).$$

Then we have that,

$$\begin{aligned} m_{\widehat{G}, S, \cap}(i+1, x) &= \int_{\mathbb{R}^m} \max_{g \in G(x, v) \cap (\mathbb{R}^n \setminus \mathcal{O})} \mathbb{I}_S(g) m_{\widehat{G}, S, \cap}(i, g) \mu(dv) \\ &= \int_{\mathbb{R}^m} \max_{g \in G(x, v)} \mathbb{I}_S(g) m_{G, S, \cap}(i, g) \mu(dv) = m_{G, S, \cap}(i+1, x). \end{aligned}$$

The result now follows by induction. \blacksquare

So given  $x \in \mathbb{R}^n \setminus \mathcal{O}$  and  $\gamma \in (0, 1)$  it follows from Proposition D.3 that there exists a  $R > 0$ , such that  $\lim_{i \rightarrow \infty} m_{\widehat{G}, \mathbb{R}^n \setminus R\mathbb{B}^\circ, \cup}(i, x) \leq \gamma/2$ . Without loss of generality we can assume that  $R > 0$  is such that  $\mathcal{O} \subset R\mathbb{B}$  and  $x \in (\mathbb{R}^n \setminus \mathcal{O}) \cap R\mathbb{B} = R\mathbb{B} \setminus \mathcal{O}$ . Then using the result of Lemma 3.1 we have that for all  $i \in \mathbb{Z}_{\geq 0}$ ,

$$m_{\widehat{G}, \mathbb{R}^n \setminus \mathcal{O}, \cap}(i, x) \leq m_{\widehat{G}, R\mathbb{B} \setminus \mathcal{O}, \cap}(i, x) + m_{\widehat{G}, \mathbb{R}^n \setminus R\mathbb{B}^\circ, \cup}(i, x).$$

Since  $x \in (\mathbb{R}^n \setminus \mathcal{O}) \cap R\mathbb{B}$ , using Lemma D.4 it follows that for all  $i \in \mathbb{Z}_{\geq 0}$ ,  $m_{\widehat{G}, \mathbb{R}^n \setminus \mathcal{O}, \cap}(i, x) = m_{G, \mathbb{R}^n \setminus \mathcal{O}, \cap}(i, x)$  and  $m_{\widehat{G}, R\mathbb{B} \setminus \mathcal{O}, \cap}(i, x) = m_{G, R\mathbb{B} \setminus \mathcal{O}, \cap}(i, x)$ . Then we have that for all  $i \in \mathbb{Z}_{\geq 0}$ ,

$$m_{G, \mathbb{R}^n \setminus \mathcal{O}, \cap}(i, x) \leq m_{G, R\mathbb{B} \setminus \mathcal{O}, \cap}(i, x) + m_{\widehat{G}, \mathbb{R}^n \setminus R\mathbb{B}^\circ, \cup}(i, x).$$

Then from Propositions D.3 and D.4 it follows that there exists  $J > 0$  such that,

$$m_{G, \mathbb{R}^n \setminus \mathcal{O}, \cap}(J, x) \leq m_{G, R\mathbb{B} \setminus \mathcal{O}, \cap}(J, x) + m_{\widehat{G}, \mathbb{R}^n \setminus R\mathbb{B}^\circ, \cup}(J, x) \leq \gamma/2 + \gamma/2 \leq \gamma.$$

So for every  $x \in \mathbb{R}^n \setminus \mathcal{O}$ , we have that  $\widehat{m}_{G, \mathbb{R}^n \setminus \mathcal{O}, \cap}(x) \leq \gamma < 1$ . Then from Proposition 3.2 it follows that for all  $x \in \mathbb{R}^n$ ,  $\widehat{m}_{G, \mathbb{R}^n \setminus \mathcal{O}, \cap}(x) = 0$ . Finally using Proposition 3.2, it follows that the set  $\mathcal{O}$  is globally recurrent for (3.1).

## D.8 Proof of Proposition 4.1

We first show that for any  $\epsilon > 0$  there exists a concave  $\Gamma \in \mathcal{K}_\infty$ ,  $M \in \mathbb{R}_{>0}$  and  $\alpha \in \mathcal{K}_\infty$  such that for all  $x \in \mathcal{X}$  we have

$$\begin{aligned} \max_{u \in K(x)} \int_{\mathcal{V}} \Gamma(V(f(x, u, v))) \mu(dv) &\leq \max_{u \in K(\{x\} + \epsilon\mathbb{B})} \int_{\mathcal{V}} \Gamma(V(f(\{x\} + \epsilon\mathbb{B}, u, v))) \mu(dv) \\ &\leq \int_{\mathcal{V}} \max_{g \in f(\{x\} + \epsilon\mathbb{B}, K(\{x\} + \epsilon\mathbb{B}), v)} \Gamma(V(g)) \mu(dv) \\ &\leq \int_{\mathcal{V}} \Gamma \left( \max_{g \in f(\{x\} + \epsilon\mathbb{B}, K(\{x\} + \epsilon\mathbb{B}), v)} V(g) \right) \mu(dv) \quad (\text{D.17}) \\ &\leq M + \alpha(|x|) < \infty. \end{aligned}$$

Since the mapping  $(x, v) \mapsto \max_{g \in f(\{x\} + \epsilon\mathbb{B}, K(\{x\} + \epsilon\mathbb{B}), v)} V(g)$  is locally bounded, then there exists  $\tilde{M} > 0$  and  $\tilde{\alpha} \in \mathcal{K}_\infty$  such that  $\max_{g \in f(\{x\} + \epsilon\mathbb{B}, K(\{x\} + \epsilon\mathbb{B}), v)} V(g) \leq \tilde{M} + \tilde{\alpha}(|v|) + \tilde{\alpha}(|x|)$ ,

therefore

$$\begin{aligned} \int_{\mathcal{V}} \Gamma \left( \max_{g \in f(\{x\} + \epsilon \mathbb{B}, K(\{x\} + \epsilon \mathbb{B}), v)} V(g) \right) \mu(dv) &\leq \int_{\mathcal{V}} \Gamma \left( \tilde{M} + \tilde{\alpha}(|v|) + \tilde{\alpha}(|x|) \right) \mu(dv) \\ &\leq \int_{\mathcal{V}} \Gamma \left( 2\tilde{\alpha}(|v|) + 2\tilde{M} \right) \mu(dv) + \Gamma(2\tilde{\alpha}(|x|)). \end{aligned} \quad (\text{D.18})$$

It follows from Lemma 4.1 that we can choose a concave  $\Gamma \in \mathcal{K}_\infty$  and  $M > 0$  such that  $\int_{\mathcal{V}} \Gamma(2\tilde{\alpha}(|v|) + 2\tilde{M})\mu(dv) \leq M$ . Then, (D.17) follows by choosing  $\alpha(s) := \Gamma(2\tilde{\alpha}(s))$  for all  $s \in \mathbb{R}_{\geq 0}$ . Therefore from now on we fix  $\Gamma \in \mathcal{K}_\infty$  satisfying (D.17)–(D.18).

Since  $K$  (4.5) is the regularization of the control law  $\kappa$ , namely for every  $x \in \mathcal{X}$  and sequence  $x_i \rightarrow x \in \mathcal{X}$ ,  $K(x)$  is the smallest closed set containing the limit points of  $\kappa(x_i)$ , for every  $x \in \mathcal{X}$  and  $u \in K(x)$ , there exists a sequence  $\{(x_i, u_i)\}_{i=1}^\infty$ , with  $(x_i, u_i) \in \mathcal{X} \times \mathcal{U}$ ,  $u_i := \kappa(x_i)$ , such that  $(x_i, u_i) \rightarrow (x, u)$ . Let  $\epsilon := 1$ . Then without loss of generality there exists a subsequence, which we do not relabel, such that  $(x_i, u_i) \in (\{x\} + \epsilon \mathbb{B}, K(\{x\} + \epsilon \mathbb{B}))$ . Therefore for all  $(x, u) \in \text{graph}(K)$  we have

$$\begin{aligned} \int_{\mathcal{V}} \Gamma(V(f(x, u, v)))\mu(dv) &= \int_{\mathcal{V}} \lim_{i \rightarrow \infty} \Gamma(V(f(x_i, u_i, v)))\mu(dv) \\ &= \lim_{i \rightarrow \infty} \int_{\mathcal{V}} \Gamma(V(f(x_i, \kappa(x_i), v)))\mu(dv) \leq \lim_{i \rightarrow \infty} \Gamma(V(x_i)) - \varrho(x_i) \\ &= \Gamma(V(x)) - \varrho(x). \end{aligned}$$

The first equation is due to the continuity of  $\Gamma(V(f(\cdot, \cdot, v)))$  for each fixed  $v$ . For the second equation, we exploit the Lebesgue Dominated Convergence Theorem as functions  $\Gamma(V(f(x_i, \kappa(x_i), \cdot)))$  are all upper bounded by  $v \mapsto \max_{g \in f(\{x\} + \mathbb{B}, K(\{x\} + \mathbb{B}), v)} \Gamma(V(g))$ , which is integrable because of the choice of  $\Gamma$ . The inequality follows from Lemma 4.2, while the last equality is due to the continuity of  $\Gamma(V)$  and  $\varrho$ . The proof follows as  $u \in K(x)$  has been chosen arbitrarily.

## D.9 Proof of Theorem 4.1

It follows from the first part of the proof of Proposition 4.1 that for any  $\Delta \geq 1$ , there exists a concave  $\Gamma \in \mathcal{K}_\infty$ ,  $M \in \mathbb{R}_{>0}$  and  $\alpha \in \mathcal{K}_\infty$  such that for all  $x \in \mathcal{X}$  we have

$$\begin{aligned} \max_{u \in K_\Delta(x)} \int_{\mathcal{V}} \max_{g \in f_\Delta(x, u, v)} \Gamma(V(g)) \mu(dv) &\leq \int_{\mathcal{V}} \max_{g \in f_\Delta(x, K_\Delta(x), v)} \Gamma(V(g)) \mu(dv) \\ &\leq \int_{\mathcal{V}} \Gamma \left( \max_{g \in f_\Delta(x, K_\Delta(x), v)} V(g) \right) \mu(dv) \quad (\text{D.19}) \\ &\leq M + \alpha(|x|) < \infty, \end{aligned}$$

where  $K_\Delta$  and  $f_\Delta$  are, respectively, defined as in (4.7) and (4.8), but with constant perturbation  $\delta(x) \equiv \Delta$ . We make this choice in order to address perturbations  $\delta \in \mathcal{PD}(\mathcal{A})$  upper bounded by  $\Delta$ . We take an arbitrary  $\Delta \geq 1$  so that  $f_\Delta(x, K_\Delta(x), v) \supseteq f(\{x\} + \mathbb{B}, K(\{x\} + \mathbb{B}), v)$  for all  $(x, v) \in \mathcal{X} \times \mathcal{V}$ , therefore any concave  $\Gamma \in \mathcal{K}_\infty$  satisfying (D.19) also satisfies (D.17) of the proof of Proposition 4.1.

Therefore from now on we fix  $\Gamma \in \mathcal{K}_\infty$  satisfying (D.19). It follows from Lemma 4.2 that for all  $x \in \mathcal{X}$  we have  $\int_{\mathcal{V}} \Gamma(V(f(x, \kappa(x), v))) \mu(dv) \leq \Gamma(V(x)) - \varrho(x)$ , for some  $\varrho \in \mathcal{PD}(\mathcal{A})$ .

We now present some preliminary results in order to finally construct an admissible, sufficiently small, perturbation  $\delta \in \mathcal{PD}(\mathcal{A})$ .

**Lemma D.5** *For each  $\Delta \geq 1$ ,  $\bar{\delta} \in \mathcal{PD}(\mathcal{A})$  such that  $\bar{\delta}(x) \leq \Delta$  for all  $x \in \mathcal{X}$ , and concave  $\Gamma \in \mathcal{K}_\infty$  satisfying (D.19), the function  $\phi : \mathcal{X} \times \mathcal{U} \rightarrow \mathbb{R}_{\geq 0}$  defined as*

$$\phi(x, u) := \int_{\mathcal{V}} \max_{g \in f_{\bar{\delta}}(x, u, v)} \Gamma(V(g)) \mu(dv)$$

*is upper semicontinuous.*

*Proof:* Consider an arbitrary sequence  $\{(x_i, u_i)\}_{i=1}^\infty$ , with  $(x_i, u_i) \rightarrow (x, u)$ . We



can assume  $(x_i, u_i) \in (\{x\} + \mathbb{B}, K(\{x\} + \mathbb{B}))$  without loss of generality. Then by using Fatou's Lemma, where the function  $\max_{g \in f(\{x\} + \mathbb{B}, K(\{x\} + \mathbb{B}), \cdot)} \Gamma(V(g))$  is integrable because of the choice of  $\Gamma$ , and continuity of  $f_{\bar{\delta}}(\cdot, \cdot, v)$  we get

$$\begin{aligned} \limsup_{i \rightarrow \infty} \phi(x_i, u_i) &= \limsup_{i \rightarrow \infty} \int_{\mathcal{V}} \max_{g \in f_{\bar{\delta}}(x_i, u_i, v)} \Gamma(V(g)) \mu(dv) \\ &\leq \int_{\mathcal{V}} \limsup_{i \rightarrow \infty} \max_{g \in f_{\bar{\delta}}(x_i, u_i, v)} \Gamma(V(g)) \mu(dv) \\ &\leq \int_{\mathcal{V}} \max_{g \in f_{\bar{\delta}}(x, u, v)} \Gamma(V(g)) \mu(dv) = \phi(x, u). \end{aligned}$$

■

**Lemma D.6** For each  $\bar{\delta} \in \mathcal{PD}(\mathcal{A})$ , let  $\phi_i : \mathcal{X} \times \mathcal{U} \rightarrow \mathbb{R}_{\geq 0}$  be a sequence of upper semicontinuous, bounded, monotonically non-increasing functions (with respect to  $i$ ), and let  $\{c_i\}_{i=1}^{\infty}$  be a bounded sequence such that  $c_i \in \mathbb{R}_{> 0}$ ,  $c_i \rightarrow 0$ . For all sequences  $\{x_i\}_{i=1}^{\infty}$  such that  $x_i \in \mathcal{X}$ ,  $x_i \rightarrow x \in \mathcal{X}$ , we have

$$\limsup_{i \rightarrow \infty} \max_{u \in K_{c_i \bar{\delta}}(x_i)} \phi_i(x_i, u) \leq \max_{u \in K(x)} \limsup_{i \rightarrow \infty} \phi_i(x, u).$$

*Proof:* For all  $i \in \mathbb{Z}_{\geq 1}$ , and  $x_i \in \mathcal{X}$ , let  $u_i$  be such that  $\max_{u \in K_{c_i \bar{\delta}}(x_i)} \phi_i(x_i, u) = \phi_i(x_i, u_i)$ . It follows from the proof of Proposition 4.2 that  $K_{c_i \bar{\delta}}$  is outer semicontinuous for each  $i \in \mathbb{Z}_{\geq 1}$ . From the proof of [42, Claim 1], we get that  $\limsup_{i \rightarrow \infty} K_{c_i \bar{\delta}}(x_i) \subseteq K(x)$  for all  $x \in \mathcal{X}$ . Therefore we can assume without loss of generality that  $u_i \rightarrow u^* \in K(x)$ , because  $K_{\bar{\delta}}$  is compact valued. Then we can just follow the proof of [55, Claim 1]. For every  $\epsilon > 0$ , there exists  $i^* \in \mathbb{Z}_{\geq 0}$  such that  $\phi_{i^*}(x, u^*) \leq \lim_{i \rightarrow \infty} \phi_i(x, u^*) + \epsilon$ . Then we

have

$$\begin{aligned}
 \limsup_{i \rightarrow \infty} \max_{u \in K_{c_i \bar{\delta}}(x_i)} \phi_i(x_i, u) &= \limsup_{i \rightarrow \infty} \phi_i(x_i, u_i) \leq \limsup_{i \rightarrow \infty} \phi_{i^*}(x_i, u_i) \\
 &\leq \phi_{i^*}(x, u^*) \leq \lim_{i \rightarrow \infty} \phi_i(x, u^*) + \epsilon \\
 &\leq \max_{u \in K(x)} \limsup_{i \rightarrow \infty} \phi_i(x, u) + \epsilon.
 \end{aligned}$$

The proof follows as  $\epsilon > 0$  is arbitrary. ■

**Lemma D.7** *For each  $\Delta \geq 1$ ,  $\bar{\delta} \in \mathcal{PD}(\mathcal{A})$  such that  $\bar{\delta}(x) \leq \Delta$  for all  $x \in \mathcal{X}$ , and each  $\Gamma$  satisfying (D.19), we have that for any compact set  $\hat{\mathcal{X}} \subset \mathcal{X}$  there exists a constant  $\hat{c} \in (0, 1]$  such that*

$$\max_{u \in K_{\hat{c}\bar{\delta}}(x)} \int_{\mathcal{V}} \max_{g \in f_{\hat{c}\bar{\delta}}(x, u, v)} \Gamma(V(g)) \mu(dv) \leq \Gamma(V(x)) - \varrho(x) \tag{D.20}$$

for all  $x \in \hat{\mathcal{X}}$ .

*Proof:* By contradiction, suppose not. Then for each  $i \in \mathbb{Z}_{\geq 1}$  there exists  $x_i \in \hat{\mathcal{X}}$  such that for  $c_i := 1/i$  we have

$$\max_{u \in K_{c_i \bar{\delta}}(x_i)} \int_{\mathcal{V}} \max_{g \in f_{c_i \bar{\delta}}(x_i, u, v)} \Gamma(V(g)) \mu(dv) > \Gamma(V(x_i)) - \varrho(x_i) + \epsilon, \tag{D.21}$$

for some  $\epsilon > 0$ . Since  $\hat{\mathcal{X}}$  is compact, without loss of generality we can assume  $x_i \rightarrow x \in \hat{\mathcal{X}}$ .

We now consider the functions  $\phi_i(x, u) := \int_{\mathcal{V}} \max_{g \in f_{c_i \bar{\delta}}(x, u, v)} \Gamma(V(g)) \mu(dv)$ ,  $i \in \mathbb{Z}_{\geq 1}$ , which are upper semicontinuous according to Lemma D.5, bounded, monotonically non-increasing (with respect to  $i$ ), and hence satisfy the conditions of Lemma D.6.

Since the function  $v \mapsto \max_{g \in f_{\bar{\delta}}(x, K_{\bar{\delta}}(x), v)} \Gamma(V(g))$  is integrable according to (D.19), we

have

$$\begin{aligned}
 \limsup_{i \rightarrow \infty} \max_{u \in K_{c_i \bar{\delta}}(x_i)} \int_{\mathcal{V}} \max_{g \in f_{c_i \bar{\delta}}(x_i, u, v)} \Gamma(V(g)) \mu(dv) &\leq \max_{u \in K(x)} \int_{\mathcal{V}} \limsup_{i \rightarrow \infty} \max_{g \in f_{c_i \bar{\delta}}(x, u, v)} \Gamma(V(g)) \mu(dv) \\
 &= \max_{u \in K(x)} \int_{\mathcal{V}} \Gamma(V(f(x, u, v))) \mu(dv) \\
 &\leq \Gamma(V(x)) - \varrho(x). \tag{D.22}
 \end{aligned}$$

The first inequality follows from Lemma D.6 and Fatou's Lemma; the second inequality follows from the second part of the proof of Proposition 4.1. The inequality (D.22) contradicts the initial assumption (D.21) for  $i$  sufficiently large.  $\blacksquare$

We now construct the state-dependent perturbation  $\delta \in \mathcal{PD}(\mathcal{A})$  starting from an arbitrary perturbation function  $\bar{\delta} \in \mathcal{PD}(\mathcal{A})$  satisfying  $\bar{\delta}(x) \leq \Delta$  for all  $x \in \mathcal{X}$ .

For each  $i \in \mathbb{Z}$ , we define the compact sets  $\mathcal{X}_i := \{x \in \mathcal{X} \mid |x|_{\mathcal{A}} \in [2^i, 2^{i+1}]\}$ . According to Lemma D.7, for each  $\mathcal{X}_i$  there exists  $c_i > 0$  such that for all  $x \in \mathcal{X}_i$  we have  $\max_{u \in K_{c_i \bar{\delta}}(x)} \int_{\mathcal{V}} \max_{g \in f_{c_i \bar{\delta}}(x, u, v)} \Gamma(V(g)) \mu(dv) \leq \Gamma(V(x)) - \varrho(x)$ . Then we define  $\tilde{\delta}(x) := c_i \bar{\delta}(x)$  for all  $x \in \mathcal{X}_i$  and  $i \in \mathbb{Z}$ , and the continuous state-dependent perturbation  $\delta(x) := \inf_{y \in \mathcal{X}} \{\tilde{\delta}(y) + |y - x|\}$ . We notice that  $\delta(x) \leq \tilde{\delta}(x) \leq \bar{\delta}(x)$  for all  $x \in \mathcal{X}$ . Since  $\tilde{\delta}$  is bounded away from zero on compact sets disjoint from  $\mathcal{A}$ , we have that  $\delta \in \mathcal{PD}(\mathcal{A})$ .

For the second statement of the theorem, we assume there exists a compact set  $\mathcal{C}$  such that  $\mu(\mathcal{V}) = \mu(\mathcal{C}) = 1$ . We can follow the same proof of the first statement with  $\Gamma(s) = s$  for all  $s \in \mathbb{R}_{\geq 0}$ . In fact, noticing that  $V$  is upper bounded by a  $\mathcal{K}_{\infty}$  function and the mapping  $x \mapsto \max_{g \in f_{\bar{\delta}}(x, K_{\bar{\delta}}(x), \mathcal{C})} V(g)$  is locally bounded, we have that (D.19) can be satisfied with  $\Gamma(s) = s$  because for all  $x \in \mathcal{X}$  we have  $\tilde{M} + \tilde{\alpha}(|v|) + \tilde{\alpha}(|x|) \leq \left( \tilde{M} + \max_{v \in \mathcal{C}} \tilde{\alpha}(|v|) \right) + \tilde{\alpha}(|x|) < \infty$ , as the set  $\mathcal{C}$  is compact.

## D.10 Proof of Proposition 4.2

The set-valued mapping  $K$  (4.5) is locally bounded as  $\kappa$  is locally bounded from Assumption 4.1; the controlled regularization  $K$  (4.5) is outer semicontinuous. For a continuous  $\delta : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ , define the set-valued mapping  $H : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$  as  $H(\xi) := \{\xi\} + \delta(\xi)\mathbb{B}$ , which is locally bounded and outer semicontinuous. From [84, Proposition 5.52 (a), (b)] we get that the composition mapping  $K \circ H$  is locally bounded and outer semicontinuous as well. Then also  $K_\delta(x) = K(H(x)) + \delta(x)\mathbb{B}$  is locally bounded and outer semicontinuous [84, Proposition 5.51 (a), (b)].

Therefore also the set-valued mapping  $\bar{H} : \mathbb{R}^n \rightrightarrows \mathbb{R}^{n+m}$  defined as

$$\bar{H}(\xi) := \left\{ \left( \kappa_\delta^\varphi \right) \mid \varphi = \xi \right\} \tag{D.23}$$

is locally bounded and outer semicontinuous. Noticing that for any  $v \in \mathcal{V}$  the function  $(x, u) \mapsto f(x, u, v)$  is continuous from Standing Assumption 4.1 and that  $G_\delta(x, u, v) = \bar{H}(H(f(H(x), u, v)))$ , according to [84, Prop. 5.52 (a), (b)] we have that for any  $v \in \mathcal{V}$  the mapping  $(x, u) \mapsto G_\delta(x, u, v)$  is outer semicontinuous; also,  $G_\delta$  is locally bounded.

Let us now prove that the mapping  $v \mapsto \text{graph}(G_\delta(\cdot, \cdot, v))$  is measurable. Since, from Standing Assumption 4.1,  $f : (\mathcal{X} \times \mathcal{U}) \times \mathcal{V} \rightarrow \mathcal{X}$  is a Caratheodory mapping, the proof of [84, Example 14.15] shows that  $v \mapsto \text{graph}(f(\cdot, \cdot, v))$  is measurable. From the proof of [42, Proposition 3], since  $v \mapsto \text{graph}(f(\cdot, \cdot, v))$  is measurable, then also the mapping  $v \mapsto \text{graph}(H(f(H(\cdot), \cdot, v)))$  is measurable, because  $H$  is locally bounded and outer semicontinuous. But then, applying the same argument again, also the mapping  $v \mapsto \text{graph}(\bar{H}(H(f(H(\cdot), \cdot, v)))) = \text{graph}(G_\delta(\cdot, \cdot, v))$  is measurable.

## D.11 Proof of Lemma 4.3

According to the definition of  $\text{graph}(K_\delta)$ , we have that  $u \in K_\delta(x) \iff (x, u) \in \text{graph}(K_\delta) \iff W(x, u) = |(x, u)|_{\text{graph}(K_\delta)} = 0$ .

Since  $K$  is outer semicontinuous, it follows that  $\text{graph}(K)$  is a closed set. Now we notice that  $\bar{\mathcal{A}} = (\mathcal{A} \times \mathbb{R}^m) \cap \text{graph}(K)$  is the intersection of two closed sets, therefore it is closed as well. Since  $K$  is locally bounded and  $\mathcal{A}$  is compact, it follows that  $K(\mathcal{A})$  is compact. Then boundedness of  $\bar{\mathcal{A}}$  follows as  $\bar{\mathcal{A}} \subseteq \mathcal{A} \times K(\mathcal{A})$  which is a compact set. The mapping  $W$  is continuous as it is the Euclidean distance to the closed set  $\text{graph}(K_\delta)$ . Therefore  $\Gamma(V) + W$  is continuous (upper semicontinuous) if  $V$  is continuous (upper semicontinuous).

Let us now prove that for any  $\alpha : \mathcal{X} \rightarrow \mathbb{R}_{\geq 0}$  radially unbounded, the function  $(x, u) \mapsto \bar{Y}(x, u) := \Gamma(\alpha(x)) + W(x, u)$  is radially unbounded as well. This claim is exploited later. We consider a sequence  $\{(x_i, u_i)\}_{i=1}^\infty$ ,  $(x_i, u_i) \in \mathbb{R}^n \times \mathbb{R}^m$ , with  $|(x_i, u_i)| \rightarrow \infty$ . If  $|x_i| \rightarrow \infty$  then  $\bar{Y}(x_i, u_i) = \Gamma(\alpha(x_i)) + W(x_i, u_i) \geq \Gamma(\alpha(x_i)) \rightarrow \infty$  because  $\alpha$  is radially unbounded and  $\Gamma \in \mathcal{K}_\infty$ . Otherwise, we can suppose there exists a subsequence, which we do not relabel, and  $M > 0$  such that  $|x_i| \leq M$  for all  $i \geq 1$ . Since  $K_\delta$  is locally bounded, the set  $\text{graph}(K_\delta) \cap (M\mathbb{B} \times \mathbb{R}^m)$  is compact. Then, as  $|x_i| \leq M$ ,  $|(x_i, u_i)|_{\text{graph}(K_\delta)} = |(x_i, u_i)|_{\text{graph}(K_\delta) \cap (M\mathbb{B} \times \mathbb{R}^m)}$ . Since  $|(x_i, u_i)| \rightarrow \infty$ , it follows that  $\bar{Y}(x_i, u_i) = \Gamma(\alpha(x_i)) + W(x_i, u_i) \geq W(x_i, u_i) \rightarrow \infty$ .

For the last statement of the lemma, we have that  $\bar{V}(x, u) = \Gamma(V(x)) + W(x, u) \geq \Gamma(\alpha_1(|x|_{\mathcal{A}})) + W(x, u)$ , where  $x \mapsto \alpha_1(|x|_{\mathcal{A}})$  is continuous and radially unbounded. This fact implies that  $\bar{Y}_1(x, u) := \Gamma(\alpha_1(|x|_{\mathcal{A}})) + W(x, u)$  is continuous and radially unbounded as well, according to the previous parts of the proof. Moreover, we notice that  $\alpha_1(|x|_{\mathcal{A}}) = 0 \iff x \in \mathcal{A}$ , and indeed we prove that  $\bar{Y}_1(x, u) = 0 \iff (x, u) \in \bar{\mathcal{A}}$ . If  $\bar{Y}_1(x, u) = 0$ , then  $\alpha_1(|x|_{\mathcal{A}}) = 0$  and  $W(x, u) = 0$ , that are equivalent to  $x \in \mathcal{A}$  and  $u \in K_\delta(x)$ . Since

$\delta(x) = 0$  for  $x \in \mathcal{A}$ , we have that  $u \in K(x)$  whenever  $x \in \mathcal{A}$ . Hence  $(x, u) \in \bar{\mathcal{A}}$ . Conversely,  $(x, u) \in \bar{\mathcal{A}}$  implies that  $x \in \mathcal{A}$  and  $u \in K(x)$ , therefore we get  $\alpha_1(|x|_{\mathcal{A}}) = 0$  and  $W(x, u) = 0$ , so that  $\bar{Y}_1(x, u) = \Gamma(\alpha_1(|x|_{\mathcal{A}})) + W(x, u) = 0$ . Due to the properties of  $\bar{Y}_1$ , there exists  $\bar{\alpha}_1 \in \mathcal{K}_\infty$  such that  $\bar{\alpha}_1(|(x, u)|_{\bar{\mathcal{A}}}) \leq \bar{Y}_1(x, u)$  for all  $(x, u) \in (\mathcal{X} \times \mathcal{U})$ . In turn we have that  $\bar{\alpha}_1(|(x, u)|_{\bar{\mathcal{A}}}) \leq \bar{Y}_1(x, u) = \Gamma(\alpha_1(|x|_{\mathcal{A}})) + W(x, u) \leq \Gamma(V(x)) + W(x, u) = \bar{V}(x, u)$  for all  $(x, u) \in (\mathcal{X} \times \mathcal{U})$ . Finally, we can follow the same arguments for  $\bar{Y}_2(x, u) := \Gamma(\alpha_2(|x|_{\mathcal{A}})) + W(x, u)$ , which is continuous, radially unbounded, and such that  $\bar{Y}_2(x, u) \iff (x, u) \in \bar{\mathcal{A}}$ , proving that there exists  $\bar{\alpha}_2 \in \mathcal{K}_\infty$  such that  $\bar{V}(x, u) \leq \bar{Y}_2(x, u) = \Gamma(\alpha_2(|x|_{\mathcal{A}})) + W(x, u) \leq \bar{\alpha}_2(|(x, u)|_{\bar{\mathcal{A}}})$  for all  $(x, u) \in (\mathcal{X} \times \mathcal{U})$ .

## D.12 Proof of Theorem 5.1

We first show that  $\Psi := \Psi(\mathbf{z})$  is nonempty. Due to the assumption that  $\mathbf{z}$  is almost surely contained in the compact set  $K$  and is complete with positive probability, it follows that there exists  $\rho > 0$  such that

$$\mathbb{P}(\tau \leq \varphi_{\tau, K}(\mathbf{z})) \geq \rho \quad \forall \tau \geq 0. \quad (\text{D.24})$$

Let  $\ell \in \mathbb{Z}_{\geq 0}$  and, using that  $K$  is compact, let the points  $z_{\ell j} \in K$ ,  $j \in \{1, \dots, N_\ell\}$ , satisfy

$$K \subset \bigcup_{j \in \{1, \dots, N_\ell\}} S_{\ell, j}, \quad S_{\ell, j} := \{z_{\ell j}\} + \frac{1}{\ell + 1} \mathbb{B}.$$

Then, combining (5.3) and (D.24), we have

$$\mathbb{P}\left(\tau \leq \sum_{j=1}^{N_\ell} \varphi_{\tau, S_{\ell, j}}(\mathbf{z})\right) \geq \rho \quad \forall (\tau, \ell) \in \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}. \quad (\text{D.25})$$

We assert that, for each  $\ell \in \mathbb{Z}_{\geq 0}$  there exists  $j_\ell \in \{1, \dots, N_\ell\}$  such that

$$\mathbb{P}\left(\tau/N_\ell \leq \varphi_{\tau, S_{\ell, j_\ell}}(\mathbf{z})\right) \geq \rho/N_\ell \quad \forall \tau \in \mathbb{R}_{\geq 0}. \quad (\text{D.26})$$

Assuming (D.26) holds, the sequence  $z_{\ell j_\ell}$  is well-defined and contains a subsequence converging to a point  $\zeta \in K$ . Given  $\varepsilon > 0$ , let  $\ell \in \mathbb{Z}_{\geq 0}$  be sufficiently large so that  $S_{\ell j_\ell} \subset \{\zeta\} + \varepsilon \mathbb{B} =: S_\varepsilon$ . Then pick  $\varrho = \rho/N_\ell$  and, given  $\Delta$ , pick  $\tau = \Delta N_\ell$ . It follows that  $\mathbb{P}(\Delta \leq \varphi_{\tau, S_\varepsilon}(\mathbf{z})) \geq \varrho$ . Therefore,  $\zeta \in \Psi$ . Let us establish (D.26). It is straightforward to verify that, with the definitions

$$\begin{aligned} \Omega_\Sigma &:= \left\{ \omega \in \Omega : \tau \leq \sum_{j=1}^{N_\ell} \varphi_{\tau, S_{\ell, j}}(\mathbf{z}(\omega)) \right\} \\ \Omega_j &:= \left\{ \omega \in \Omega : \tau/N_\ell \leq \varphi_{\tau, S_{\ell, j}}(\mathbf{z}(\omega)) \right\} \end{aligned}$$

that  $\Omega_\Sigma \subset \bigcup_{j=1}^{N_\ell} \Omega_j$ . If (D.26) doesn't hold for some  $j_\ell \in \{1, \dots, N_\ell\}$  then

$$\mathbb{P}(\Omega_\Sigma) \leq \mathbb{P}\left(\bigcup_{j=1}^{N_\ell} \Omega_j\right) \leq \sum_{j=1}^{N_\ell} \mathbb{P}(\Omega_j) < N_\ell(\rho/N_\ell) = \rho$$

which contradicts (D.25) and thus establishes (D.26).

Next we show that  $\Psi$  is compact. Let the sequence  $\{\zeta_\ell\}_{\ell=0}^\infty$  satisfy  $\zeta_\ell \in \Psi$  for all  $\ell \in \mathbb{Z}_{\geq 0}$  and be convergent to some point  $\zeta \in \mathbb{R}^n$ . Let  $\varepsilon > 0$  be given. Pick  $\ell$  sufficiently large so that

$$S_\ell := \{\zeta_\ell\} + \frac{1}{\ell+1} \mathbb{B} \subset \{\zeta\} + \varepsilon \mathbb{B} =: S_\varepsilon.$$

Using that  $\zeta_\ell \in \Psi$ , there exists  $\varrho > 0$  and for each  $\Delta > 0$  there exists  $\tau > 0$  such that  $\mathbb{P}(\Delta \leq \varphi_{\tau, S_\ell}(\mathbf{z})) \geq \varrho$ . It follows that  $\mathbb{P}(\Delta \leq \varphi_{\tau, S_\varepsilon}(\mathbf{z})) \geq \varrho$ , which implies that  $\zeta \in \Psi$ .

It is immediate from the compactness of  $\Psi$ ,  $K$  and the assumption on  $\mathbf{z}$  that  $\Psi \subset K$ .

We show that  $\Psi \subset K_\infty$ . Since  $\Psi$  is compact, it is enough to show that for each  $\xi \in K$  and some  $\varepsilon > 0$  such that  $S_\varepsilon := (\{\xi\} + \varepsilon\mathbb{B}) \cap K_\infty = \emptyset$ , we have the following property: for each  $\varrho > 0$  there exists  $\Delta > 0$  such that

$$\mathbb{P}(\Delta \leq \varphi_{\tau, S_\varepsilon}(\mathbf{z})) \leq \varrho \quad \forall \tau \geq 0. \quad (\text{D.27})$$

Due to the assumption that almost every complete solution converges to the compact set  $K_\infty$ , for each  $\varrho > 0$  there exists  $\hat{\tau} > 0$  such that

$$\mathbb{P}(\text{graph}(\mathbf{z}) \cap (\Gamma_{\geq \hat{\tau}} \times S_\varepsilon) \neq \emptyset) \leq \varrho.$$

It turn, it follows with  $\Delta := \hat{\tau} + 1$  that (D.27) holds. This fact establishes the claim.

Next we establish that almost every complete solution converges to  $\Psi$ . If this claim fails then, using the almost sure uniform continuity of  $\mathbf{z}_\omega(\cdot, j)$ , where  $\mathbf{z}_\omega := \mathbf{z}(\omega)$ , (due to the local boundedness of  $F$  and the fact that almost every solution remains in  $K$  for all time), it follows that there exists a compact set  $\hat{K}_\infty \subset K_\infty$  such that  $\hat{K}_\infty \cap \Psi = \emptyset$  and  $\rho > 0$  and for each  $\Delta > 0$  there exists  $\tau > 0$  such that

$$\mathbb{P}\left(\Delta \leq \varphi_{\tau, \hat{K}_\infty}(\mathbf{z})\right) \geq \rho.$$

The relationship between  $\Delta$  and  $\tau$  implies the existence of a function  $\alpha \in \mathcal{K}_\infty$  such that

$$\mathbb{P}\left(\alpha(\tau) \leq \varphi_{\tau, \hat{K}_\infty}(\mathbf{z})\right) \geq \rho \quad \forall \tau \geq 0. \quad (\text{D.28})$$

Noticing the similarity between (D.28) and (D.24), it is now possible to follow the calculations above that show that  $\Psi$  is nonempty to show that there exist  $\zeta \in \hat{K}_\infty \cap \Psi$ , which contradicts  $\hat{K}_\infty \cap \Psi = \emptyset$ .



Finally, we establish that  $\Psi$  is weakly totally recurrent in probability. That is, we show that each  $\zeta \in \Psi$  is weakly recurrently in probability relative to  $\Psi$  for  $\mathcal{H}$ ; more specifically, for each  $\zeta \in \Psi$ ,  $\varepsilon > 0$  there exists  $\varrho > 0$  and for each  $\Delta > 0$  there exist  $\tau > 0$  and  $\mathbf{x} \in \mathcal{S}_r^\varepsilon(\Psi + \varepsilon\mathbb{B})$  such that, with the definition  $S_\varepsilon := \{\zeta\} + \varepsilon\mathbb{B}$ , (5.5) holds. Let  $\hat{\varrho} > 0$  be generated by  $\zeta$  and  $\varepsilon$  via the definition of the recurrent in probability set for  $\mathbf{z}$ . Using that almost every complete sample path of  $\mathbf{z}$  converges to  $\Psi$ , let  $\hat{\tau} > 0$  be such that

$$\mathbb{P}(\text{graph}(\mathbf{z}) \cap (\Gamma_{>\hat{\tau}} \times (\mathbb{R}^n \setminus (\Psi + \varepsilon\mathbb{B}))) \neq \emptyset) \leq \hat{\varrho}/2. \quad (\text{D.29})$$

Define

$$\Omega_C := \{\omega \in \Omega : \emptyset \neq (\text{graph}(\mathbf{z}(\omega)) \cap (\Gamma_{\geq\hat{\tau}+1} \times \mathbb{R}^n)) \subset (\mathbb{R}^2 \times (\Psi + \varepsilon\mathbb{B}))\}. \quad (\text{D.30})$$

We claim that for each  $\Delta > 0$  there exists  $\tau > 0$  such that

$$\mathbb{P}((\Delta \leq \varphi_{\hat{\tau}+1, \tau, S_\varepsilon}(\mathbf{z})) \cap \Omega_C) \geq \hat{\varrho}/2. \quad (\text{D.31})$$

Indeed, (D.29) and the opposite of (D.31) imply that there exists  $\Delta > 0$  such that  $\mathbb{P}(\Delta \leq \varphi_{\hat{\tau}+1, \tau, S_\varepsilon}(\mathbf{z})) < \hat{\varrho} \quad \forall \tau > 0$  and in turn  $\mathbb{P}(\Delta + \hat{\tau} + 1 \leq \varphi_{\tau, S_\varepsilon}(\mathbf{z})) < \hat{\varrho} \quad \forall \tau > 0$  which is a contradiction to the assumption that (5.5) holds with  $\hat{\varrho}$  in place of  $\varrho$ .

Next we define

$$\mathbf{T}_i(\omega) := \inf \{t \in \pi_1(\text{graph}_{\leq i}(\mathbf{z}(\omega)) \cap (\Gamma_{\geq\hat{\tau}+1} \times \mathbb{R}^n))\} \quad (\text{D.32a})$$

$$\mathbf{J}_i(\omega) := \inf \{j \in \pi_2(\text{graph}_{\leq i}(\mathbf{z}(\omega)) \cap (\Gamma_{\geq\hat{\tau}+1} \times \mathbb{R}^n))\}. \quad (\text{D.32b})$$

According to [25, Proposition 2.1, (2e)-(2g)],  $\mathbf{T}_i$  and  $\mathbf{J}_i$  are  $\mathcal{F}_i$ -measurable, as is the mapping  $\omega \mapsto p_i(\omega) := \mathbf{z}_\omega(\mathbf{T}_i(\omega), \mathbf{J}_i(\omega))$  where  $\mathbf{z}_\omega := \mathbf{z}(\omega)$ . Note that  $\mathbf{T}_i(\omega)$  and  $\mathbf{J}_i(\omega)$

are infinite when the intersections used to define them are empty. If the intersections are nonempty for some  $i^* \in \mathbb{Z}_{\geq 0}$  then they are nonempty for all  $i \in \mathbb{Z}_{\geq i^*}$  and for such  $i$  do not vary with  $i$  and satisfy  $\hat{\tau} + 1 \leq \mathbf{T}_i(\omega) + \mathbf{J}_i(\omega) \leq \hat{\tau} + 2$ , and  $\mathbf{J}_i(\omega) \leq \lceil \hat{\tau} \rceil + 1$ . For  $i \in \{0, \dots, \lceil \hat{\tau} \rceil + 1\}$ , we define

$$\Omega_i := \{\omega \in \Omega : \mathbf{J}_i(\omega) = i\} \tag{D.33a}$$

$$\Omega_{i,C} := \{\omega \in \Omega_i : \text{graph}(\mathbf{z}(\omega)) \cap (\mathbb{R}_{\geq \mathbf{T}_i(\omega)} \times \mathbb{Z}_{\geq \mathbf{J}_i(\omega)} \times \mathbb{R}^n) \subset \mathbb{R}^2 \times (\Psi + \varepsilon\mathbb{B})\}. \tag{D.33b}$$

Note that the sets  $\Omega_{i,C}$  are disjoint and

$$\bigcup_{i=0}^{\lceil \hat{\tau} \rceil + 1} \Omega_{i,C} = \Omega_C. \tag{D.34}$$

Using [98, Lemma 7, p. 411], there exists a measurable function  $\gamma$ , such that and  $\mathbf{p}_i(\omega) = \gamma(\mathbf{v}_1(\omega), \dots, \mathbf{v}_i(\omega))$ . For each  $\omega \in \Omega_i$ , define  $\mathbf{z}_i(\omega)$  to be the hybrid arc satisfying

$$\text{graph}(\mathbf{z}_i(\omega)) = \text{graph}(\mathbf{z}(\omega)) - (\mathbf{T}_i(\omega), \mathbf{J}_i(\omega), 0). \tag{D.35}$$

Then, for each  $i \in \{0, \dots, \lceil \hat{\tau} \rceil + 1\}$  and conditioned on  $\mathcal{F}_i$ ,  $\mathbf{z}_i$  is a random solution with inputs  $(\mathbf{v}_{i+1}, \mathbf{v}_{i+2}, \dots)$  starting at  $\mathbf{p}_i(\omega)$ . Since the sequence  $\mathbf{v}$  is i.i.d, the statistics of  $\mathbf{z}_i$  are unaffected by the shift in the inputs; cf. [98, Section 22.2]. We claim that for each  $\Delta > 0$  there exist  $\tau > 0$   $i \in \{0, \dots, \lceil \hat{\tau} \rceil + 1\}$  and values  $(\mathbf{v}_1, \dots, \mathbf{v}_i)$  such that

$$\mathbb{P} \left( ((\Delta \leq \varphi_{\tau, S_\varepsilon}(\mathbf{z}_i)) \wedge (\text{graph}(\mathbf{z}_i) \subset \mathbb{R}^2 \times (\Psi + \varepsilon\mathbb{B}))) \mid \mathcal{F}_i \right) (\omega) \geq \varrho/2$$

which would establish the weak total recurrence in probability. Suppose not. Then

$$\begin{aligned}
 \varrho/2 &\leq \mathbb{P}((\Delta \leq \varphi_{\hat{\tau}+1, \tau, S_\varepsilon}(\mathbf{z})) \cap \Omega_C) \\
 &= \sum_{i=0}^{\lceil \hat{\tau} \rceil + 1} \mathbb{P}((\Delta \leq \varphi_{\hat{\tau}+1, \tau, S_\varepsilon}(\mathbf{z})) \cap \Omega_{i, C}) \\
 &= \sum_{i=0}^{\lceil \hat{\tau} \rceil + 1} \mathbb{E} [\mathbb{P}(\Delta \leq \varphi_{\tau, S_\varepsilon}(\mathbf{z}_i)) \wedge (\text{graph}(\mathbf{z}_i) \subset \mathbb{R}^2 \times (\Psi + \varepsilon \mathbb{B})) \mid \mathcal{F}_i] \mathbb{I}_{\Omega_{i, C}} \\
 &< \sum_{i=0}^{\lceil \hat{\tau} \rceil + 1} \mathbb{P}(\Omega_{i, C}) \varrho/2 = \mathbb{P}(\Omega_C) \varrho/2 \leq \varrho/2.
 \end{aligned}$$

This contradiction establishes the result.

## D.13 Proof of Theorem 5.2

We present the proof of Theorem 5.2 in three parts. The first part establishes convergence of complete sample paths of the almost surely bounded random solution to the set  $K$ . The second part proves convergence to the level set of the Krasovkii-LaSalle function. Finally, in the third part we establish convergence to the largest weakly totally recurrent in probability set contained in the level set of the Krasovkii-LaSalle function.

### D.13.1 Convergence to $K$

In this section we prove that for every solution  $\mathbf{x}$  that is almost surely contained in  $\Lambda$ , the complete sample paths of  $\mathbf{x}$  converges to the set  $K$ . This result is a consequence of the Krasovskii-LaSalle function satisfying strict decrease conditions almost surely during flows and in expected value during jumps along solutions outside  $K$ .

**Proposition D.5** *If there exists a stochastic Krasovskii-LaSalle function relative to  $(K, \Lambda)$ , then for every random solution  $\mathbf{x}$  generated from the data  $(C \cap \Lambda, F, D \cap \Lambda, G \cap \Lambda, \mu)$*

almost all complete solutions converges to  $K$ .

*Proof:* Let  $\mathbf{x}$  be a random solution generated by  $(C \cap \Lambda, F, D \cap \Lambda, G \cap \Lambda, \mu)$  from initial condition  $x \in \Lambda$ . For every  $i \in \mathbb{Z}_{\geq 0}$ , the maximum of  $\varphi_{i,\kappa}(\mathbf{x}(\omega))$  (defined in Section VI) is achieved by some pair  $(\mathbf{t}_i(\omega), \mathbf{j}_i(\omega))$  where

$$\begin{aligned} \mathbf{t}_i(\omega) &:= \sup\{t \in \pi_1(\text{graph}(\mathbf{x}(\omega)) \cap (\Gamma_{\leq i} \times \mathbb{R}^n))\} \\ \mathbf{j}_i(\omega) &:= \sup\{j \in \pi_2(\text{graph}(\mathbf{x}(\omega)) \cap (\Gamma_{\leq i} \times \mathbb{R}^n))\}. \end{aligned} \tag{D.36}$$

From now on we suppress the dependence of  $\omega$  on the random variables for simplicity. The  $\mathcal{F}$ -measurability of  $\mathbf{t}_i, \mathbf{j}_i$  follows from [25, Prop 2.1(2i,2j)]. The continuity of  $V$  and [25, Prop 2.1 (k)] imply  $\mathcal{F}$ -measurability of  $V(\mathbf{x}(\mathbf{t}_i, \mathbf{j}_i))$ . We now establish that for every  $i \in \mathbb{Z}_{\geq 0}$ ,

$$\mathbb{E}[\varphi_{i,\kappa}(\mathbf{x})] \leq V(x) - \mathbb{E}[V(\mathbf{x}(\mathbf{t}_i, \mathbf{j}_i))]. \tag{D.37}$$

We observe that the bound (D.37) is similar to the bound in the proof of [14, Thm 3.18] for non-stochastic hybrid systems. The proof proceeds by induction. For  $i = 0$ ,  $\mathbf{t}_i = 0$  and  $\mathbf{j}_i = 0$  almost surely. It also follows from the definition that

$$0 = \mathbb{E}[\varphi_{0,\kappa}(\mathbf{x})] \leq V(x) - \mathbb{E}[V(x)].$$

We now assume that (D.37) is true for some  $i \in \mathbb{Z}_{\geq 0}$  with times  $\mathbf{t}_i$  and  $\mathbf{j}_i$ . For  $\varphi_{i+1,\kappa}(\mathbf{x})$ , let the corresponding times be denoted by  $(\mathbf{t}_{i+1}, \mathbf{j}_{i+1})$ . We now observe that

$$\varphi_{i+1,\kappa}(\mathbf{x}) = \varphi_{i,\kappa}(\mathbf{x}) + \int_{\mathbf{t}_i}^{\mathbf{t}_{i+1}} \kappa(\mathbf{x}(s, \mathbf{j}_{i+1})) ds + \sum_{k=\mathbf{j}_i+1}^{\mathbf{j}_{i+1}} \kappa(\mathbf{x}(\mathbf{t}_i, k-1)).$$

The above equality arises due to the total hybrid time increased by one unit which permits at most one jump and one period of flow. From the definitions it also follows that  $\mathbf{j}_{i+1} - \mathbf{j}_i \leq 1$  almost surely. Let  $\mathcal{F}_{\mathbf{j}_i}$  be the sigma algebra generated by the random variable  $\mathbf{j}_i$ . Then,

$$\begin{aligned} \mathbb{E}[\varphi_{i+1,\kappa}(\mathbf{x})] &= \mathbb{E}[\varphi_{i,\kappa}(\mathbf{x})] + \mathbb{E}\left[\sum_{k=\mathbf{j}_i+1}^{\mathbf{j}_{i+1}} \kappa(\mathbf{x}(\mathbf{t}_i, k-1)) + \int_{\mathbf{t}_i}^{\mathbf{t}_{i+1}} \kappa(\mathbf{x}(s, \mathbf{j}_{i+1})) ds\right] \\ &\leq V(x) - \mathbb{E}[V(\mathbf{x}(\mathbf{t}_i, \mathbf{j}_i))] + \mathbb{E}[V(\mathbf{x}(\mathbf{t}_i, \mathbf{j}_{i+1})) - V(\mathbf{x}(\mathbf{t}_{i+1}, \mathbf{j}_{i+1}))] \\ &\quad + \mathbb{E}[V(\mathbf{x}(\mathbf{t}_i, \mathbf{j}_i)) - \mathbb{E}[V(\mathbf{x}(\mathbf{t}_i, \mathbf{j}_{i+1})) | \mathcal{F}_{\mathbf{j}_i}]] \\ &= V(x) - \mathbb{E}[V(\mathbf{x}(\mathbf{t}_{i+1}, \mathbf{j}_{i+1}))]. \end{aligned}$$

Hence we proved that there exists a  $\Delta > 0$  such that for every  $i \in \mathbb{Z}_{\geq 0}$ ,  $\mathbb{E}[\varphi_{i,\kappa}(\mathbf{x})] \leq \Delta$  where  $\Delta := \max_{x \in \Lambda} V(x)$ . Then without loss of generality it follows from Corollary 5.3 that almost every complete sample path of the random solution  $\mathbf{x}$  converges to  $\kappa^{-1}(0) \subset K$ . ■

### D.13.2 Convergence to level sets of $V$

In this section we will establish that complete sample paths of random solution  $\mathbf{x}$  that is almost surely contained in  $\Lambda$  converge to level sets of  $V$ . The convergence to level sets of  $V$  is primarily due to the non-increasing on average nature of  $V$ .

For  $a, b \in \mathbb{R}_{\geq 0}$  define  $\mathcal{S}_1^a := \{x \in \Lambda : V(x) \in [c_1, a]\}$  and  $\mathcal{S}_2^b := \{x \in \Lambda : V(x) \in [b, c_2]\}$ . The sets  $\mathcal{S}_1^a, \mathcal{S}_2^b$  are compact for every  $a, b \in \mathbb{R}_{\geq 0}$ . The proof ideas in this section are motivated by [99, Chp. VII]. So, we define a sequence of times to keep track of the upcrossings of  $V(\mathbf{x}_\omega(t, j))$  through intervals of the form  $[a, b]$ . Define  $\mathbf{R}_0^t(\omega) \equiv 0$  and  $\mathbf{R}_0^j(\omega) \equiv 0$  for every  $\omega \in \Omega$ . Then, for  $c_1 < a < b < c_2$  and  $k \in \mathbb{Z}_{\geq 1}$  define the random variables  $\mathbf{S}_k^t, \mathbf{S}_k^j, \mathbf{R}_k^t$  and  $\mathbf{R}_k^j$  inductively as

$$\begin{aligned}
 \mathbf{S}_k^t(\omega) &:= \inf\{t \in \pi_1(\text{graph}(\mathbf{x}(\omega))) \cap (\mathbb{R}_{\geq \mathbf{R}_{k-1}^t(\omega)} \times \mathbb{Z}_{\geq 0} \times \mathcal{S}_1^a)\} \\
 \mathbf{S}_k^j(\omega) &:= \inf\{j \in \pi_2(\text{graph}(\mathbf{x}(\omega))) \cap (\mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq \mathbf{R}_{k-1}^j(\omega)} \times \mathcal{S}_1^a)\} \\
 \mathbf{R}_k^t(\omega) &:= \inf\{t \in \pi_1(\text{graph}(\mathbf{x}(\omega))) \cap (\mathbb{R}_{\geq \mathbf{S}_k^t(\omega)} \times \mathbb{Z}_{\geq 0} \times \mathcal{S}_2^b)\} \\
 \mathbf{R}_k^j(\omega) &:= \inf\{j \in \pi_2(\text{graph}(\mathbf{x}(\omega))) \cap (\mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq \mathbf{S}_k^j(\omega)} \times \mathcal{S}_2^b)\}.
 \end{aligned}$$

The superscripts  $t, j$  are used to indicate the flow time and jump time respectively and we assign the value  $\infty$  to the variables if the intersection is empty. For  $c_1 < a < b < c_2$ , let  $\mathbf{U}_{[a,b]}(\tau, \omega)$  denote the number of upcrossings within hybrid time  $\tau \in \mathbb{R}_{\geq 0}$  for  $\mathbf{x}(\omega)$ . The number of upcrossings  $\mathbf{U}_{[a,b]}(\tau, \omega)$  denotes the number of times in which  $\mathbf{x}(\omega)$  reaches the set  $\mathcal{S}_2^b$  starting from  $\mathcal{S}_1^a$  within hybrid time  $\tau$ . An equivalent characterization is given by

$$\mathbf{U}_{[a,b]}(\tau, \omega) = \max\{k : \mathbf{R}_k^t(\omega) + \mathbf{R}_k^j(\omega) \leq \tau\}. \tag{D.38}$$

For simplicity, we will suppress the dependence of the number of upcrossings on the function  $V$  and the random solution  $\mathbf{x}$ . Let  $\mathbf{U}_{[a,b]}(\infty, \omega)$  denote the number of upcrossings in the limit as hybrid time  $\tau$  tends to  $\infty$ . So  $\mathbf{U}_{[a,b]}(\infty, \omega) := \lim_{\tau \rightarrow \infty} \mathbf{U}_{[a,b]}(\tau, \omega)$ . The limit is well defined (although may not be finite) since the mapping  $\tau \mapsto \mathbf{U}_{[a,b]}(\tau, \omega)$  is monotone with respect to  $\tau$ . We also note that the number of upcrossings  $\mathbf{U}_{[a,b]}(\infty, \omega) = \infty$  only if for each  $k \in \mathbb{Z}_{\geq 1}$ ,  $\mathbf{R}_k^t(\omega) + \mathbf{R}_k^j(\omega) < \infty$  and  $\lim_{k \rightarrow \infty} (\mathbf{R}_k^t(\omega) + \mathbf{R}_k^j(\omega)) = \infty$ . Hence, only complete sample paths can achieve  $\mathbf{U}_{[a,b]}(\infty, \omega) = \infty$ . We now establish that the times related to the upcrossings are random variables.

**Lemma D.8** *For every  $k \in \mathbb{Z}_{\geq 1}$ ,  $\tau \geq 0$  and  $c_1 < a < b < c_2$ , the mappings  $\omega \mapsto \mathbf{S}_k^t(\omega), \omega \mapsto \mathbf{S}_k^j(\omega), \omega \mapsto \mathbf{R}_k^t(\omega), \omega \mapsto \mathbf{R}_k^j(\omega), \omega \mapsto \mathbf{U}_{[a,b]}(\tau, \omega), \omega \mapsto \mathbf{U}_{[a,b]}(\infty, \omega)$  are  $\mathcal{F}$ -*

measurable.

*Proof:* We first observe that  $\mathcal{F}$ -measurability of  $\mathbf{S}_1^t, \mathbf{S}_1^j$  follows from [25, Prop 2.1, (2e), (2f)]. Next we note that if  $\mathbf{h} : \Omega \rightarrow \mathbb{R}_{\geq 0}$  is  $\mathcal{F}$ -measurable then the set valued mapping  $\mathbf{H}(\omega) := \mathbb{R}_{\geq \mathbf{h}(\omega)}$  is also  $\mathcal{F}$ -measurable. This follows from [84, Thm 14.13(a)](with  $\mathbf{H} = \mathbf{M} \circ \mathbf{h}$ , with  $\mathbf{M}(r) = [r, \infty)$  a continuous set valued mapping). The  $\mathcal{F}$ -measurability of  $\mathbf{R}_1^t, \mathbf{R}_1^j$  now follows from [84, Prop 14.11 (a),(d)] and [25, Prop 2.1]. The proof then proceeds by iteration over  $k > 1$  on the mappings  $\omega \mapsto \mathbf{S}_k^t(\omega), \omega \mapsto \mathbf{S}_k^j(\omega), \omega \mapsto \mathbf{R}_k^t(\omega), \omega \mapsto \mathbf{R}_k^j(\omega)$ .

From [84, Prop 14.11 (c)] we know that for every  $k \in \mathbb{Z}_{\geq 1}$ , the mapping  $\omega \mapsto \hat{\mathbf{R}}_k(\omega) := \mathbf{R}_k^t(\omega) + \mathbf{R}_k^j(\omega)$  is  $\mathcal{F}$ -measurable. Then, we define a set valued mapping  $\mathbf{C}^\tau(\omega) := \{k : \hat{\mathbf{R}}_k(\omega) \leq \tau\}$ . By definition  $\mathbf{C}$  has closed values and is measurable for every  $\tau \geq 0$ . Then define the function  $f^\tau(k, \omega) := -k + \delta_{\mathbf{C}^\tau(\omega)}(k)$  where  $\delta_{\mathbf{C}^\tau(\omega)}(k) = 0$  if  $k \in \mathbf{C}(\omega)$  and  $\delta_{\mathbf{C}^\tau(\omega)}(k) = \infty$  otherwise. Then  $f^\tau$  is normal integrand from [84, Example 14.32] and  $\mathbf{U}(\tau, \omega) = \operatorname{argmin} f^\tau(\cdot, \omega)$ . Then,  $\mathcal{F}$ -measurability of  $\mathbf{U}(\tau, \omega)$  then follows from [84, Thm 14.37]. Without loss of generality we can consider  $\mathbf{U}_{[a,b]}(\infty, \omega) = \lim_{i \rightarrow \infty} \mathbf{U}_{[a,b]}(i, \omega)$  where  $i \in \mathbb{Z}_{\geq 0}$ . Since  $\omega \mapsto \mathbf{U}_{[a,b]}(i, \omega)$  is measurable for each  $i$  and the limit of sequence of measurable functions is measurable ([98, § 2.3, Corollary 12]), it follows that  $\omega \mapsto \mathbf{U}_{[a,b]}(\infty, \omega)$  is  $\mathcal{F}$ -measurable. ■

The next result is a hybrid version of Doob's optional stopping theorem [99, Chp VII, Thm 2.2].

**Proposition D.6** *If  $\mathbf{x}$  is a random solution that is almost surely contained in  $\Lambda$ ,  $V$  is a stochastic Krasovskii-LaSalle function with respect to  $(K, \Lambda)$ ,  $\mathbf{T}_i : \Omega \rightarrow \mathbb{R}_{\geq 0}$ ,  $\mathbf{S}_i : \Omega \rightarrow \mathbb{Z}_{\geq 0}$  are  $\mathcal{F}$ -measurable for each  $i \in \{1, 2\}$ ,  $\mathbf{S}_1, \mathbf{S}_2$  are stopping times with respect to  $\{\mathcal{F}_i\}_{i \in \mathbb{Z}_{\geq 0}}$ ,  $\mathbf{S}_2 \leq n$  almost surely for some  $n \in \mathbb{Z}_{\geq 0}$ ,  $\mathbf{T}_1 \leq \mathbf{T}_2$ ,  $\mathbf{S}_1 \leq \mathbf{S}_2$  almost surely and  $(\mathbf{T}_i(\omega), \mathbf{S}_i(\omega)) \in \operatorname{dom}(\mathbf{x}(\omega))$  for each  $i \in \{1, 2\}$  then  $\mathbb{E}[V(\mathbf{x}(\mathbf{T}_2, \mathbf{S}_2))] \leq \mathbb{E}[V(\mathbf{x}(\mathbf{T}_1, \mathbf{S}_1))]$ .*

*Proof:* For  $i \in \mathbb{Z}_{[0,n]}$ , define  $\hat{\mathbf{t}}_i(\omega) := \inf\{t \in \pi_1(\text{graph}(\mathbf{x}(\omega)) \cap (\mathbb{R}_{\geq 0} \times \{i\} \times \mathbb{R}^n))\}$ . Then, let  $\mathbf{t}_i(\omega) = \hat{\mathbf{t}}_i(\omega)\mathbb{I}_{\Omega_{i_1}}(\omega) + \mathbf{T}_2(\omega)\mathbb{I}_{\Omega_{i_2}}(\omega) + \mathbf{T}_1(\omega)\mathbb{I}_{\Omega_{i_3}}(\omega)$  where  $\Omega_{i_1} := \{\omega : \mathbf{S}_1(\omega) < i \leq \mathbf{S}_2(\omega)\}$ ,  $\Omega_{i_2} := \{\omega : i > \mathbf{S}_2(\omega)\}$  and  $\Omega_{i_3} := \{\omega : i \leq \mathbf{S}_1(\omega)\}$ . It follows from measurability of  $\mathbf{S}_2, \mathbf{S}_1$  that  $\Omega_{i_1}, \Omega_{i_2}$  and  $\Omega_{i_3}$  are measurable sets and since the indicator of measurable set is measurable ([98, § 2.2, Corollary 10]), we have that  $\mathbf{t}_i$  is  $\mathcal{F}$ -measurable for each  $i \in \mathbb{Z}_{[0,n]}$ . We observe that since we have almost sure non-increase of  $V$  during flows,

$$\begin{aligned} V(\mathbf{x}(\mathbf{T}_2, \mathbf{S}_2)) &= V(\mathbf{x}(\mathbf{T}_1, \mathbf{S}_1)) + \sum_{j=\mathbf{S}_1}^{\mathbf{S}_2} (V(\mathbf{x}(\mathbf{t}_{j+1}, j)) - V(\mathbf{x}(\mathbf{t}_j, j))) \\ &\quad + \sum_{j=\mathbf{S}_1}^{\mathbf{S}_2-1} (V(\mathbf{x}(\mathbf{t}_{j+1}, j+1)) - V(\mathbf{x}(\mathbf{t}_{j+1}, j))) \\ &\leq V(\mathbf{x}(\mathbf{T}_1, \mathbf{S}_1)) + \sum_{j=\mathbf{S}_1}^{\mathbf{S}_2-1} \left[ V(\mathbf{x}(\mathbf{t}_{j+1}, j+1)) - V(\mathbf{x}(\mathbf{t}_{j+1}, j)) \right] \\ &\leq V(\mathbf{x}(\mathbf{T}_1, \mathbf{S}_1)) \\ &\quad + \sum_{j=0}^n \left[ V(\mathbf{x}(\mathbf{t}_{j+1}, j+1)) - V(\mathbf{x}(\mathbf{t}_{j+1}, j)) \right] \mathbb{I}_{[0,j]}(\mathbf{S}_1) \mathbb{I}_{(j,n]}(\mathbf{S}_2). \end{aligned}$$

Let  $\mathcal{F}_{\mathbf{S}_1}$  be the sigma algebra generated by  $\mathbf{S}_1$ . Then, taking expectations conditioned on  $\mathcal{F}_{\mathbf{S}_1}$  on both sides and using the stopping time property of  $\mathbf{S}_1, \mathbf{S}_2$  and non-increase of



$V$  on average during jumps we have

$$\begin{aligned}
 \mathbb{E}[V(\mathbf{x}(\mathbf{T}_2, \mathbf{S}_2)) | \mathcal{F}_{\mathbf{S}_1}] &\leq \mathbb{E}[V(\mathbf{x}(\mathbf{T}_1, \mathbf{S}_1)) | \mathcal{F}_{\mathbf{S}_1}] \\
 &\quad + \sum_{j=0}^n \left[ \mathbb{E}[V(\mathbf{x}(\mathbf{t}_{j+1}, j+1)) \right. \\
 &\quad \quad \left. - V(\mathbf{x}(\mathbf{t}_{j+1}, j)) \mathbb{I}_{[0,j]}(\mathbf{S}_1) \mathbb{I}_{(j,n]}(\mathbf{S}_2) | \mathcal{F}_{\mathbf{S}_1}] \right] \\
 &\leq \mathbb{E}[V(\mathbf{x}(\mathbf{T}_1, \mathbf{S}_1)) | \mathcal{F}_{\mathbf{S}_1}] \\
 &\quad + \sum_{j=0}^n \left[ \mathbb{I}_{[0,j]}(\mathbf{S}_1) \mathbb{E}[V(\mathbf{x}(\mathbf{t}_{j+1}, j+1)) \right. \\
 &\quad \quad \left. - V(\mathbf{x}(\mathbf{t}_{j+1}, j)) \mathbb{I}_{(j,n]}(\mathbf{S}_2) | \mathcal{F}_j] \right] \\
 &\leq \mathbb{E}[V(\mathbf{x}(\mathbf{T}_1, \mathbf{S}_1)) | \mathcal{F}_{\mathbf{S}_1}] \\
 &\quad + \sum_{j=0}^n \left[ \mathbb{I}_{[0,j]}(\mathbf{S}_1) \mathbb{I}_{(j,n]}(\mathbf{S}_2) \mathbb{E}[V(\mathbf{x}(\mathbf{t}_{j+1}, j+1)) \right. \\
 &\quad \quad \left. - V(\mathbf{x}(\mathbf{t}_{j+1}, j)) | \mathcal{F}_j] \right] \\
 &\leq \mathbb{E}[V(\mathbf{x}(\mathbf{T}_1, \mathbf{S}_1)) | \mathcal{F}_{\mathbf{S}_1}].
 \end{aligned}$$

Taking expectations on both sides, we get  $\mathbb{E}[V(\mathbf{x}(\mathbf{T}_2, \mathbf{S}_2))] \leq \mathbb{E}[V(\mathbf{x}(\mathbf{T}_1, \mathbf{S}_1))]$ . ■

Next we establish a hybrid version of Doob's upcrossing lemma similar to [99, Chp VII, Thm 3.3] and [98, Chp 24, Lemma 18]. In order to prove Doob's upcrossing lemma we first prove that the assumptions of Proposition D.6 are satisfied by certain random variables related to the upcrossing times. For every  $i \in \mathbb{Z}_{\geq 0}$  and  $k \in \mathbb{Z}_{\geq 1}$  define

$$\begin{aligned}
 \mathbf{S}_k^t(i, \omega) &:= \min\{\mathbf{S}_k^t(\omega), \mathbf{t}_i(\omega)\}, \quad \mathbf{S}_k^j(i, \omega) := \min\{\mathbf{S}_k^j(\omega), \mathbf{j}_i(\omega)\} \\
 \mathbf{R}_k^t(i, \omega) &:= \min\{\mathbf{R}_k^t(\omega), \mathbf{t}_i(\omega)\}, \quad \mathbf{R}_k^j(i, \omega) := \min\{\mathbf{R}_k^j(\omega), \mathbf{j}_i(\omega)\}.
 \end{aligned}$$

where  $\mathbf{t}_i, \mathbf{j}_i$  are from (D.36).

**Lemma D.9** *For every  $i \in \mathbb{Z}_{\geq 0}$  and  $k \in \mathbb{Z}_{\geq 1}$ ,  $\mathbf{S}_k^t(i, \omega) \leq \mathbf{R}_k^t(i, \omega)$  and  $\mathbf{S}_k^j(i, \omega) \leq \mathbf{R}_k^j(i, \omega)$  almost surely, the random variables  $\mathbf{S}_k^j(i, \omega), \mathbf{R}_k^j(i, \omega)$  are stopping times and  $\mathbf{R}_k^j(i, \omega) \leq i$  almost surely.*

*Proof:* It follows from the definition that for every  $i \in \mathbb{Z}_{\geq 0}$   $\mathbf{S}_k^t(i, \omega) \leq \mathbf{R}_k^t(i, \omega)$ ,  $\mathbf{S}_k^j(i, \omega) \leq \mathbf{R}_k^j(i, \omega)$  and  $\mathbf{R}_k^j(i, \omega) \leq i$  almost surely. Now fix  $i \in \mathbb{Z}_{\geq 0}$ . We claim that for every  $k \in \mathbb{Z}_{\geq 1}$   $\mathbf{S}_k^j(i, \omega), \mathbf{R}_k^j(i, \omega)$  are stopping times. We establish the claim for  $k = 1$  and the rest of the proof follows by iteration. We first note that  $\mathbf{j}_i, \mathbf{S}_1^j$  are stopping times with respect to the filtration  $\{\mathcal{F}_n\}_{n \in \mathbb{Z}_{\geq 0}}$  since the event  $\mathbf{j}_i = n$  and  $\mathbf{S}_1^j = n$  depend only on the mapping  $\omega \mapsto \text{graph}(\mathbf{x}(\omega))_{\leq n}$  which is  $\mathcal{F}_n$  measurable. The same argument then applies to the event  $\mathbf{R}_1^j = n$  with the additional constraint that  $\mathbf{S}_1^j \leq n$  (which is  $\mathcal{F}_n$  measurable since  $\mathbf{S}_1^j$  is a stopping time). Then, since the minimum of two stopping times is also a stopping time ([98, § 11.3, Prop 6]) the proof follows.  $\blacksquare$

**Lemma D.10** *Let  $\mathbf{x}$  be a random solution that is almost surely contained in  $\Lambda$ ,  $V$  a stochastic Krasovskii-LaSalle function with respect to  $(K, \Lambda)$ . Then, for every  $c_1 < a < b < c_2$ ,  $\mathbb{E}[\mathbf{U}_{[a,b]}(\infty, \cdot)] < \infty$ .*

*Proof:* For  $N \in \mathbb{Z}_{\geq 1}$ ,  $i \in \mathbb{Z}_{\geq 0}$ , define  $\mathbf{U}_{[a,b]}(i, N, \omega) := \min\{\mathbf{U}_{[a,b]}(i, \omega), N\}$ . Then, we have that  $\mathbf{U}_{[a,b]}(i, N, \cdot)$  is  $\mathcal{F}$ -measurable. Define

$$\mathbf{y}(i, N, \omega) := \sum_{k=1}^N [V(\mathbf{x}(\mathbf{R}_k^t(i, \omega), \mathbf{R}_k^j(i, \omega))) - V(\mathbf{x}(\mathbf{S}_k^t(i, \omega), \mathbf{S}_k^j(i, \omega)))]. \quad (\text{D.39})$$

The function  $\mathbf{y}$  is well defined from the continuity of  $V$  and  $\mathcal{F}$ -measurability of  $\mathbf{y}$  follows from the definition. We now observe from Lemma D.9 and Proposition D.6 that  $\mathbb{E}[\mathbf{y}(i, N, \cdot)] \leq 0$ . The function  $\mathbf{y}$  is used to keep track of the number of upcrossings. When an upcrossing is complete the difference of the term in the summation is greater than  $b - a$ , since there are  $N$  terms there can a maximum of  $\mathbf{U}_{[a,b]}(i, N, \omega)$  upcrossings

possible and there can be at most one uncompleted upcrossing. Then, it follows from the construction that

$$\begin{aligned}
 a - c_1 &\geq \mathbb{E}[\mathbf{y}(i, N, \cdot)] + a - c_1 \\
 &\geq (b - a)\mathbb{E}[\mathbf{U}_{[a,b]}(i, N, \cdot)] + \mathbb{E}[V(\mathbf{x}(\mathbf{t}_i, \mathbf{j}_i))] - c_1 \\
 &\geq (b - a)\mathbb{E}[\mathbf{U}_{[a,b]}(i, N, \cdot)].
 \end{aligned}$$

We note that for every  $i \in \mathbb{Z}_{\geq 0}$ , and almost every  $\omega \in \Omega$ ,  $\mathbf{U}_{[a,b]}(i, N, \omega) \leq \mathbf{U}_{[a,b]}(i, N + 1, \omega)$  by definition and  $\lim_{N \rightarrow \infty} \mathbf{U}_{[a,b]}(i, N, \omega) = \mathbf{U}_{[a,b]}(i, \omega)$ . Then it follows from the monotone convergence theorem ([98, § 4.3, Thm 11]) that  $\lim_{N \rightarrow \infty} \mathbb{E}[\mathbf{U}_{[a,b]}(i, N, \cdot)] = \mathbb{E}[\lim_{N \rightarrow \infty} \mathbf{U}_{[a,b]}(i, N, \cdot)] = \mathbb{E}[\mathbf{U}_{[a,b]}(i, \cdot)]$ . Using the monotone convergence theorem again we establish that

$$(b - a)\mathbb{E}[\mathbf{U}_{[a,b]}(\infty, \cdot)] = (b - a) \lim_{i \rightarrow \infty} \mathbb{E}[\mathbf{U}_{[a,b]}(i, \cdot)] \leq a - c_1.$$

■

Finally, we establish have almost sure convergence of complete sample paths to the level set of the Krasovskii-LaSalle function  $V$ .

**Lemma D.11** *If  $\mathbb{E}[\mathbf{U}_{[a,b]}(\infty, \cdot)] < \infty$  for every  $c_1 < a < b < c_2$ , then almost every complete sample path of  $\mathbf{x}$  converges to a level set of  $V$ .*

*Proof:* Since  $\mathbb{E}[\mathbf{U}_{[a,b]}(\infty, \cdot)] < \infty$  for every  $c_1 < a < b < c_2$ , it follows that  $\mathbb{P}(\mathbf{U}_{[a,b]}(\infty, \omega) = \infty) = 0$  for every  $c_1 < a < b < c_2$ . Let  $\Omega_c$  denote the set of  $\omega \in \Omega$  for which sample paths of  $\mathbf{x}$  are complete. We can establish that  $\Omega_c \in \mathcal{F}$  similar to [61, Prop 2]. Let  $\Omega^* := \{\omega \in \Omega : \lim_{t+j \rightarrow \infty} V(\mathbf{x}_\omega(t, j)) \text{ exists}\}$ . Then  $\Omega^* \subset \Omega_c$ .

We first show that if  $\lim_{t+j \rightarrow \infty} V(\mathbf{x}_\omega(t, j))$  does not exist for some  $\omega^* \in \Omega_c$ , then necessarily for some  $c_1 < a < b < c_2$ , we have  $\mathbf{U}_{[a,b]}(\infty, \omega^*) = \infty$ . Since we do not

have convergence, it follows that  $\liminf_{t+j \rightarrow \infty} V(\mathbf{x}_{\omega^*}(t, j)) < \limsup_{t+j \rightarrow \infty} V(\mathbf{x}_{\omega^*}(t, j))$ . Since  $\mathbb{Q}$  is dense in  $\mathbb{R}$  there exists  $a, b \in \mathbb{Q}$  such that  $\liminf_{t+j \rightarrow \infty} V(\mathbf{x}_{\omega^*}(t, j)) < a < b < \limsup_{t+j \rightarrow \infty} V(\mathbf{x}_{\omega^*}(t, j))$ . Then we can define a sequence of upcrossing times with respect to such  $a, b$  and from the definition of  $\limsup, \liminf$ , we have that for  $k \in \mathbb{Z}_{\geq 1}$ ,  $\mathbf{R}_k^t(\omega^*) + \mathbf{R}_k^j(\omega^*) < \infty$  and  $\lim_{k \rightarrow \infty} (\mathbf{R}_k^t(\omega^*) + \mathbf{R}_k^j(\omega^*)) = \infty$ . Hence, we have that  $\mathbf{U}_{[a,b]}(\infty, \omega^*) = \infty$ . Similarly, we can conclude that if  $\lim_{t+j \rightarrow \infty} V(\mathbf{x}_{\omega}(t, j))$  exists then necessarily  $\mathbf{U}_{[a,b]}(\infty, \omega) < \infty$  for every  $a < b$ .

Let  $\hat{\Omega}_{a,b} := \{\omega \in \Omega_c : \mathbf{U}_{[a,b]}(\infty, \omega) < \infty\}$ . Then,  $\hat{\Omega}_{a,b} \in \mathcal{F}$  and  $\Omega^* = \bigcap_{a,b \in \mathbb{Q} \cap (c_1, c_2), a < b} \hat{\Omega}_{a,b}$  and  $\Omega^* \in \mathcal{F}$ . We now show that  $\mathbb{P}(\Omega_c \setminus \Omega^*) = 0$ . Let  $\Omega_{a,b} := \{\omega \in \Omega_c : \mathbf{U}_{[a,b]}(\infty, \omega) = \infty\}$  where  $c_1 < a < b < c_2$ . Then  $\Omega_{a,b} \in \mathcal{F}$ . Then it follows that  $\Omega_c \setminus \Omega^* \subset \bigcup_{a,b \in \mathbb{Q} \cap (c_1, c_2), a < b} \Omega_{a,b}$ . Since  $\mathbb{P}(\mathbf{U}_{[a,b]}(\infty, \omega) = \infty) = 0$  for every  $c_1 < a < b < c_2$ , it follows that  $\mathbb{P}(\Omega_{a,b}) = 0$ . Then since  $\mathbb{Q}$  is countable we have that  $\mathbb{P}(\Omega_c \setminus \Omega^*) \leq \sum_{a,b \in \mathbb{Q} \cap (c_1, c_2), a < b} \mathbb{P}(\Omega_{a,b}) = 0$ . Then, from the continuity of  $V$  it follows that for almost every  $\omega \in \Omega_c$ , there exists  $c(\omega) \in [c_1, c_2]$  such that  $\lim_{t+j \rightarrow \infty} |\mathbf{x}_{\omega}(t, j)|_{L^V(c(\omega))} = 0$ . ■

### D.13.3 Convergence to largest weakly totally recurrent in probability sets inside level sets

The proofs presented in this section are an extension of the results established in [61, Section X.D] for a class of stochastic difference inclusions.

**Lemma D.12** *Let the compact sets  $K_1, K_2 \subset \mathbb{R}^n$  satisfy  $K_1 \subset K_2$  and let  $\tau > 0$ . For the solution  $\mathbf{x}$ , let  $\Omega_a$  denote the set of  $\omega \in \Omega$  such that  $\mathbf{x}(\omega)$  is complete, belongs to  $K_2$  for all time in its domain, and belongs to  $K_1$  for those times in its domain that are greater than  $\tau$ . Let  $\Omega_b \subset \Omega_a$  be those  $\omega \in \Omega_a$  for which  $\mathbf{x}(\omega)$  converges to the largest weakly totally recurrent in probability set contained in  $K_1$ . Then  $\mathbb{P}(\Omega_a) = \mathbb{P}(\Omega_b)$ .*

*Proof:* If  $\mathbb{P}(\Omega_a) = 0$  the statement of the lemma holds trivially. Thus we assume that  $\mathbb{P}(\Omega_a) > 0$ . Define a new solution  $\mathbf{z}$  from the original solution  $\mathbf{x}$  by truncating  $\mathbf{x}(\omega)$  to  $K_2$ , like in [25, Prop 2.1, (2d)], at the infimum over times such that the intersection of the graph of  $\mathbf{x}(\omega)$  with the open set

$$(\mathbb{R}^2 \times (\mathbb{R}^n \setminus K_2)) \cup (\Gamma_{>\tau} \times (\mathbb{R}^n \setminus K_1))$$

is nonempty. This truncation produces a mapping that satisfies the conditions for a solution. Moreover,  $\mathbf{z}(\omega)$  is complete if and only if  $\omega \in \Omega_a$ . In fact,  $\mathbf{z}$  has the properties assumed in Theorem 5.1 with  $K_\infty = K_1$ . It follows from Theorem 5.1 that the recurrent in probability set for  $\mathbf{z}$ , denoted  $\Psi(\mathbf{z})$ , is nonempty, compact, contained in  $K_1$ , weakly totally recurrent in probability, and almost every complete sample path of  $\mathbf{z}$  converges to  $\Psi(\mathbf{z})$  and thus to the largest weakly totally recurrent set contained in  $K_1$ . Since  $\mathbf{z}(\omega)$  is complete for all  $\omega \in \Omega_a$ , it follows  $\mathbb{P}(\Omega_a) = \mathbb{P}(\Omega_b)$ . ■

The next result relies on the previous lemma.

**Lemma D.13** *Let the compact sets  $K_\infty, \hat{K} \subset \mathbb{R}^n$  satisfy  $K_\infty \subset \hat{K}$ . For the solution  $\mathbf{x}$ , let  $\Omega_a$  denote the set of  $\omega \in \Omega$  for which  $\mathbf{x}(\omega)$  is complete, remains in  $\hat{K}$  for all time, and converges to  $K_\infty$ ; let  $\Omega_b \subset \Omega_a$  denote the set of  $\omega \in \Omega_a$  for which  $\mathbf{x}(\omega)$  converges to the largest weakly totally recurrent in probability set contained in  $K_\infty$ . Then  $\mathbb{P}(\Omega_a) = \mathbb{P}(\Omega_b)$ .*

*Proof:* If  $\mathbb{P}(\Omega_a) = 0$  the statement of the lemma holds trivially. Thus, we assume that  $\mathbb{P}(\Omega_a) > 0$ . For each  $i \in \mathbb{Z}_{\geq 1}$ , let  $\Psi_i$  denote the largest weakly totally recurrent in probability set contained in  $K_\infty + i^{-1}\mathbb{B}$ . Due to Lemma D.12 and the assumption that the probability of converging to  $K_\infty$  while remaining in  $\hat{K}$  is positive, it follows that  $\Psi_i$  is non-empty for each  $i \in \mathbb{Z}_{\geq 1}$ . Moreover, by construction,  $\Psi_j \subset \Psi_i$  for all  $j \geq i$ . Thus,  $\Psi := \lim_{i \rightarrow \infty} \Psi_i$  is well-defined, nonempty, compact, and contained in  $K_\infty$ . We also claim that it is weakly totally recurrent in probability. Indeed, by [84, Theorem 4.10(a),(b)],

for each  $\varepsilon > 0$  and  $x \in \Psi$  there exist  $i^* \in \mathbb{Z}_{\geq 1}$  and  $\{x_i\}_{i=1}^{\infty}$  with  $x_i \in \Psi_i$  for all  $i \in \mathbb{Z}_{\geq 1}$  such that

$$\Psi_i + 0.5\varepsilon\mathbb{B} \subset \Psi + \varepsilon\mathbb{B} \quad \forall i \geq i^* \tag{D.40a}$$

$$\{x_i\} + 0.5\varepsilon\mathbb{B} \subset \{x\} + \varepsilon\mathbb{B} \quad \forall i \geq i^* \tag{D.40b}$$

the latter following from  $x \in \{x_i\} + 0.5\varepsilon\mathbb{B}$  for all  $i \geq i^*$ . Thus, weak total recurrence of  $\Psi$  follows from weak total recurrence of  $\Psi_{i^*}$ . We let  $\widehat{\Psi}$  denote the largest weakly totally recurrent in probability set contained in  $K_{\infty}$ .

For each  $\varepsilon > 0$ , let  $\Omega_{\varepsilon,\tau,a} \subset \Omega$  denote the set of  $\omega \in \Omega$  for which  $\mathbf{x}(\omega)$  is complete, remains in  $\widehat{K}$ , and belongs to  $K_{\infty} + \varepsilon\mathbb{B}$  for all time greater than  $\tau > 0$ . By construction,

$$\Omega_a = \bigcap_{i=1}^{\infty} \bigcup_{j=1}^{\infty} \Omega_{1/i,j,a} . \tag{D.41}$$

Let  $\Omega_{1/i,j,b} \subset \Omega_{1/i,j,a}$  denote the subset of  $\omega \in \Omega_{1/i,j,a}$  for which  $\mathbf{x}(\omega)$  converges to  $\Psi_i$ , i.e., the largest weakly totally recurrent in probability set contained in  $K_{\infty} + i^{-1}\mathbb{B}$ . By Lemma D.12,

$$\mathbb{P}(\Omega_{1/i,j,b}) = \mathbb{P}(\Omega_{1/i,j,a}) \quad \forall (i, j) \in \mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 1}. \tag{D.42}$$

Define

$$\widehat{\Omega}_b := \bigcap_{i=1}^{\infty} \bigcup_{j=1}^{\infty} \Omega_{1/i,j,b}. \tag{D.43}$$

For each  $\omega \in \widehat{\Omega}_b$  let  $j_i$  be such that  $\omega \in \Omega_{1/i,j_i,b}$  for all  $i \in \mathbb{Z}_{\geq 1}$ . Then  $\mathbf{x}(\omega)$  converges to  $\Psi_i$  for each  $i \in \mathbb{Z}_{\geq 1}$  and, in turn, to  $\Psi \subset \widehat{\Psi}$ . It follows that  $\omega \in \Omega_b$ ; in other words

$\widehat{\Omega}_b \subset \Omega_b$ . Finally, using (D.41)-(D.43),

$$\begin{aligned} \mathbb{P}(\Omega_b) &\geq \mathbb{P}(\widehat{\Omega}_b) = \mathbb{P}\left(\bigcap_{i=1}^{\infty} \bigcup_{j=1}^{\infty} \Omega_{1/i,j,b}\right) \\ &= \lim_{i \rightarrow \infty} \lim_{j \rightarrow \infty} \mathbb{P}(\Omega_{1/i,j,b}) = \lim_{i \rightarrow \infty} \lim_{j \rightarrow \infty} \mathbb{P}(\Omega_{1/i,j,a}) \\ &= \mathbb{P}(\Omega_a) \end{aligned}$$

which establishes the result. ■

**Theorem D.1** *Let  $\mathbf{x}$  be a random solution that is almost surely contained in the compact set  $\Lambda$ . Then, almost every complete sample path of  $\mathbf{x}$  that converges to a level set of  $V$  converges to a level set that contains a weakly totally recurrent in probability set and converges to the largest weakly totally recurrent in probability set contained in the level set.*

*Proof:* Let  $\mathbb{Q}$  denote the rational numbers. For each  $(q, j) \in \mathbb{Q} \times \mathbb{Z}_{\geq 1}$ , define

$$S_{qj} := \{x \in \Lambda : V(x) \in \{q\} + j^{-1}\mathbb{B}\} \tag{D.44a}$$

$$\mu_{qj} := \mathbb{P}\left(\lim_{\tau+k \rightarrow \infty} |\mathbf{x}(\tau, k)|_{S_{qj}} = 0\right). \tag{D.44b}$$

Observe that, for each  $q \in \mathbb{Q}$ , and  $j \leq k$ ,  $S_{qk} \subset S_{qj}$  so that  $j \mapsto \mu_{qj}$  is non-increasing.

Let  $\mathcal{I}$  denote those  $c \in [c_1, c_2]$  with the following property:

- (P) there exists a sequence  $\{(q_i, j_i)\}_{i=0}^{\infty}$  with  $(q_i, j_i) \in \mathbb{Q} \times \mathbb{Z}_{\geq 1}$  for all  $i \in \mathbb{Z}_{\geq 0}$  such that  $\lim_{i \rightarrow \infty} q_i = c$ ,  $\lim_{i \rightarrow \infty} j_i = \infty$ , and  $\mu_{q_i j_i} > 0$  for all  $i \in \mathbb{Z}_{\geq 0}$ .

**Claim D.4** *The set  $\mathcal{I} \subset [c_1, c_2]$  is compact.*

*Proof:* The set  $\mathcal{I}$  is bounded since it is a subset of  $[c_1, c_2]$ . To see that it is closed, suppose  $c_k \in \mathcal{I}$  for all  $k \in \mathbb{Z}_{\geq 0}$  and  $\lim_{k \rightarrow \infty} c_k = c$ . Necessarily  $c \in [c_1, c_2]$ . For each

$k \in \mathbb{Z}_{\geq 0}$  let the sequence  $\{(q_{i,k}, j_{i,k})\}_{i=0}^{\infty}$  verify property (P) for  $c_k$ . For each  $k \in \mathbb{Z}_{\geq 0}$  let  $i_k$  be sufficiently large so that  $|q_{i_k,k} - c_k| \leq k^{-1}$  and  $j_{i_k} \geq k$ . It is now straightforward to verify that the sequence  $\{(q_{i_k,k}, j_{i_k,k})\}_{k=0}^{\infty}$  verifies property (P) for  $c$ .  $\blacksquare$

Given a compact set  $\mathcal{J} \subset \mathbb{R}$ , Let  $V^{-1}(\mathcal{J}) := \{x \in \Lambda : V(x) \in \mathcal{J}\}$ . Let  $\Omega_a \subset \Omega$  denote the set of  $\omega \in \Omega$  for which the corresponding sample path is complete and converges to a level set of  $V$ . Let  $\Omega_b \subset \Omega_a$  denote the set of  $\omega \in \Omega_a$  for which  $\lim_{t+j \rightarrow \infty} |\mathbf{x}_\omega(t, j)|_{V^{-1}(\mathcal{I})} = 0$ .

**Claim D.5**  $\mathbb{P}(\Omega_b) = \mathbb{P}(\Omega_a)$ .

*Proof:* Let  $\{K_i\}_{i=0}^{\infty}$  be a nested sequence of closed subsets of  $\mathbb{R} \setminus \mathcal{I}$  that cover the open set  $\mathbb{R} \setminus \mathcal{I}$ . We will prove that, with  $\Omega_i \subset \Omega_a$  denoting the set of  $\omega \in \Omega_a$  for which  $\lim_{t+j \rightarrow \infty} |\mathbf{x}_\omega(t, j)|_{V^{-1}([c_1, c_2] \cap K_i)} = 0$ , that  $\mathbb{P}(\Omega_i) = 0$ . The result then follows from the fact that

$$\mathbb{P}(\Omega_b) = \mathbb{P}(\Omega_a) - \lim_{i \rightarrow \infty} \mathbb{P}(\Omega_i). \quad (\text{D.45})$$

For each  $c \in [c_1, c_2] \setminus \mathcal{I}$ , define  $q(c) \in \mathbb{Q}$  and  $j(c) \in \mathbb{Z}_{\geq 1}$  as follows: let  $\{(q_i, j_i)\}_{i=1}^{\infty}$  with  $(q_i, j_i) \in \mathbb{Q} \times \mathbb{Z}_{\geq 1}$  for all  $i \in \mathbb{Z}_{\geq 0}$  be such that  $\lim_{i \rightarrow \infty} q_i = c$  (such a sequence exists since the rational numbers are dense in the set of real numbers),  $\lim_{i \rightarrow \infty} j_i = \infty$ , and  $c \in \{q_i\} + j_i^{-1}\mathbb{B}$  for all  $i \in \mathbb{Z}_{\geq 0}$ , let  $i^* \in \mathbb{Z}_{\geq 0}$  be the smallest nonnegative integer such that  $u_{q_i j_i} = 0$  for all  $i \geq i^*$  (such an integer exists for, otherwise, property (P) would hold as verified by an appropriate subsequence of  $\{(q_i, j_i)\}_{i=1}^{\infty}$ ) and then define  $q(c) := q_{i^*}$  and  $j(c) := j_{i^*}$ . Note that

$$[c_1, c_2] \cap K_i \subset [c_1, c_2] \setminus \mathcal{I} \subset \bigcup_{c \in [c_1, c_2] \setminus \mathcal{I}} (\{q(c)\} + j(c)^{-1}\mathbb{B}). \quad (\text{D.46})$$



Define

$$\mathbb{Q}_0 := \{q \in \mathbb{Q} : q(c) = q \text{ for some } c \in [c_1, c_2] \setminus \mathcal{I}\}. \quad (\text{D.47})$$

For each  $q \in \mathbb{Q}_0$ , define

$$j_q := \min \{j : j = j(c) \text{ for some } c \in [c_1, c_2] \setminus \mathcal{I} \text{ s.t. } q(c) = q\}.$$

It is evident from these definitions and (D.46) that

$$[c_1, c_2] \cap K_i \subset [c_1, c_2] \setminus \mathcal{I} \subset \bigcup_{q \in \mathbb{Q}_0} (\{q\} + j_q^{-1}\mathbb{B}). \quad (\text{D.48})$$

We also claim that

$$\mu_{qj_q} = 0 \quad \forall q \in \mathbb{Q}_0. \quad (\text{D.49})$$

This fact follows from the fact that  $\mu_{q(c)j(c)} = 0$  for each  $c \in [c_1, c_2] \setminus \mathcal{I}$  and the fact that  $j \mapsto \mu_{qj}$  is monotonically non-increasing for each  $q \in \mathbb{Q}$ .

It follows from (D.48), (D.44), and (D.49) that, for each  $i \in \mathbb{Z}_{\geq 0}$ ,

$$\mathbb{P}(\Omega_i) \leq \sum_{q \in \mathbb{Q}_0} \mu_{qj_q} = 0. \quad (\text{D.50})$$

This bound and (D.45) establishes the claim. ■

For each  $c \in \mathcal{I}$ , let  $\Psi_c$  denote the largest weakly totally recurrent in probability set contained in the set  $\{x \in \Lambda : V(x) = c\}$ . Let  $\Omega_0 \subset \Omega_b$  denote the subset of  $\omega \in \Omega_b$  for which  $\lim_{t+j \rightarrow \infty} |\mathbf{x}_\omega(t, j)|_{\Psi_c} = 0$  for some  $c \in \mathcal{I}$ .

**Claim D.6** *The set  $\Omega_0 \in \mathcal{F}$  and  $\mathbb{P}(\Omega_0) = \mathbb{P}(\Omega_b)$ .*

*Proof:* We first establish that  $\Omega_0 \in \mathcal{F}$ . For each  $q \in \mathbb{Q}$  and  $\varepsilon > 0$ , let  $\Psi_{q,\varepsilon}$  denote the largest weakly totally recurrent in probability set contained in the set  $\{x \in \Lambda : V(x) \in \{q\} + \varepsilon\mathbb{B}\}$ . Define  $\Omega_{q,\varepsilon} \subset \Omega_b$  to be the subset of  $\omega \in \Omega_b$  for which  $\lim_{t+j \rightarrow \infty} |\mathbf{x}_\omega(t, j)|_{\Psi_{q,\varepsilon}} = 0$ . For each  $(q, \varepsilon) \in \mathbb{Q} \times \mathbb{R}_{>0}$ , since  $\Psi_{q,\varepsilon}$  is compact it follows that  $\Omega_{q,\varepsilon} \in \mathcal{F}$ . Define  $\Omega_\varepsilon := \cup_{q \in \mathbb{Q}} \Omega_{q,\varepsilon}$ . Since  $\mathbb{Q}$  is countable, it follows that  $\Omega_\varepsilon \in \mathcal{F}$  for each  $\varepsilon > 0$ . It is evident from the definition of  $\Psi_{q,\varepsilon}$  that  $\Omega_{\varepsilon_1} \subset \Omega_{\varepsilon_2}$  for each  $0 < \varepsilon_1 \leq \varepsilon_2$ . The next two paragraphs establish that  $\Omega_0 = \cap_{i=1}^\infty \Omega_{1/i}$ .

First we establish that  $\Omega_0 \subset \cap_{i=1}^\infty \Omega_{1/i}$ . Suppose  $\omega \in \Omega_0$  and let  $c \in \mathcal{I}$  be such that the corresponding sample path converges to  $\Psi_c$ . Let the sequence  $q_k \in \mathbb{Q}$  be such that  $\lim_{k \rightarrow \infty} q_k = c$ . Let the unbounded sequence  $i_k \in \mathbb{Z}_{\geq 1}$  be such that  $c \in \{q_k\} + i_k^{-1}\mathbb{B}$  for all  $k \in \mathbb{Z}_{\geq 1}$ . It follows that  $\Psi_c \subset \Psi_{q_k, i_k^{-1}}$  for all  $k \in \mathbb{Z}_{\geq 1}$ . Therefore,  $\omega \in \Omega_{q_k, i_k^{-1}} \subset \Omega_{i_k^{-1}}$  for all  $k \in \mathbb{Z}_{\geq 1}$ . In other words,  $\omega \in \cap_{k=1}^\infty \Omega_{i_k^{-1}}$ . Since the sequence  $\{i_k\}_{k \in \mathbb{Z}_{\geq 1}}$  is unbounded and the sets  $\Omega_{1/i}$  are nested, it follows that  $\omega \in \cap_{i=1}^\infty \Omega_{1/i}$ , i.e.,  $\Omega_0 \subset \cap_{i=1}^\infty \Omega_{1/i}$ .

Next we establish that  $\cap_{i=1}^\infty \Omega_{1/i} \subset \Omega_0$ . Suppose that  $\omega \in \cap_{i=1}^\infty \Omega_{1/i}$  and let the sequence  $\{q_i\}_{i \in \mathbb{Z}_{\geq 1}}$  with  $q_i \in \mathbb{Q}$  for each  $i \in \mathbb{Z}_{\geq 1}$  be such that the corresponding sample path converges to  $\Psi_{q_i, 1/i}$ . It follows that the corresponding sample path converges to the limit of any convergent subsequence of the sequence of sets  $\{\Psi_{q_i, 1/i}\}_{i \in \mathbb{Z}_{\geq 1}}$ . Let us use  $\{\Psi_k\}_{k \in \mathbb{Z}_{\geq 1}}$  for such a converging subsequence and let us use  $\Psi$  for the limit. The set  $\Psi$  is contained in  $\{x \in \Lambda : V(x) = c\}$  for some  $c \in \mathcal{I}$ . We claim that  $\Psi$  is weakly totally recurrent, and thus contained in the largest weakly totally recurrent in probability set contained in  $\{x \in \Lambda : V(x) = c\}$ , i.e.,  $\omega \in \Omega_0$ . Indeed, by [84, Theorem 4.10(a),(b)], for each  $\varepsilon > 0$  and  $x \in \Psi$  there exists  $k^*$  and  $\{x_k\}_{k=1}^\infty$  with  $x_k \in \Psi_k$  for each  $k \in \mathbb{Z}_{\geq 1}$  such

that

$$\Psi_k + 0.5\varepsilon\mathbb{B} \subset \Psi + \varepsilon\mathbb{B} \quad \forall k \geq k^* \quad (\text{D.51a})$$

$$\{x_k\} + 0.5\varepsilon\mathbb{B} \subset \{x\} + \varepsilon\mathbb{B} \quad \forall k \geq k^* \quad (\text{D.51b})$$

the latter following from  $x \in \{x_k\} + 0.5\varepsilon\mathbb{B}$  for all  $k \geq k^*$ . Thus, weak total recurrence of  $\Psi$  follows from weak total recurrence of  $\Psi_{k^*}$ .

Next we claim that  $\mathbb{P}(\Omega_{1/i}) = \mathbb{P}(\Omega_b)$  for each  $i \in \mathbb{Z}_{\geq 1}$ . To see this, we extract a finite cover of the compact set  $\mathcal{I}$  from the countable cover  $\{\{q\} + i^{-1}\mathbb{B}\}_{q \in \mathbb{Q}}$ . Let  $\mathbb{Q}_0 \subset \mathbb{Q}$  denote the indices of the cover. Let  $\Omega_{1,q,i}$  denote the subset of  $\omega \in \Omega_b$  such that  $\mathbf{x}(\omega)$  converges to  $\{x \in \Lambda : V(x) \in \{q\} + i^{-1}\mathbb{B}\} = S_{qi}$  and let  $\Omega_{2,q,i}$  denote the subset of  $\omega \in \Omega_{1,q,i}$  for which  $\mathbf{x}(\omega)$  converges to the largest weakly totally recurrent in probability set contained in  $S_{qi}$ . By the definition of  $\Omega_b$  and that the fact that neighborhoods of size  $1/i$  of the points in  $\mathbb{Q}_0$  provide a cover for  $\mathcal{I}$ , it follows that  $\Omega_b = \cup_{q \in \mathbb{Q}_0} \Omega_{1,q,i}$ . By the definition of  $\Omega_{1/i}$ , it follows that  $\cup_{q \in \mathbb{Q}_0} \Omega_{2,q,i} \subset \Omega_{1/i}$ . By Lemma D.13,  $\mathbb{P}(\Omega_{1,q,i}) = \mathbb{P}(\Omega_{2,q,i})$  for each  $q \in \mathbb{Q}_0$ . It now follows from the next claim and  $\Omega_{1/i} \subset \Omega_b$  that  $\mathbb{P}(\Omega_{1/i}) = \mathbb{P}(\Omega_b)$  for each  $i \in \mathbb{Z}_{\geq 1}$ , and in turn that  $\mathbb{P}(\Omega_0) = \lim_{i \rightarrow \infty} \mathbb{P}(\Omega_{1/i}) = \mathbb{P}(\Omega_b)$ , which concludes the proof. ■

**Claim D.7** *If, for  $j \in \{1, \dots, n\}$ ,  $R_j \subset S_j$  and  $\mathbb{P}(R_j) = \mathbb{P}(S_j)$  then  $\mathbb{P}(\cup_{j=1}^n R_j) = \mathbb{P}(\cup_{j=1}^n S_j)$ .*

*Proof:* For general  $n$ , the result follows by induction after establishing the result for  $n = 2$ . Since  $R_j \subset S_j$  for  $j = 1, 2$ , it follows that  $\mathbb{P}(R_1 \cup R_2) \leq \mathbb{P}(S_1 \cup S_2)$ . Now

observe that, using  $R_j \subset S_j$  and  $\mathbb{P}(R_j) = \mathbb{P}(S_j)$  for  $j = 1, 2$ ,

$$\begin{aligned} \mathbb{P}(S_1 \cup S_2) &= \mathbb{P}(R_1 \cup (S_1 \setminus R_1) \cup R_2 \cup (S_2 \setminus R_2)) \\ &\leq \mathbb{P}(R_1 \cup R_2) + \mathbb{P}(S_1 \setminus R_1) + \mathbb{P}(S_2 \setminus R_2) \\ &= \mathbb{P}(R_1 \cup R_2) + \mathbb{P}(S_1) - \mathbb{P}(R_1) + \mathbb{P}(S_2) - \mathbb{P}(R_2) \\ &= \mathbb{P}(R_1 \cup R_2) \end{aligned}$$

which establishes the result for  $n = 2$ , and thus for general  $n$ . ■

The theorem now follows from the combination of Claims D.4 and D.6. ■

## D.14 Proof of Lemma 5.1

We first note that the set of solutions starting from  $(K, F, K, K)$  is closed when  $F$  satisfies Standing Assumption 1. Without loss of generality we consider two cases. If  $\phi_i$  is not generated by  $(K, F, K, K)$ , then by the definition of  $\varphi$ ,  $\limsup_{i \rightarrow \infty} \varphi_{\tau, S}(\phi_i) = 0$  and hence  $\limsup_{i \rightarrow \infty} \varphi_{\tau, S}(\phi_i) \leq \varphi_{\tau, S}(\phi)$  holds trivially. If  $\phi_i$  is generated by  $(K, F, K, K)$  then the limit  $\phi$  is also generated from  $(K, F, K, K)$ . We then establish the result by contradiction. Suppose not, then for every  $N \in \mathbb{Z}_{>0}$ , there exists  $i \in \mathbb{Z}_{\geq N}$  and  $\varepsilon > 0$  such that

$$\varphi_{\tau, S}(\phi_i) \geq \varphi_{\tau, S}(\phi) + \varepsilon. \tag{D.52}$$

We consider two possible consequences of (D.52) and establish that in both the cases (D.52) is not true for  $N$  arbitrarily large.

For  $j \in \{0, \dots, \lfloor \tau \rfloor\}$ , let  $t_i(j)$  be the smallest time  $t$  such that  $(t, j) \in \text{dom}(\phi_i)$ . Similarly,  $t(j)$  is defined for the solution  $\phi$ . We first consider the case, where for some

$j \in \{0, \dots, \lfloor \tau \rfloor\}$ ,  $\phi(t(j), j) \notin S$  and  $\phi_i(t_i(j), j) \in S$  for some  $i$  arbitrarily large. Since the set  $\mathbb{R}^n \setminus S$  is open and by convergence of hybrid arcs,  $t_i(j) \rightarrow t(j)$ , it follows that  $\phi_i(t_i(j), j) \notin S$  for sufficiently large  $i$  and hence the above scenario during jumps cannot occur for  $N$  sufficiently large.

If for some  $j \in \{0, \dots, \lfloor \tau \rfloor\}$ , there exists  $0 \leq T_1 < T_2$  such that  $\phi(s, j) \notin S$  for  $s \in [T_1, T_2]$  and  $s + j \leq \tau$ . Let  $\epsilon > 0$  be such that  $\phi(s, j) + \epsilon \mathbb{B} \notin S$  for  $s \in [T_1, T_2]$ . This is possible since  $\mathbb{R}^n \setminus S$  is open. Then, from [14, Thm 5.25], there exists  $i_0$  such that for all  $i \geq i_0$ ,  $\phi_i$  and  $\phi$  are  $(\tau, \epsilon)$  close. Since  $\phi_i$  converges graphically to  $\phi$ , it follows that in the limit as  $i \rightarrow \infty$ ,  $\phi_i(s, j) \notin S$  for  $s \in [T_1, T_2]$ . This argument ensures that (D.52) cannot occur for  $N$  sufficiently large.

## D.15 Proof of Theorem 5.3

The proof of the theorem follows directly from the next lemma.

**Lemma D.14** *Let the compact set  $K_\infty$  be given. For the solution  $\mathbf{x}$ , let  $\Omega_a$  denote the set of  $\omega \in \Omega$ , such that  $\mathbf{x}(\omega)$  is complete and converges to  $K_\infty$ . Let  $\Omega_b \subset \Omega_a$  denote the set of  $\omega \in \Omega_a$  such that  $\mathbf{x}(\omega)$  converges to the largest weakly totally recurrent in probability set contained in the set  $K_\infty$ . Then  $\mathbb{P}(\Omega_a) = \mathbb{P}(\Omega_b)$ .*

*Proof:* For every  $i \in \mathbb{Z}_{\geq 1}$ , define the compact set  $K_i := K_\infty + i\mathbb{B}$  and let the solution  $\mathbf{x}_i$  be the truncated version of the solution  $\mathbf{x}$  restricted to the set  $K_i$ . Apply Lemma D.13 to the solution  $\mathbf{x}_i$  with  $K = K_i$  to get  $\Omega_{a_i}$  and an  $\Omega_{b_i}$  satisfying  $\mathbb{P}(\Omega_{a_i}) = \mathbb{P}(\Omega_{b_i})$ . Then, it follows that  $\Omega_a = \cup_i \Omega_{a_i}$  and  $\Omega_b = \cup_i \Omega_{b_i}$ . Consequently, we have  $\mathbb{P}(\Omega_a) = \lim_{i \rightarrow \infty} \mathbb{P}(\Omega_{a_i}) = \lim_{i \rightarrow \infty} \mathbb{P}(\Omega_{b_i}) = \mathbb{P}(\Omega_b)$ . ■

## D.16 Proof of Proposition 6.3

We recall the definition of  $\tilde{m}$  in (6.6) related to the largest viability probabilities for a closed set  $S$ :

$$\tilde{m}_{\subset S}(\ell, \xi) := \sup_{\mathbf{x} \in \mathcal{S}_r(\xi)} \mathbb{P}(\text{graph}(\mathbf{x}) \cap (\Gamma_{<\ell} \times \mathbb{R}^n) \subset (\mathbb{R}^2 \times S)).$$

Next, we observe from the definition of  $\hat{m}$  in (6.3), and (6.2), (6.6) that for any closed set  $S \subset \mathbb{R}^n$  and  $x \in \mathbb{R}^n$ ,

$$\hat{m}_{\subset S}(x) = \lim_{\tau \rightarrow \infty} \tilde{m}_{\subset S}(\tau, x).$$

We now establish using sequential compactness results in [85] that the supremum in the characterization of  $\tilde{m}$  is achieved for some random solution. Let  $\ell \geq 0$ . For hybrid arcs  $\phi$ , define the function  $\varphi$  such that  $\varphi(\phi) = 1$  if  $\text{graph}(\phi) \cap (\Gamma_{<\ell} \times \mathbb{R}^n) \subset (\mathbb{R}^2 \times S)$  and  $\varphi(\phi)$  is equal to 0 otherwise. Let  $x \in \mathbb{R}^n$  and  $\tilde{m}_{\subset S}(\ell, x) = \Delta \geq 0$ . If  $\Delta = 0$ , then every solution  $\mathbf{x} \in \mathcal{S}_r(x)$  achieves the supremum. We now consider the case when  $\Delta > 0$ . Let  $\Delta_i < \Delta, i \in \mathbb{Z}_{\geq 0}$  be a sequence that converges to the value  $\Delta$ . Then, there exists a sequence of solution  $\mathbf{x}_i$  such that  $\mathbb{E}[\varphi(\mathbf{x}_i)] \geq \Delta_i$ . Then, from [85, Thm 1], it follows that there exists a random solution  $\mathbf{x}^*$  such that  $\mathbb{E}[\varphi(\mathbf{x}^*)] \geq \Delta$ . Since  $\Delta$  is the supremum, it follows that  $\mathbb{E}[\varphi(\mathbf{x}^*)] = \Delta$  which establishes the result.

Next, we show that for every  $x \in \mathbb{R}^n$  and closed set  $S \subset \mathbb{R}^n$ ,

$$\lim_{i \rightarrow \infty} \sup_{\mathbf{x} \in \mathcal{S}_r(x)} \mathbb{P}\left(\text{graph}(\mathbf{x}) \cap (\Gamma_{<i} \times \mathbb{R}^n) \subset (\mathbb{R}^2 \times S)\right) = \sup_{\mathbf{x} \in \mathcal{S}_r(x)} \mathbb{P}\left(\text{graph}(\mathbf{x}) \subset (\mathbb{R}^2 \times S)\right). \tag{D.53}$$

We first observe that

$$\begin{aligned} \sup_{\mathbf{x} \in \mathcal{S}_r(x)} \mathbb{P}\left(\text{graph}(\mathbf{x}) \subset (\mathbb{R}^2 \times S)\right) &= \sup_{\mathbf{x} \in \mathcal{S}_r(x)} \lim_{i \rightarrow \infty} \mathbb{P}\left(\text{graph}(\mathbf{x}) \cap (\Gamma_{<i} \times \mathbb{R}^n) \subset (\mathbb{R}^2 \times S)\right) \\ &\leq \lim_{i \rightarrow \infty} \sup_{\mathbf{x} \in \mathcal{S}_r(x)} \mathbb{P}\left(\text{graph}(\mathbf{x}) \cap (\Gamma_{<i} \times \mathbb{R}^n) \subset (\mathbb{R}^2 \times S)\right). \end{aligned} \tag{D.54}$$

For  $x \in \mathbb{R}^n$  and  $i \in \mathbb{Z}_{\geq 0}$ , let  $\Delta_i = \tilde{m}_{CS}(i, x)$  and  $\Delta = \lim_{i \rightarrow \infty} \tilde{m}_{CS}(i, x)$ . Then, the sequence  $\Delta_i$  converges to  $\Delta$ . For hybrid arcs  $\phi$ , define the function  $\varphi_i$  such that  $\varphi_i(\phi) = 1$  if  $\text{graph}(\phi) \cap (\Gamma_{<i} \times \mathbb{R}^n) \subset (\mathbb{R}^2 \times S)$  and  $\varphi_i(\phi)$  is equal to 0 otherwise. The function  $\varphi$  is defined such that  $\varphi(\phi) = 1$  if  $\text{graph}(\phi) \subset (\mathbb{R}^2 \times S)$  and  $\varphi(\phi)$  is equal to 0 otherwise. It follows from the above discussion that there exists a random solution  $\mathbf{x}_i \in \mathcal{S}_r(x)$  such that  $\mathbb{E}[\varphi_i(\mathbf{x}_i)] = \Delta_i$ . Then, from [85, Thm 1], there exists a solution  $\mathbf{x}$  such that  $\mathbb{E}[\varphi(\mathbf{x})] \geq \Delta$ . Hence, we have

$$\begin{aligned} \lim_{i \rightarrow \infty} \sup_{\mathbf{x} \in \mathcal{S}_r(x)} \mathbb{P}\left(\text{graph}(\mathbf{x}) \cap (\Gamma_{<i} \times \mathbb{R}^n) \subset (\mathbb{R}^2 \times S)\right) &= \Delta = \mathbb{E}[\varphi(\mathbf{x})] \\ &\leq \sup_{\mathbf{x} \in \mathcal{S}_r(x)} \lim_{i \rightarrow \infty} \mathbb{P}\left(\text{graph}(\mathbf{x}) \cap (\Gamma_{<i} \times \mathbb{R}^n) \subset (\mathbb{R}^2 \times S)\right). \end{aligned} \tag{D.55}$$

The bound (D.53) now follows from (D.54) and (D.55). Hence, we have

$$\lim_{i \rightarrow \infty} \sup_{\mathbf{x} \in \mathcal{S}_r(x)} \mathbb{P}\left(\text{graph}(\mathbf{x}) \cap (\Gamma_{<i} \times \mathbb{R}^n) \subset (\mathbb{R}^2 \times S)\right) = \sup_{\mathbf{x} \in \mathcal{S}_r(x)} \mathbb{P}\left(\text{graph}(\mathbf{x}) \subset (\mathbb{R}^2 \times S)\right).$$

The proof of existence of a random solution  $\mathbf{x}^* \in \mathcal{S}_r(x)$  such that

$$\mathbb{P}\left(\text{graph}(\mathbf{x}^*) \subset (\mathbb{R}^2 \times S)\right) = \sup_{\mathbf{x} \in \mathcal{S}_r(x)} \mathbb{P}\left(\text{graph}(\mathbf{x}) \subset (\mathbb{R}^2 \times S)\right)$$

follows along the same lines as the proof for the random solution achieving the supremum in the definition of  $\tilde{m}_{CS}(\ell, x)$  and is thus omitted.

## D.17 Proof of Proposition 6.4

Let  $x \in \mathbb{R}^n$  and  $\mathbf{x} \in \mathcal{S}_r(x)$ . Let  $\Omega_\infty^\mathbf{x} := \{\omega : \text{graph}(\mathbf{x}(\omega)) \subset \mathbb{R}^2 \times S\}$ . Define

$$\begin{aligned} \mathbf{T}(\omega) &:= \sup\{t \in \pi_1(\text{graph}(\mathbf{x}(\omega)) \cap (\Gamma_{\leq 1} \times \mathbb{R}^n))\} \\ \mathbf{J}(\omega) &:= \sup\{j \in \pi_2(\text{graph}(\mathbf{x}(\omega)) \cap (\Gamma_{\leq 1} \times \mathbb{R}^n))\}. \end{aligned} \tag{D.56}$$

It follows from [25, Prop 2.1] that  $\mathbf{T}, \mathbf{J}$  are  $\mathcal{F}_1$  measurable. Define the process  $\mathbf{y}$  such that  $\text{graph}(\mathbf{y}(\omega)) = \text{graph}(\mathbf{x}(\omega)) - (\mathbf{T}(\omega), \mathbf{J}(\omega), 0)$ . Without loss of generality the hybrid time domain of  $\mathbf{x}(\omega)$  restricted to at most one jump is given by  $\bigcup_{i=0}^1 ([\mathbf{t}_i(\omega), \mathbf{t}_{i+1}(\omega)] \times \{i\})$  for  $\mathcal{F}_{i-1}$  measurable random variables  $\mathbf{t}_i$  for  $i = \{1, 2\}$  and  $\mathbf{t}_0 \equiv 0$ . See [25, Section 9.1] for more details. For hybrid arc  $\phi$ , the function  $\varphi$  is defined such that  $\varphi(\phi) = 1$  if  $\text{graph}(\phi) \subset (\mathbb{R}^2 \times S)$  and  $\varphi(\phi)$  is equal to 0 otherwise. We then have from [25, eqn (77)]

$$\begin{aligned} \mathbb{P}(\Omega_\infty^\mathbf{x}) &= \mathbb{E} \left[ \max_{i \in \{0,1\}} \prod_{j=0}^i \mathbb{I}_{C\mathbb{R}^2 \times S}(\text{graph}_{j,1}(\mathbf{x})) \mathbb{I}_{\Gamma \cap \mathbb{R}^2 \times \mathbb{R}^n}(\text{graph}_i(\mathbf{x})) \right. \\ &\quad \left. \mathbb{I}_{\mathbb{R}_{\leq 0}}(1 - \mathbf{t}_{i+1} - i) \mathbb{E}[\varphi(\mathbf{y}) | \mathcal{F}_1] \right] \end{aligned}$$

where

$$\begin{aligned} \text{graph}_i(\mathbf{x}) &:= \text{graph}(\mathbf{x}) \cap (\mathbb{R} \times \{i\} \times \mathbb{R}^n) \\ \text{graph}_{j,1}(\mathbf{x}) &:= \text{graph}_j(\mathbf{x}) \cap (\Gamma_{\leq 1} \times \mathbb{R}^n). \end{aligned}$$



We now establish that for every  $k \in \mathbb{Z}_{\geq 0}$  and  $x \in S$ ,

$$\widehat{m}_{CS}(x) \leq \gamma m_{CS}(k, x). \quad (\text{D.57})$$

The bound holds for  $k = 0$  and every  $x \in S$  since  $\widehat{m}_{CS}(x) \leq \gamma$  and  $m_{CS}(0, x) = 1$  for every  $x \in S$ . We assume that the bound holds for some  $k$  and every  $x \in S$ . Then, let  $x \in S$  and  $\mathbf{x} \in \mathcal{S}_r(x)$ . Then, from [25, Prop 9.1, eqn(77)] we have

$$\begin{aligned} \mathbb{P}(\Omega_\infty^{\mathbf{x}}) &= \mathbb{E} \left[ \max_{i \in \{0,1\}} \prod_{j=0}^i \mathbb{I}_{\mathbb{C}\mathbb{R}^2 \times S}(\text{graph}_{j,1}(\mathbf{x})) \mathbb{I}_{\mathbb{R}^2 \times \mathbb{R}^n}(\text{graph}_i(\mathbf{x})) \right. \\ &\quad \left. \mathbb{I}_{\mathbb{R}_{\leq 0}}(1 - \mathbf{t}_{i+1} - i) \mathbb{E}[\varphi(\mathbf{y}) | \mathcal{F}_1] \right] \\ &\leq \mathbb{E} \left[ \max_{i \in \{0,1\}} \prod_{j=0}^i \mathbb{I}_{\mathbb{C}\mathbb{R}^2 \times S}(\text{graph}_{j,1}(\mathbf{x})) \mathbb{I}_{\mathbb{R}^2 \times \mathbb{R}^n}(\text{graph}_i(\mathbf{x})) \right. \\ &\quad \left. \mathbb{I}_{\mathbb{R}_{\leq 0}}(1 - \mathbf{t}_{i+1} - i) \widehat{m}_{CS}(\mathbf{y}(0, 0)) \right] \\ &\leq \gamma \mathbb{E} \left[ \max_{i \in \{0,1\}} \prod_{j=0}^i \mathbb{I}_{\mathbb{C}\mathbb{R}^2 \times S}(\text{graph}_{j,1}(\mathbf{x})) \mathbb{I}_{\mathbb{R}^2 \times \mathbb{R}^n}(\text{graph}_i(\mathbf{x})) \right. \\ &\quad \left. \mathbb{I}_{\mathbb{R}_{\leq 0}}(1 - \mathbf{t}_{i+1} - i) m_{CS}(k, \mathbf{y}(0, 0)) \right] \\ &\leq \gamma m_{CS}(k + 1, x). \end{aligned}$$

Since this is true for any  $x \in S$  and  $\mathbf{x} \in \mathcal{S}_r(x)$ , it follows from Proposition 6.3 that

$$\widehat{m}_{CS}(x) \leq \gamma m_{CS}(k + 1, x).$$

The bound (D.57) holds by induction. Then as  $k \rightarrow \infty$  we have

$$\widehat{m}_{CS}(x) \leq \gamma \widehat{m}_{CS}(x).$$

Since  $\gamma < 1$ , it implies that  $\sup_{x \in S} \widehat{m}_{CS}(x) = 0$ .

## D.18 Proof of Proposition 6.5

Let  $x \in \mathbb{R}^n$  and  $\mathbf{x} \in \mathcal{S}_r(x)$ . For  $x \in \mathbb{R}^n \setminus S_1$ , the bound holds automatically due to the mapping  $\tau \mapsto m_{\subset S_0}(\tau, x)$  being non-increasing. Now, we consider the case when  $x \in S_1$  and  $\mathbf{x} \in \mathcal{S}_r(x)$ . We first show that

$$\begin{aligned} \mathbb{P}(\text{graph}(\mathbf{x}) \cap (\Gamma_{\leq k_1+k_2} \times \mathbb{R}^n) \subset \mathbb{R}^2 \times S_0) &\leq \mathbb{P}(\text{graph}(\mathbf{x}) \cap (\Gamma_{\leq k_1} \times \mathbb{R}^n) \subset \mathbb{R}^2 \times S_1) \\ &\quad + \sup_{\xi \in \mathbb{R}^n \setminus S_1} m_{\subset S_0}(k_2, \xi). \end{aligned}$$

Let  $\Omega_0 := \{\omega : \text{graph}(\mathbf{x}) \cap (\Gamma_{\leq k_1+k_2} \times \mathbb{R}^n) \subset \mathbb{R}^2 \times S_0\}$ ,  $\Omega_1 := \{\omega : \text{graph}(\mathbf{x}) \cap (\Gamma_{\leq k_1} \times \mathbb{R}^n) \subset \mathbb{R}^2 \times S_1\}$  and  $\Omega_2 := \{\omega : \omega \in \Omega_0, \text{graph}(\mathbf{x}) \cap (\Gamma_{\leq k_1} \times \mathbb{R}^n \setminus S_1) \neq \emptyset\}$ . We claim that  $\Omega_0 \subset (\Omega_1 \cup \Omega_2)$ . If not, there exists  $\omega \in \Omega_0$  such that  $\omega \notin \Omega_1$  and  $\omega \notin \Omega_2$ . Since  $\omega \in \Omega_0$  and  $\omega \notin \Omega_1$ , then necessarily for some  $(t, j)$ ,  $\mathbf{x}_\omega(t, j) \in \mathbb{R}^n \setminus S_1$  and  $t + j \leq k_1$  and  $\mathbf{x}(\omega)$  remains in  $S_0$  till hybrid time  $k_1 + k_2$ . Hence,  $\omega \in \Omega_2$ . This leads to a contradiction and establishes the claim. Define

$$\begin{aligned} \mathbf{T}(\omega) &:= \inf\{t \in \pi_1(\text{graph}(\mathbf{x}(\omega)) \cap (\Gamma_{\leq k_1} \times \mathbb{R}^n \setminus S_1))\} \\ \mathbf{J}(\omega) &:= \inf\{j \in \pi_2(\text{graph}(\mathbf{x}(\omega)) \cap (\Gamma_{\leq k_1} \times \mathbb{R}^n \setminus S_1))\} \end{aligned}$$

It follows from [25, Prop 2.1] that  $\mathbf{T}, \mathbf{J}$  are  $\mathcal{F}_{k_1}$  measurable random variables. Then, we have

$$\begin{aligned}
 \mathbb{P}(\Omega_0) &\leq \mathbb{P}(\Omega_1 \cup \Omega_2) \leq \mathbb{P}(\Omega_1) + \mathbb{P}(\Omega_2) \\
 &\leq \mathbb{P}(\Omega_1) + \mathbb{E}[m_{\subset S_0}(k_2, \mathbf{x}(\mathbf{T}, \mathbf{J}))] \\
 &\leq \mathbb{P}(\Omega_1) + \sup_{\xi \in \overline{\mathbb{R}^n \setminus S_1}} m_{\subset S_0}(k_2, \xi).
 \end{aligned}$$

Consequently, we have

$$\mathbb{P}(\text{graph}(\mathbf{x}) \subset \Gamma_{\leq k_1+k_2} \times S_0) \leq \mathbb{P}(\text{graph}(\mathbf{x}) \subset \Gamma_{\leq k_1} \times S_1) + \sup_{\xi \in \overline{\mathbb{R}^n \setminus S_1}} m_{\subset S_0}(k_2, \xi).$$

The result of the proposition now follows as

$$\begin{aligned}
 m_{\subset S_0}(k_1 + k_2, x) &= \sup_{\mathbf{x} \in \mathcal{S}_r(x)} \mathbb{P}(\text{graph}(\mathbf{x}) \subset \Gamma_{\leq k_1+k_2} \times S_0) \\
 &\leq \sup_{\mathbf{x} \in \mathcal{S}_r(x)} \mathbb{P}(\text{graph}(\mathbf{x}) \subset \Gamma_{\leq k_1} \times S_1) + \sup_{\xi \in \overline{\mathbb{R}^n \setminus S_1}} m_{\subset S_0}(k_2, \xi) \\
 &= m_{\subset S_1}(k_1, x) + \sup_{\xi \in \overline{\mathbb{R}^n \setminus S_1}} m_{\subset S_0}(k_2, \xi).
 \end{aligned}$$

## D.19 Proof of Proposition 6.6

The bound holds true for any  $x \in \mathbb{R}^n \setminus S$  trivially. We now prove the result for  $x \in S$ .

We claim that for every  $x \in S$  and  $\mathbf{x} \in \mathcal{S}_r(x)$ ,

$$\begin{aligned}
 \mathbb{P}(\text{graph}(\mathbf{x}) \cap (\Gamma_{\leq \tau} \times \mathbb{R}^n) \subset \mathbb{R}^2 \times S) &\leq \mathbb{P}(\text{graph}(\mathbf{x}) \cap (\Gamma_{\leq \tau} \times \mathbb{R}^n) \subset \mathbb{R}^2 \times S_1) \\
 &\quad + \mathbb{P}(\text{graph}(\mathbf{x}) \cap (\Gamma_{\leq \tau} \times S_2) \neq \emptyset).
 \end{aligned}$$

Let  $\Omega_0 := \{\omega : \text{graph}(\mathbf{x}(\omega)) \cap (\Gamma_{\leq \tau} \times \mathbb{R}^n) \subset \mathbb{R}^2 \times S\}$ ,  $\Omega_1 := \{\omega : \text{graph}(\mathbf{x}(\omega)) \cap (\Gamma_{\leq \tau} \times \mathbb{R}^n) \subset \mathbb{R}^2 \times S_1\}$   $\Omega_2 := \{\omega : \text{graph}(\mathbf{x}(\omega)) \cap (\Gamma_{\leq \tau} \times S_2) \neq \emptyset\}$ . We claim that

$\Omega_0 \subset (\Omega_1 \cup \Omega_2)$ . If not, for some  $\omega \in \Omega_0$  we have  $\omega \notin \Omega_1$  and  $\omega \notin \Omega_2$ . If  $\omega \notin \Omega_1$ , it implies that  $\text{graph}(\mathbf{x}(\omega)) \cap (\Gamma_{\leq \tau} \times (S \setminus S_1)) \neq \emptyset$ . Since  $S \subset S_1 \cup S_2$ , this means that  $\text{graph}(\mathbf{x}(\omega)) \cap (\Gamma_{\leq \tau} \times S_2) \neq \emptyset$  and hence  $\omega \in \Omega_2$ . This leads to a contradiction. Then,  $\mathbb{P}(\Omega_0) \leq \mathbb{P}(\Omega_1) + \mathbb{P}(\Omega_2)$ . The result of the proposition then follows from taking the supremum over all possible random solutions from  $x$  on both sides.

## D.20 Proof of Proposition 6.2

1)  $\Rightarrow$  2) Since  $\mathcal{O}$  is globally recurrent for  $\mathcal{H}$ , from Lemma 6.2,  $\mathcal{O}$  is globally recurrent for  $\widehat{\mathcal{H}}$ . Hence, for every  $x \in \mathbb{R}^n$  and  $\mathbf{x} \in \mathcal{S}_r(x)$  (generated by  $\widehat{\mathcal{H}}$ ), we have

$$\mathbb{P}(\text{graph}(\mathbf{x}) \cap (\mathbb{R}^2 \times \mathcal{O})) = 1.$$

Then, it follows that for every  $x \in \mathbb{R}^n$  and  $\mathbf{x} \in \mathcal{S}_r(x)$  (generated by  $\widehat{\mathcal{H}}$ ), we have

$$\mathbb{P}(\text{graph}(\mathbf{x}) \subset (\mathbb{R}^2 \times \mathbb{R}^n \setminus \mathcal{O})) = 0.$$

It follows from Proposition 6.3 that for every  $x \in \mathbb{R}^n$ , we have  $\widehat{m}_{\mathbb{C}\mathbb{R}^n \setminus \mathcal{O}}(x) = 0$  for the SHS  $\widehat{\mathcal{H}}$ . Since the solutions of  $\mathcal{H}$  are also solutions of  $\widehat{\mathcal{H}}$ , it follows that  $m_{\mathbb{C}\mathbb{R}^n \setminus \mathcal{O}, \mathcal{H}}(\tau, x) \leq m_{\mathbb{C}\mathbb{R}^n \setminus \mathcal{O}, \widehat{\mathcal{H}}}(\tau, x)$  for every  $(\tau, x) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n$  and consequently for every  $x \in \mathbb{R}^n$ ,  $\widehat{m}_{\mathbb{C}\mathbb{R}^n \setminus \mathcal{O}, \mathcal{H}}(x) = 0$ .

2)  $\Rightarrow$  3) Follows from the proof of [42, Prop 5] using the upper semicontinuity of  $(\tau, x) \mapsto m_{\mathbb{C}\mathbb{R}^n \setminus \mathcal{O}}(\tau, x)$ .

3)  $\Rightarrow$  1) Follows from the definition of  $m_{\mathbb{C}\mathbb{R}^n \setminus \mathcal{O}}(\tau, x)$  in (6.2), the definition of uniform global recurrence and Proposition 6.1.

## D.21 Proof of Proposition 6.8

We first claim that for each  $(\ell, \rho) \in \mathbb{Z}_{\geq 0} \times \mathbb{R}_{>0}$  and  $K \subset \mathbb{R}^n$  compact there exists a  $\varepsilon > 0$  such that, for every  $x \in K$  compact and  $\mathbf{x} \in \mathcal{S}_r(x)$ ,

$$\mathbb{P}(\text{graph}(\mathbf{x}) \cap (\Gamma_{<\ell} \times \mathbb{R}^n) \subset \mathbb{R}^2 \times (S + \varepsilon\mathbb{B})) \leq \max_{\xi \in K} \tilde{m}_{CS}(\ell, \xi) + \rho. \quad (\text{D.58})$$

If the claim is not true, then there exists  $(\ell, \rho)$  and a compact set  $K$  such that for every  $i \in \mathbb{Z}_{\geq 1}$  we have for some  $x_i \in K$  and  $\mathbf{x}_i \in \mathcal{S}_r(x_i)$

$$\mathbb{P}(\text{graph}(\mathbf{x}_i) \cap (\Gamma_{<\ell} \times \mathbb{R}^n) \subset \mathbb{R}^2 \times (S + 1/i\mathbb{B})) > \max_{\xi \in K} \tilde{m}_{CS}(\ell, \xi) + \rho.$$

For hybrid arcs  $\phi$ , define the function  $\varphi_i$  such that  $\varphi_i(\phi) = 1$  if  $\text{graph}(\phi) \cap (\Gamma_{<\ell} \times \mathbb{R}^n) \subset \mathbb{R}^2 \times (S + (1/i)\mathbb{B})$  and 0 otherwise. Similarly, the function  $\varphi$  is defined using set  $S$ . Then,

$$\mathbb{E}[\varphi_i(\mathbf{x}_i)] > \max_{\xi \in K} \tilde{m}_{CS}(\ell, \xi) + \rho.$$

From [85, Theorem 1], we can establish that there exists a random solution  $\mathbf{x}$  from  $K$  such that

$$\mathbb{E}[\varphi(\mathbf{x})] \geq \max_{\xi \in K} \tilde{m}_{CS}(\ell, \xi) + \rho.$$

Since  $\mathbb{E}[\varphi(\mathbf{x})] \leq \max_{\xi \in K} \tilde{m}_{CS}(\ell, \xi)$  and  $\rho > 0$ , it leads to a contradiction that establishes the result. Then, we observe that

$$\mathbb{P}(\text{graph}(\mathbf{x}) \cap (\Gamma_{\leq \ell} \times \mathbb{R}^n) \subset \mathbb{R}^2 \times (S + \varepsilon\mathbb{B})) \leq \mathbb{P}(\text{graph}(\mathbf{x}) \cap (\Gamma_{<\ell} \times \mathbb{R}^n) \subset \mathbb{R}^2 \times (S + \varepsilon\mathbb{B})).$$

The result now follows from the bound (D.58) by taking the supremum over all possible random solutions from the initial condition  $x$  for the SHS  $\widehat{\mathcal{H}}$ .

## D.22 Proof of Theorem 6.1

Let  $\ell_0 \in \mathbb{Z}_{>0}$  be such that  $\tilde{m}_{\mathbb{C}\mathbb{R}^n \setminus \mathcal{O}}(\ell_0, x) \leq 0.25/2$  for all  $x \in \overline{\mathcal{O} + \mathbb{B}^o}$ . This bound follows from the uniform global recurrence of the set  $\mathcal{O}$ . We now use the result of Proposition 6.8 with  $K := \overline{\mathcal{O} + \mathbb{B}^o}$  and  $S = \mathbb{R}^n \setminus \mathcal{O}$ . Then, there exists  $\tilde{\varepsilon} \in (0, 1)$  such that, for every  $x \in K$ ,

$$m_{\mathbb{C}(\mathbb{R}^n \setminus \mathcal{O}) + \tilde{\varepsilon}\mathbb{B}}(\ell_0, x) \leq \max_{\xi \in \overline{\mathcal{O} + \mathbb{B}^o}} \tilde{m}_{\mathbb{C}\mathbb{R}^n \setminus \mathcal{O}}(\ell_0, \xi) + 0.25/2.$$

Define the open, bounded set  $\widehat{\mathcal{O}} := \mathbb{R}^n \setminus ((\mathbb{R}^n \setminus \mathcal{O}) + \tilde{\varepsilon}\mathbb{B})$ . Hence, it follows that for  $\varepsilon = \tilde{\varepsilon}/2 > 0$ ,  $\widehat{\mathcal{O}} + \varepsilon\mathbb{B} \subset \mathcal{O}$ . Since  $\tilde{\varepsilon} < 1$ ,  $\mathcal{O} \subset \widehat{\mathcal{O}} + \mathbb{B}^o$ . Then, for all  $x \in \overline{\widehat{\mathcal{O}} + \mathbb{B}^o}$ ,

$$m_{\mathbb{C}\mathbb{R}^n \setminus \widehat{\mathcal{O}}}(\ell_0, x) \leq m_{\mathbb{C}(\mathbb{R}^n \setminus \mathcal{O}) + \tilde{\varepsilon}\mathbb{B}}(\ell_0, x) \leq 0.25.$$

We now complete the proof as follows. Let  $S_i := \mathbb{R}^n \setminus (\widehat{\mathcal{O}} + i\mathbb{B}^o)$  be a sequence of closed sets for  $i \in \mathbb{Z}_{\geq 0}$ . Since  $\mathcal{O} \subset \widehat{\mathcal{O}} + \mathbb{B}^o$ , it follows from uniform global recurrence of the set  $\mathcal{O}$  that for all  $\xi \in \overline{\mathbb{R}^n \setminus S_{i+1}}$  there exists  $\ell_i \in \mathbb{Z}_{>0}$  such that  $m_{\mathbb{C}S_1}(\ell_i, \xi) \leq 0.25$ . Then, for all  $i \in \mathbb{Z}_{\geq 1}$ ,  $x \in \overline{\mathbb{R}^n \setminus S_{i+1}}$ , we have from Proposition 6.5

$$\begin{aligned} m_{\mathbb{C}S_0}(\ell_i + \ell_0, x) &\leq m_{\mathbb{C}S_1}(\ell_i, x) + \sup_{\xi \in \overline{\mathbb{R}^n \setminus S_1}} m_{\mathbb{C}S_0}(\ell_0, \xi) \\ &\leq 0.5. \end{aligned}$$

Then, from the monotonicity of the viability probabilities we have that for every  $x \in \mathbb{R}^n$ ,  $\widehat{m}_{\mathbb{C}\mathbb{R}^n \setminus \widehat{\mathcal{O}}}(x) \leq 0.5$ . Hence, it follows from Proposition 6.4 that for all  $x \in \mathbb{R}^n$ ,  $\widehat{m}_{\mathbb{C}\mathbb{R}^n \setminus \widehat{\mathcal{O}}}(x) =$

0. This equality implies that the set  $\widehat{\mathcal{O}}$  is globally recurrent for  $\widehat{\mathcal{H}}$  from Proposition 6.2.

## D.23 Proof of Theorem 6.2

We denote the probabilities generated by the system  $\widehat{\mathcal{H}}_\nu$  with the subscript  $\nu$ . Let  $\mathcal{O}$  be the recurrent set. Now let  $S_i := \mathbb{R}^n \setminus (\mathcal{O} + i\mathbb{B}^o)$  be a sequence of closed sets and  $0 < \varepsilon_i \leq (\frac{1}{2})^{i+2}$  for all  $i \in \mathbb{Z}_{\geq 0}$ . Then, for every  $i \in \mathbb{Z}_{\geq 0}$ , choose  $\ell_i$  such that

$$m_{\subset S_i}(\ell_i, x) \leq \frac{1}{2}\varepsilon_i \quad \forall x \in \overline{\mathbb{R}^n \setminus S_{i+1}}. \quad (\text{D.59})$$

This bound follows from the uniform global recurrence of the set  $\mathcal{O}$  for the system  $\widehat{\mathcal{H}}$ . Let  $\beta_i \in \mathbb{Z}_{\geq 0}$ . Then, choose  $\beta_i \geq i + 1$  such that, with  $\nu(s) = s$  for all  $s \geq 0$ ,

$$m_{\nu, \cap S_{\beta_i}}(\ell_i, x) \leq \frac{1}{2}\varepsilon_i \quad \forall x \in \overline{\mathbb{R}^n \setminus S_{i+1}}. \quad (\text{D.60})$$

The values  $\beta_i$  exists according to Proposition 6.7. Without loss of generality we can assume the function  $i \mapsto \beta_i$  is strictly increasing and unbounded. Define the compact set  $K_i := (\overline{\mathcal{O} + \beta_i\mathbb{B}^o}) \setminus (\mathcal{O} + i\mathbb{B}^o)$ .

We use the fact that under the conditions of Standing Assumption 6.1 the infinite time reachable set ([14, Sec 6.3.2]) from  $\overline{\mathcal{O}}$  denoted by  $\Gamma$  is bounded for solutions of  $\dot{x} \in F(x), x \in C$  (from [31, Prop. 2]). Uniform global recurrence of  $\mathcal{O}$  implies that there exists a time  $J_i > 0$  such that each solution of  $\dot{x} \in F(x), x \in C$  from  $\overline{\mathcal{O} + \beta_i\mathbb{B}^o}$  reaches the set  $\mathcal{O}$ , or stops, within time  $J_i$ . Then, the reachable set in infinite time for  $\dot{x} \in F(x), x \in C$  from  $\overline{\mathcal{O} + \beta_i\mathbb{B}^o}$ , is given by  $\mathcal{R}(\overline{\mathcal{O} + \beta_i\mathbb{B}^o}) = \mathcal{R}_{\leq J_i}(\overline{\mathcal{O} + \beta_i\mathbb{B}^o}) \cup \Gamma$  where  $\mathcal{R}_{\leq J_i}(\overline{\mathcal{O} + \beta_i\mathbb{B}^o})$  is the reachable set within time  $J_i$ . It follows from [14, Lemma 6.16] that  $\mathcal{R}(\widehat{\mathcal{O}} + \beta_i\mathbb{B}^o)$  is bounded.

We define  $\gamma_i := \sup_{x \in \mathcal{R}(\overline{\mathcal{O} + \beta_i\mathbb{B}^o})} |x - x^*|$  and  $r_i := \inf_{y \in \partial(\overline{\mathcal{O} + i\mathbb{B}^o})} |y - x^*|$ . Let  $\nu \in \mathcal{K}_\infty$

with  $\nu(s) < s$  for all  $s > 0$  and satisfy  $\nu(\gamma_i) < r_i/2$  for all  $i \in \mathbb{Z}_{\geq 0}$ .

Next we claim that for all  $x \in \overline{\mathbb{R}^n \setminus S_{\beta_i}}$  and all  $\tau \in \mathbb{Z}_{\geq 0}$ ,

$$m_{\nu, \subset K_i}(\tau, x) = m_{\subset K_i}(\tau, x).$$

The proof of the above result follows along the same line as [55, Thm 4] using induction and dynamic programming from [25, Sec 9] and is thus omitted.

Now using Proposition 6.6, we have that for all  $i \in \mathbb{Z}_{\geq 0}$  and every  $x \in \overline{\mathbb{R}^n \setminus S_{i+1}}$ ,

$$\begin{aligned} m_{\nu, \subset S_i}(\ell_i, x) &\leq m_{\nu, \cap S_{\beta_i}}(\ell_i, x) + m_{\nu, \subset K_i}(\ell_i, x) \\ &\leq \frac{1}{2}\varepsilon_i + m_{\subset K_i}(\ell_i, x) \leq \frac{1}{2}\varepsilon_i + m_{\subset S_i}(\ell_i, x) \\ &\leq \varepsilon_i. \end{aligned}$$

Given  $x \in \mathbb{R}^n$ , let  $i \in \mathbb{Z}_{\geq 1}$  be such that  $x \in \overline{\mathbb{R}^n \setminus S_{i+1}}$ . Then, we have  $m_{\nu, \subset S_i}(\ell_i, x) \leq \varepsilon_i$ , and similarly we have that for every  $k \in \{0, \dots, i-1\}$ ,  $\sup_{\xi \in \overline{\mathbb{R}^n \setminus S_{k+1}}} m_{\nu, \subset S_k}(\ell_k, \xi) \leq \varepsilon_k$ .

Then, from Proposition 6.5 it follows that

$$\begin{aligned} m_{\nu, \subset S_0} \left( \sum_{j=0}^i \ell_j, x \right) &\leq m_{\nu, \subset S_1} \left( \sum_{j=1}^i \ell_j, x \right) + \sup_{\xi \in \overline{\mathbb{R}^n \setminus S_1}} m_{\nu, \subset S_0}(\ell_0, \xi) \\ &\leq m_{\nu, \subset S_i}(\ell_i, x) + \sum_{k=0}^{i-1} \sup_{\xi \in \overline{\mathbb{R}^n \setminus S_{k+1}}} m_{\nu, \subset S_k}(\ell_k, \xi) \\ &\leq \sum_{k=0}^i \varepsilon_k \leq 0.5. \end{aligned}$$

Hence, from the monotonicity of the viability probabilities we can conclude that  $\widehat{m}_{\nu, \subset S_0}(x) \leq 0.5$  for all  $x \in \mathbb{R}^n$ . Then, from Proposition 6.4 it follows that  $\widehat{m}_{\nu, \subset \mathbb{R}^n \setminus \mathcal{O}}(x) = 0$  for every  $x \in \mathbb{R}^n$ . Global recurrence of  $\mathcal{O}$  for  $\mathcal{H}_\nu$  follows from Proposition 6.2.



## D.24 Proof of Proposition 6.9

We first claim that for each  $(\ell, \rho) \in \mathbb{Z}_{\geq 0} \times \mathbb{R}_{>0}$  and  $K \subset \mathbb{R}^n$  compact there exists a  $\delta > 0$  such that, for every  $x \in K$  compact and  $\mathbf{x} \in \mathcal{S}_r^\delta(x)$ ,

$$\mathbb{P}(\text{graph}(\mathbf{x}) \cap (\Gamma_{<\ell} \times \mathbb{R}^n) \subset \mathbb{R}^2 \times S) \leq \max_{\xi \in K} \tilde{m}_{CS}(\ell, \xi) + \rho \quad (\text{D.61})$$

where  $\mathcal{S}_r^\delta(x)$  refers to the set of random solutions generated by the  $\widehat{\mathcal{H}}_\delta$  system. If the claim is not true, then there exists  $(\ell, \rho)$  and compact set  $K$  such that for every  $i \in \mathbb{Z}_{\geq 1}$  we have for some  $x_i \in K$  and  $\mathbf{x}_i \in \mathcal{S}_r^{(1/i)}(x_i)$

$$\mathbb{P}(\text{graph}(\mathbf{x}_i) \cap (\Gamma_{<\ell} \times \mathbb{R}^n) \subset \mathbb{R}^2 \times S) > \max_{\xi \in K} \tilde{m}_{CS}(\ell, \xi) + \rho.$$

Define the function  $\varphi_i$  for hybrid arc  $\phi$  such that  $\varphi(\phi) = 1$  if  $\text{graph}(\phi) \cap (\Gamma_{<\ell} \times \mathbb{R}^n) \subset \mathbb{R}^2 \times S$  and 0 otherwise. Then,

$$\mathbb{E}[\varphi(\mathbf{x}_i)] \geq \max_{\xi \in K} \tilde{m}_{CS}(\ell, \xi) + \rho.$$

From [85, Thm 1] we have that there exists a random solution  $\mathbf{x}$  from  $K$  for the nominal system such that

$$\mathbb{E}[\varphi(\mathbf{x})] \geq \max_{\xi \in K} \tilde{m}_{CS}(\ell, \xi) + \rho.$$

Since  $\mathbb{E}[\varphi(\mathbf{x})] \leq \max_{\xi \in K} \tilde{m}_{CS}(\ell, \xi)$  and  $\rho > 0$ , it leads to a contradiction that establishes the claim. The result now follows from the bound (D.61) by taking the supremum over all possible random solutions from the initial condition  $x$  for the system  $\widehat{\mathcal{H}}_\delta$ .

## D.25 Proof of Theorem 6.3

For  $i \in \mathbb{Z}_{\geq 0}$ , let  $0 < \varepsilon_i \leq (1/2)^{i+2}$  and  $S_i = \mathbb{R}^n \setminus (\mathcal{O} + i\mathbb{B}^o)$ . It follows from uniform global recurrence of  $\mathcal{O}$  for  $\widehat{\mathcal{H}}$  that there exists  $\ell_i$  be such that  $\sup_{\xi \in \overline{\mathbb{R}^n \setminus S_{i+1}}} \tilde{m}_{\subset S_i}(\ell_i, \xi) \leq \varepsilon_i/2$ . Then, let  $\delta_i > 0$  come from the application of Proposition 6.9 with the compact set  $K = \overline{\mathbb{R}^n \setminus S_{i+1}}$  and  $\rho = \varepsilon_i/2$ .

Given  $x \in \mathbb{R}^n$ , define  $i(x) := \min_{j \geq 1} \{j : x \in \overline{\mathbb{R}^n \setminus S_{j+1}}\}$ . Then, we have the viability probabilities satisfying  $m_{\delta_{i(x)}, \subset S_{i(x)}}(\ell_{i(x)}, x) \leq \varepsilon_i$ . Similarly we have that for every  $k \in \{0, \dots, i(x) - 1\}$  there exists  $\delta_k, \ell_k > 0$  such that  $\sup_{\xi \in \overline{\mathbb{R}^n \setminus S_{k+1}}} m_{\delta_k, \subset S_k}(\ell_k, \xi) \leq \varepsilon_k$ . Then, define a continuous state dependent perturbation  $\delta : \mathbb{R}^n \rightarrow \mathbb{R}_{>0}$  as follows,

$$\widehat{\delta}(x) := \min_{k \in \{0, 1, \dots, i(x)\}} \delta_k, \quad \delta(x) := \inf_{\xi \in \mathbb{R}^n} \left( \widehat{\delta}(\xi) + |\xi - x| \right).$$

Then, similar to the proof of Theorem 6.2 it follows from Proposition 6.5 that

$$\begin{aligned} m_{\delta, \subset S_0} \left( \sum_{j=0}^{i(x)} \ell_j, x \right) &\leq m_{\delta, \subset S_1} \left( \sum_{j=1}^{i(x)} \ell_j, x \right) + \sup_{\xi \in \mathbb{R}^n \setminus S_1} m_{\delta, \subset S_0}(\ell_0, \xi) \\ &\leq m_{\delta, \subset S_{i(x)}}(\ell_{i(x)}, x) + \sum_{k=0}^{i(x)-1} \sup_{\xi \in \mathbb{R}^n \setminus S_{k+1}} m_{\delta, \subset S_k}(\ell_k, \xi) \\ &\leq \sum_{k=0}^{i(x)} \varepsilon_k \leq 0.5. \end{aligned}$$

Hence, from monotonicity of the viability probabilities we can conclude that  $\widehat{m}_{\delta, \subset S_0}(x) \leq 0.5$  for all  $x \in \mathbb{R}^n$ . Then, it follows from Proposition 6.4 that for all  $x \in \mathbb{R}^n$ ,  $\widehat{m}_{\delta, \subset \mathbb{R}^n \setminus \mathcal{O}}(x) = 0$ . The result now follows from Proposition 6.2.

# Bibliography

- [1] A. Lyapunov, *The general problem of motion stability*, *Annals of Mathematics Studies* **17** (1892).
- [2] J. P. La Salle, *An invariance principle in the theory of stability*. New York: Academic Press, 1967.
- [3] N. Krasovskii, *Problems of the theory of stability of motion*. Stanford Univ. Press, 1963, translation of Russian edition, Moscow, 1959.
- [4] V. Matrosov, *On the stability of motion*, *Journal of Applied Mathematics and Mechanics* **26** (1962), no. 5 1337–1353.
- [5] A. Loria, E. Panteley, D. Popović, and A. R. Teel, *An extension of Matrosov’s theorem with application to stabilization of nonholonomic control systems*, in *Proceedings of the 41st IEEE Conference on Decision and Control*, vol. 2, pp. 1528–1533, IEEE, 2002.
- [6] A. R. Teel and L. Praly, *A smooth Lyapunov function from a class-KL estimate involving two positive semidefinite functions*, *ESAIM: Control, Optimization and Calculus of Variations* **5** (2000) 313–367.
- [7] C. M. Kellett and A. R. Teel, *Smooth Lyapunov functions and robustness of stability for difference inclusions*, *Systems & Control Letters* **52** (2004), no. 5 395–405.
- [8] G. Grimm, M. J. Messina, S. E. Tuna, and A. R. Teel, *Model predictive control: for want of a local control lyapunov function, all is not lost*, *IEEE Transactions on Automatic Control* **50** (2005), no. 5 546–558.
- [9] Z.-P. Jiang and Y. Wang, *A converse Lyapunov theorem for discrete-time systems with disturbances*, *Systems & Control Letters* **45** (2002), no. 1 49–58.
- [10] E. D. Sontag and Y. Wang, *On characterizations of the input-to-state stability property*, *Systems & Control Letters* **24** (1995), no. 5 351–359.

- [11] Y. Lin, E. D. Sontag, and Y. Wang, *A smooth converse Lyapunov theorem for robust stability*, *SIAM Journal on Control and Optimization* **34** (1996), no. 1 124–160.
- [12] F. H. Clarke, Y. S. Ledyaev, and R. J. Stern, *Asymptotic stability and smooth Lyapunov functions*, *Journal of Differential Equations* **149** (1998), no. 1 69–114.
- [13] C. M. Kellett and A. R. Teel, *On the robustness of KL-stability for difference inclusions: Smooth discrete-time Lyapunov functions*, *SIAM Journal on Control and Optimization* **44** (2005), no. 3 777–800.
- [14] R. Goebel, R. G. Sanfelice, and A. R. Teel, *Hybrid dynamical systems: modeling, stability, and robustness*. Princeton University Press, 2012.
- [15] R. G. Sanfelice and A. R. Teel, *Asymptotic stability in hybrid systems via nested Matrosov functions.*, *IEEE Transactions on Automatic Control* **54** (2009), no. 7 1569–1574.
- [16] C. Cai and A. R. Teel, *Characterizations of input-to-state stability for hybrid systems*, *Systems & Control Letters* **58** (2009), no. 1 47–53.
- [17] F. Kozin, *A survey of stability of stochastic systems*, *Automatica* **5** (1969), no. 1 95–112.
- [18] S. I. Resnick, *A probability path*. Springer Science & Business Media, 2013.
- [19] H. J. Kushner, *The concept of invariant set for stochastic dynamical systems, and applications to stochastic stability*, .
- [20] R. Khasminskii, *Stochastic stability of differential equations*, vol. 66. Springer Science & Business Media, 2011.
- [21] H. J. Kushner, *A partial history of the early development of continuous-time nonlinear stochastic systems theory*, *Automatica* **50** (2013), no. 2 303–334.
- [22] S. P. Meyn and R. L. Tweedie, *Markov chains and stochastic stability*. Cambridge University Press, 2009.
- [23] G. G. Yin and C. Zhu, *Hybrid switching diffusions: properties and applications*, vol. 63. Springer, 2009.
- [24] M. H. Davis, *Piecewise-deterministic Markov processes: A general class of non-diffusion stochastic models*, *Journal of the Royal Statistical Society. Series B (Methodological)* (1984) 353–388.
- [25] A. R. Teel, *Lyapunov conditions certifying stability and recurrence for a class of stochastic hybrid systems*, *Annual Reviews in Control* **37** (2013), no. 1 1–24.

- [26] J. Lygeros and M. Prandini, *Stochastic hybrid systems: a powerful framework for complex, large scale applications*, *European Journal of Control* **16** (2010), no. 6 583–594.
- [27] A. R. Teel, A. Subbaraman, and A. Sferlazza, *Stability analysis for stochastic hybrid systems: a survey*, *Automatica* **50** (2014), no. 10 2435–2456.
- [28] R. Goebel, R. G. Sanfelice, and A. R. Teel, *Hybrid dynamical systems*, *IEEE Control Systems Magazine* **29** (2009), no. 2 28–93.
- [29] P. Collins, *A trajectory-space approach to hybrid systems*, in *Proceedings of the 16th MTNS*, 2004.
- [30] M. S. Branicky, V. S. Borkar, and S. K. Mitter, *A unified framework for hybrid control: Model and optimal control theory*, *IEEE Transactions on Automatic Control* **43** (1998), no. 1 31–45.
- [31] A. Subbaraman and A. R. Teel, *On the equivalence between global recurrence and the existence of a smooth Lyapunov function for hybrid systems*, *Systems and Control Letters (Accepted)* (2015).
- [32] J.-P. Aubin and H. Frankowska, *Set-valued analysis*. Springer Science & Business Media, 2009.
- [33] H. Frankowska, *How set-valued maps pop up in control theory*, in *Systems and Control in the Twenty-First Century*, pp. 155–169. Springer, 1997.
- [34] E. O. Roxin, *Modern Optimal Control: A Conference in Honor of Solomon Lefschetz and Joseph P. La Salle*, vol. 119. CRC Press, 1989.
- [35] J.-P. Aubin and H. Frankowska, *Introduction: Set-valued analysis in control theory*, *Set-Valued Analysis* **8** (2000), no. 1 1–9.
- [36] J. Poveda, A. Teel, and D. Nesic, *Flexible Nash seeking using stochastic difference inclusions*, in *Proceedings of the American Control Conference*, 2015.
- [37] A. Subbaraman, M. Hartman, and A. R. Teel, *A stochastic hybrid algorithm for robust global almost sure synchronization on the circle: all-to-all communication*, in *Proceedings of the 52nd IEEE Conference on Decision and Control*, pp. 600–605, IEEE, 2013.
- [38] J. S. Shamma and K.-Y. Tu, *Set-valued observers and optimal disturbance rejection*, *IEEE Transactions on Automatic Control* **44** (1999), no. 2 253–264.
- [39] N. Seube, R. Moitie, and G. Leitmann, *Aircraft take-off in windshear: a viability approach*, *Set-Valued Analysis* **8** (2000), no. 1-2 163–180.

- [40] P. Frasca, R. Carli, and F. Bullo, *Multiagent coverage algorithms with gossip communication: control systems on the space of partitions*, in *Proceedings of the American Control Conference*, pp. 2228–2235, IEEE, 2009.
- [41] R. Goebel and A. R. Teel, *Solutions to hybrid inclusions via set and graphical convergence with stability theory applications*, *Automatica* **42** (2006), no. 4 573–587.
- [42] A. R. Teel, J. Hespanha, and A. Subbaraman, *A converse Lyapunov theorem and robustness for asymptotic stability in probability*, *IEEE Transactions on Automatic Control* (2014) 2426–2441.
- [43] M. Vidyasagar, *Nonlinear systems analysis*, vol. 42. SIAM, 2002.
- [44] H. K. Khalil, *Nonlinear systems*. Prentice hall Upper Saddle River, 3 ed., 2002.
- [45] C. Cai, A. R. Teel, and R. Goebel, *Smooth Lyapunov functions for hybrid systems Part I: existence is equivalent to robustness*, *IEEE Transactions on Automatic Control* **52** (2007), no. 7 1264–1277.
- [46] C. Cai, R. Goebel, and A. R. Teel, *Smooth Lyapunov functions for hybrid systems Part II: (pre) asymptotically stable compact sets*, *IEEE Transactions on Automatic Control* **53** (2008), no. 3 734–748.
- [47] J. B. Pomet and L. Praly, *A result on robust boundedness*, *Systems & Control Letters* **10** (1988), no. 2 83–92.
- [48] R. G. Sanfelice, R. Goebel, and A. Teel, *Invariance principles for hybrid systems with connections to detectability and asymptotic stability*, *IEEE Transactions on Automatic Control* **52** (2007), no. 12 2282–2297.
- [49] N. S. Goel and N. Richter-Dyn, *Stochastic models in biology*. Elsevier, 2013.
- [50] D. J. Wilkinson, *Stochastic modelling for systems biology*. CRC press, 2011.
- [51] P. R. Kumar and P. Varaiya, *Stochastic systems: estimation, identification and adaptive control*. Prentice-Hall, Inc., 1986.
- [52] D. A. Kendrick, *Stochastic control for economic models*. McGraw-Hill New York, 1981.
- [53] K. J. Åström, *Introduction to stochastic control theory*. Courier Corporation, 2012.
- [54] D. P. Bertsekas, *Dynamic programming and stochastic control*. Academic Press, Inc., 1976.
- [55] A. Subbaraman and A. R. Teel, *A converse Lyapunov theorem for strong global recurrence*, *Automatica* **49** (2013), no. 10 2963–2974.

- [56] A. Subbaraman and A. R. Teel, *A Matrosov theorem for strong global recurrence*, *Automatica* **49** (2013), no. 11 3390–3395.
- [57] L. E. Blume, *New techniques for the study of stochastic equilibrium processes*, *Journal of Mathematical Economics* **9** (1982), no. 1 61–70.
- [58] C. Aliprantis and K. Border, *Infinite Dimensional Analysis*. Springer, Berlin, 1999.
- [59] M. Kisielewicz, *Stochastic differential inclusions and applications*. Springer, 2013.
- [60] H. Fritsch, *A necessary extremality condition for a set-valued stochastic control problem*, *Mathematical Methods of Operations Research* **46** (1997), no. 1 29–49.
- [61] A. R. Teel, *A recurrence principle for stochastic difference inclusions*, *IEEE Transactions on Automatic Control* **60** (2015), no. 2 420–435.
- [62] W. Wonham, *Liapunov criteria for weak stochastic stability*, *Journal of Differential Equations* **2** (1966), no. 2 195–207.
- [63] R. Tweedie, *Criteria for classifying general Markov chains*, *Advances in Applied Probability* (1976) 737–771.
- [64] A. Cesaroni, *A converse Lyapunov theorem for almost sure stabilizability*, *Systems & Control Letters* **55** (2006), no. 12 992–998.
- [65] H. Kushner, *Converse theorems for stochastic Liapunov functions*, *SIAM Journal on Control* **5** (1967), no. 2 228–233.
- [66] C. M. Kellett and A. R. Teel, *On the robustness of KL-stability for difference inclusions: Smooth discrete-time Lyapunov functions*, *SIAM Journal on Control and Optimization* **44** (2005), no. 3 777–800.
- [67] N. Krasovskii, *Problems of the theory of stability of motion*. Stanford Univ. Press, 1963, translation of Russian edition, Moscow, 1959.
- [68] J. P. La Salle, *An invariance principle in the theory of stability*. New York: Academic Press, 1967.
- [69] A. Loría, E. Panteley, D. Popović, and A. R. Teel, *A nested Matrosov theorem and persistency of excitation for uniform convergence in stable nonautonomous systems*, *IEEE Transactions on Automatic Control* **50** (2005), no. 2 183–198.
- [70] T.-C. Lee, D.-C. Liaw, and B.-S. Chen, *A general invariance principle for nonlinear time-varying systems and its applications*, *IEEE Transactions on Automatic Control* **46** (2001), no. 12 1989–1993.
- [71] A. R. Teel, *A Matrosov theorem for adversarial Markov decision processes*, *IEEE Transactions on Automatic Control* **58** (2013), no. 8 2142–2148.

- [72] S. Grammatico, A. Subbaraman, and A. R. Teel, *Discrete-time stochastic control systems: A continuous Lyapunov function implies robustness to strictly causal perturbations*, *Automatica* **49** (2013), no. 10 2939–2952.
- [73] E. S. Meadows, M. A. Henson, J. W. Eaton, and J. B. Rawlings, *Receding horizon control and discontinuous state feedback stabilization*, *International Journal of Control* **62** (1995), no. 5 1217–1229.
- [74] D. Q. Mayne, J. B. Rawlings, C. V. Rao, and P. O. Scokaert, *Constrained model predictive control: Stability and optimality*, *Automatica* **36** (2000), no. 6 789–814.
- [75] J. Rawlings and D. Mayne, *Model predictive control: Theory and design*, Madison, WI: Nob Hill Publishing, LLC (2009).
- [76] D. Chatterjee and J. Lygeros, *On stability and performance of stochastic predictive control techniques*, *IEEE Transactions on Automatic Control* **60** (2015), no. 2 509–514.
- [77] G. Grimm, M. J. Messina, S. E. Tuna, and A. R. Teel, *Examples when nonlinear model predictive control is nonrobust*, *Automatica* **40** (2004), no. 10 1729–1738.
- [78] M. L. Bujorianu and J. Lygeros, *Toward a general theory of stochastic hybrid systems*, in *Stochastic Hybrid Systems*, pp. 3–30. Springer, 2006.
- [79] W. Glover and J. Lygeros, *A stochastic hybrid model for air traffic control simulation*, in *Hybrid Systems: Computation and Control*, pp. 372–386. Springer, 2004.
- [80] J. P. Hespanha, *Stochastic hybrid systems: Application to communication networks*, in *Hybrid systems: computation and control*, pp. 387–401. Springer, 2004.
- [81] J. Hu, W.-C. Wu, and S. Sastry, *Modeling subtilin production in bacillus subtilis using stochastic hybrid systems*, in *Hybrid Systems: Computation and Control*, pp. 417–431. Springer, 2004.
- [82] A. R. Teel, *Stochastic hybrid inclusions with diffusive flows*, in *Proceedings of the 53rd IEEE Conference on Decision and Control*, pp. 3071–3076, 2014.
- [83] A. Subbaraman and A. R. Teel, *Recurrence principles and their application to stability theory for a class of stochastic hybrid systems*, *IEEE Transactions on Automatic Control* (Conditionally accepted) (2015).
- [84] R. T. Rockafellar and R. J.-B. Wets, *Variational analysis*, vol. 317. Springer, 1998.
- [85] A. R. Teel, *On sequential compactness of solutions for a class of stochastic hybrid systems*, in *Proceedings of the American Control Conference*, pp. 4512–4517, 2014.



- [86] A. R. Teel, *On a recurrence principle for a class of stochastic hybrid systems*, in *Proceedings of the American Control Conference*, pp. 4518–4523, 2014.
- [87] A. Subbaraman and A. R. Teel, *Robustness of recurrence for a class of stochastic hybrid systems*, in *Analysis and Design of Hybrid Systems*, pp. 304–309, 2015.
- [88] G. Leoni, *A first course in Sobolev spaces*, vol. 105. American Mathematical Society Providence, RI, 2009.
- [89] R. L. Tweedie, *The robustness of positive recurrence and recurrence of Markov chains under perturbations of the transition probabilities*, *Journal of Applied Probability* (1975) 744–752.
- [90] R. Tweedie, *Perturbations of countable Markov chains and processes*, *Annals of the Institute of Statistical Mathematics* **32** (1980), no. 1 283–290.
- [91] C. E. de Souza, *Robust stability and stabilization of uncertain discrete-time Markovian jump linear systems*, *IEEE Transactions on Automatic Control* **51** (2006), no. 5 836–841.
- [92] L. Zhang and E.-K. Boukas, *Stability and stabilization of Markovian jump linear systems with partly unknown transition probabilities*, *Automatica* **45** (2009), no. 2 463–468.
- [93] L. Zhang and J. Lam, *Necessary and sufficient conditions for analysis and synthesis of Markov jump linear systems with incomplete transition descriptions*, *IEEE Transactions on Automatic Control* **55** (2010), no. 7 1695–1701.
- [94] M. Karan, P. Shi, and C. Y. Kaya, *Transition probability bounds for the stochastic stability robustness of continuous-and discrete-time Markovian jump linear systems*, *Automatica* **42** (2006), no. 12 2159–2168.
- [95] J. Xiong, J. Lam, H. Gao, and D. W. Ho, *On robust stabilization of Markovian jump systems with uncertain switching probabilities*, *Automatica* **41** (2005), no. 5 897–903.
- [96] S. Boyd and L. Vandenberghe, *Convex optimization*. Cambridge university press, 2004.
- [97] R. G. Sanfelice and A. R. Teel, *Asymptotic stability in hybrid systems via nested Matrosov functions.*, *IEEE Transactions on Automatic Control* **54** (2009), no. 7 1569–1574.
- [98] B. E. Fristedt and L. F. Gray, *A modern approach to probability theory*. Springer, 1997.
- [99] J. L. Doob, *Stochastic processes*, vol. 101. New York Wiley, 1953.