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Output Regulation for Linear Hybrid Systems with Periodic Jump Times

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Abstract

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The goal of this dissertation is to present a framework and regulator design for output regulation of linear hybrid systems with periodic jump times. The term output regulation is normally used in regards to the problem of regulating an error variable of a system in the presence of an exogenous system (exosystem). This problem comes up in the context of tracking a trajectory or rejecting a disturbance that can be modeled as the output of a dynamical system (the exosystem).

We begin by defining output regulation for this framework and developing a set of hybrid regulation equations and a hybrid internal model property. Following this we provide guidelines for the design of the regulator. The regulator should include an internal model capable of reproducing the output of the exosystem, as well as a stabilizer unit that is designed to make the closed loop system stable. The stabilizer unit used in this dissertation is a high gain stabilizer that utilizes a high gain observer to track unmeasured plant variables. The high gain

methods are based on their continuous time counterparts. The internal model is designed with an eye towards general applicability and thus takes advantage of a property called "visibility," so as to reproduce the steady-state trajectory of the exosystem, as opposed to the entire state, which is all that turns out to be necessary in order to achieve output regulation.

This framework of output regulation can be useful in attempting to asymptotically track trajectories that cannot be produced by continuous-time dynamical system, such as a spline trajectory, for which an example is provided. Furthermore, the use of an internal model allows one to achieve robust output regulation. In this context, robust output regulation means maintaining output regulation despite uncertain parameters in the plant.

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Chapter 1

Introduction

In recent years hybrid systems have taken a prominent role in control systems research. A hybrid system is a system that has both continuous time aspects and discrete time aspects. A physical example of a hybrid system is the bouncing ball. We can view the bouncing ball as a simple hybrid system that is a continuous time system while it is in the air and flowing according to a differential equation, but as a discrete time system at impacts, where it immediately changes directions according to a difference equation. By using a framework that accounts for both types of systems more general theory pertaining to dynamical systems can be developed. For example hybrid theory allows one to achieve global asymptotic stability of the inverted position of a pendulum on a cart ([53]). More information on hybrid systems and applications can be found in [29], [28], [37], [51]. Here we use the framework from [29] and [28].

Traditionally, most research in output regulation has dealt with either discrete time systems or continuous time systems, but not the combination of the two. The initial strides into robust output regulation were made by Francis in [22], and by Francis and Wonham in [23] and [24]. In their seminal works Francis and Wonham developed the internal model principle for continuous time dynamical systems. The internal model principle is the guideline for designing output regulators with structural stability, the robust component of output regulation. The main goal of robust output regulation is to utilize an internal model to achieve output regulation in the presence of an exosystem, while the plant may have uncertain parameters. In the terminology of output regulation, the exosystem (or exogenous system) is a dynamical system that affects the plant via disturbances or via tracking goals. For example one may want to reject a disturbance that can be generated by a dynamical system, or track a trajectory that can be generated by a dynamical system. In most cases for the purposes of output regulation the exosystem parameters are assumed to be known. Furthermore, the solvability of the output regulation problem is tied to the solvability of a set of regulation equations.

More recently research into output regulation and the use of internal models has been done in many areas. The primary areas of research have been in nonlinear systems, switched systems, and more recently hybrid systems. There has also been work pertaining to situations when the exosystem is unknown in at least some respects.

Isidori and Byrnes present the conditions needed to achieve output regulation for a fixed nonlinear plant. Most importantly they present the regulator equations analogous to those needed in the linear setting (see [34]). Later on, along with Priscoli and Kang, they further address issues pertaining to structural stability and robust regulation for nonlinear systems (see [7]). They provide an overview of their work in [3]. For more recent work by Isidori and Byrnes see [4] and [5]. Work on nonlinear minimum phase systems is done by Serrani, Isidori and Marconi in [43] (see also [40]). Further reading for nonlinear output regulation can be found in the book [6].

Recently there has been a surge in the research of switched systems and hybrid systems in relation to output regulation. The work most closely related to that presented here in switched systems is probably that of Gazi in [27], which presents a simplified version of the general hybrid output regulation equations of [39] by restricting applicable framework to not consider jumps in as broad of a sense, thus allowing for equations more similar to the ones presented by Francis and Wonham. Gazi's work has in mind the idea of tracking trajectories for robots, or robots in formation, see [26]. In other works the authors look at output regulation for linear periodic discrete time systems ([30]) and sampled-data systems ([31] and [36]). Also, in the vein of switched systems see [25]. With regards to more general hybrid systems, Marconi and Teel present results on hybrid regulation equations in [38], and again in [39], which incorporates results on robustness, relative degree and stability also discussed in [17], [18] and [13]. Further works relating to hybrid output regulation are presented by Carnevale, Galeani and Menini with regards to trajectory generation and stabilization ([9] and [10]). They also discuss an application for output tracking for a spinning and bouncing disk in [11].

In the remainder of this dissertation works pertaining to output regulation for linear hybrid systems with periodic jumps are presented. The main sources for the material are [38], [39], [17], [13], [18], [15] and [16]. The dissertation is organized as follows.

Chapter 2 defines hybrid output regulation as it will be framed in this dissertation and presents a set of hybrid regulation equations, which are shown to be similar to those presented by Francis and Wonham for continuous time linear systems. The regulation equations presented are differential Lyapunov equations, but sometimes it turns out that a time-invariant solution to these equations exists. Here a first attempt at internal model design is also made, but with the caveat that it is based on a technical assumption with respect to the solution of the hybrid regulation equations. Furthermore, Chapter 2 will discuss the relation of hybrid output regulation to the relative degree of the system and present stabilization results for relative degree one systems. Chapter 2 also preliminarily discusses the problem of robust output regulation, in particular giving an intuitive reason regarding the difficulties in achieving robustness as compared to the continuous-time setting.

Chapter 3 presents a method that achieves closed loop stability for the hybrid regulator using high-gain feedback. This is achieved for systems with relative degree greater than one by using a high-gain observer. For most of the chapter the system is assumed to be in Brunovsky's canonical form, as presented in Chapter 2, which aids in the discussion of relative degree, and the minimum phase assumptions.

Chapter 4 discusses the application of hybrid output regulation to asymptotically tracking spline trajectories. Utilizing an internal model to achieve this tracking goal allows for robust tracking, but here it is shown that the methods of internal model design presented in Chapter 2 are not sufficient for constructing an internal model capable of solving the hybrid output regulation problem for tracking splines. This topic is part of the motivation for the internal model design methods presented in Chapter 5.

A general method for internal model design for a class of MIMO linear hybrid systems is presented in Chapter 5. This work notably extends previous work to a class of MIMO systems in addition to addressing the problem of spline tracking presented in the previous chapter. It accomplishes the latter via avoiding the technical assumption present in Chapter 2 regarding the solution to the hybrid regulation equations. The internal model is constructed as a kind of observer capable of reproducing the output of the exosystem, though not necessarily the entire state. This is accomplished using continuous time visibility properties of the exosystem, while taking into account states that may only affect the output of the exosystem through jumps. Furthermore, the internal model designed in Chapter 5 is able to achieve robust hybrid output regulation with respect to parametric uncertainties in the plant.

Chapter 2

Defining Hybrid Output Regulation and the Hybrid Internal Model

The concepts of hybrid output regulation and the hybrid internal model are first laid out by Marconi and Teel in [38]. The goal of the paper is to present a set of hybrid regulation equations and a hybrid internal model property for linear systems with periodic jumps. The material in this paper is later extended to further include concepts of robustness and to tie the existence of solutions to the hybrid regulation equations to the relative degree of the plant in its journal form [39].

Here, we recall the major findings of these papers to help lay the groundwork for the hybrid regulation problems discussed later on. In doing so, we will touch on the concepts of continuous-time output regulation discussed by Francis and Wonham in [23] and [24] that are analogous to the hybrid concepts. We begin with the problem statement, including a precise definition of hybrid output regulation for linear hybrid systems with a periodic dwell time constraint. Then we present the hybrid regulation equations and the hybrid internal model property. Following this we show how the relative degree of the plant affects the solution to the hybrid regulation equations. Finally, we discuss the regulator design presented in [38] and [39]. The last two sections will also tie in closely to the work done in [18]. We present a short discussion of robust internal model design here, but leave the bulk of it until a later chapter, when the general methodology for designing such internal models is presented.

2.1 Problem Statement

Consider the hybrid linear system that flows according to

$$\begin{aligned} \dot{\tau} &= 1, \quad \dot{w} = Sw, \\ \dot{x} &= Ax + Bu + Pw, \end{aligned} \tag{2.1}$$

whenever $((\tau, w), x, u) \in \mathcal{W} \times \mathbb{R}^n \times \mathbb{R}$ and jumps according to

$$\begin{aligned} \tau^+ &= 0, \quad w^+ = Jw, \\ x^+ &= Nw + Mx, \end{aligned} \tag{2.2}$$

whenever $((\tau, w), x, u) \in (\mathcal{W} \bigcap (\{\tau_{\max}\} \times \mathbb{R}^s)) \times \mathbb{R}^n \times \mathbb{R}$, where

$$\mathcal{W} := \{(\tau, w) : \tau \in [0, \tau_{\max}], w \in W(\tau)\},$$

and where the set-valued mapping $\tau \to W(\tau) \subset \mathbb{R}^s$ is continuous with compact values. The scalar value $\tau_{\max}$ is a known constant representing the dwell time of the system between consecutive jumps. The cascade system can be seen as being broken into a clock (τ), exosystem (w) and plant (x). Furthermore we associate the *regulation error*,

$$e = Cx + Qw,\ e \in \mathbb{R}, \tag{2.3}$$

with the system (2.1)-(2.2), which jumps according to $e^+ = (CN+QJ)w + CMx$ (see (2.2)) whenever the system jumps.

Furthermore, consider a regulator of the form,

$$\begin{aligned} \dot{\xi} &= \Phi(\tau)\xi + \Lambda(\tau)e, \quad && (\tau, \xi, e) \in [0, \tau_{\max}] \times \mathbb{R}^m \times \mathbb{R}, \\ \xi^+ &= \Sigma\xi + \Delta e, && (\tau, \xi, e) \in \{\tau_{\max}\} \times \mathbb{R}^m \times \mathbb{R}, \\ u &= \Gamma(\tau)\xi + K(\tau)e, \end{aligned} \tag{2.4}$$

where $\Phi : [0, \tau_{\max}] \to \mathbb{R}^{m\times m}$, $\Lambda : [0, \tau_{\max}] \to \mathbb{R}^{m\times 1}$, $\Gamma : [0, \tau_{\max}] \to \mathbb{R}^{1\times m}$ and $K : [0, \tau_{\max}] \to \mathbb{R}$ are continuous functions.

The closed loop system resulting from (2.1)-(2.4) can be described by

$$
\begin{aligned}
\dot{\tau} &= 1, \quad \dot{w} = Sw,\\
\dot{x} &= (A + BK(\tau)C)x + B\Gamma(\tau)\xi + (p + BK(\tau)Q)w,\\
\dot{\xi} &= \Lambda(\tau)Cx + \Phi(\tau)\xi + \Lambda(\tau)Qw,\\
\tau^{+} &= 0, \quad w^{+} = Jw,\\
x^{+} &= Mx + Nw,\\
\xi^{+} &= \Delta Cx + \Sigma\xi + \Delta Qw,
\end{aligned}
\tag{2.5}
$$

where the jumps occur when $\tau = \tau_{\max}$. For brevity, we can rewrite the closed loop system (2.5) with $\zeta = \mathrm{col}(x, \xi)$ as

$$
\begin{aligned}
\dot{\tau} &= 1, \quad \dot{w} = Sw,\\
\dot{\zeta} &= \mathcal{H}_{\mathrm{cl}}(\tau)\zeta + \mathcal{L}_{\mathrm{cl}}(\tau)w,\\
\tau^{+} &= 0, \quad w^{+} = Jw,\\
\zeta^{+} &= \mathcal{J}_{\mathrm{cl}}\zeta + \mathcal{M}_{\mathrm{cl}}w.
\end{aligned}
\tag{2.6}
$$

Furthermore, let $\phi_{\mathrm{cl}}(\tau)$ be the state transition matrix of the time-varying system $\dot{\zeta} = \mathcal{H}_{\mathrm{cl}}(\tau)\zeta$.

In this framework, the goal of hybrid output regulation is to design the regulator, (2.4), such that the closed loop system, (2.5), with initial conditions in $[0, \tau_{\max}] \times \mathbb{R}^{s} \times \mathbb{R}^{n} \times \mathbb{R}^{m}$ has bounded trajectories, and is such that $\lim\limits_{t+j\to\infty} e(t, j) = 0$.

The accomplishment of this goal entails removing the effect of the exogenous system on the regulation error, where the exogenous system often represents some disturbances to be rejected or references to be tracked.

2.2 Hybrid Regulator Equations and Internal Model Property

Here we present necessary and sufficient conditions for solving the problem of hybrid output regulation as defined above. The conditions can be easily seen as an extension of the conditions for continuous-time systems (see [23]).

Theorem 1 *Suppose that the regulator (2.4) is such that the resulting closed-loop system (2.6) satisfies the following two requirements:*

- Stability Requirement *(SR):* $\mathrm{eig}(\mathcal{J}_{\mathrm{cl}}\phi_{\mathrm{cl}}(\tau_{\max})) \in \mathcal{D}_1$.
- Non-Resonance Requirement *(NR):* $\mathrm{eig}(\mathcal{J}_{\mathrm{cl}}\phi_{\mathrm{cl}}(\tau_{\max})) \bigcap \mathrm{eig}(J\exp(S\tau_{\max})) = \emptyset$.

Assume, in addition, that the set $\mathcal{W}$ is forward and backward invariant for the hybrid system described by the first two equations in (2.1)-(2.2). If $\lim\limits_{t+j\to\infty} e(t,j) = 0$ uniformly over compact sets of initial conditions, then necessarily there exist *continuous functions* $\Pi_x : [0, \tau_{\max}] \mapsto \mathbb{R}^{n\times s}$ *and* $R : [0, \tau_{\max}] \mapsto R^{1\times s}$ *that are solutions of*

$$\begin{aligned} \frac{d\Pi_x(\tau)}{d\tau} &= A\Pi_x(\tau) - \Pi_x(\tau)S + P + BR(\tau), \\ 0 &= M\Pi_x(\tau_{\max}) - \Pi_x(0)J + N, \\ 0 &= C\Pi_x(\tau) + Q, \end{aligned} \tag{2.7}$$

and $\Pi_\xi : [0, \tau_{\max}] \mapsto \mathbb{R}^{m\times s}$ *that is a solution of*

$$\begin{aligned} \frac{d\Pi_\xi(\tau)}{d\tau} &= \Phi(\tau)\Pi_\xi(\tau) - \Pi_\xi(\tau)S, \\ 0 &= \Sigma\Pi_\xi(\tau_{\max}) - \Pi_\xi(0)J, \\ R(\tau) &= \Gamma(\tau)\Pi_\xi(\tau). \end{aligned} \tag{2.8}$$

Conversely, if there exist continuous functions $\Pi_x : [0, \tau_{\max}] \mapsto \mathbb{R}^{n\times s}$, $R : [0, \tau_{\max}] \mapsto R^{1\times s}$ *and* $\Pi_\xi : [0, \tau_{\max}] \mapsto \mathbb{R}^{m\times s}$ *that solve (2.7) and (2.8) then the given controller (2.4) solves the problem of output regulation.*

See [39], Appendix B, for the proof of this theorem.

In imitation of the continuous-time terminology, we refer to (2.7) as the *Hybrid Regulation Equations* and we say that a regulator (2.4) satisfying (2.8) has the *Hybrid Internal Model Property*. To draw the direct analogy, we can consider the case where $J = I_s$, $N = 0$, and $M = I_n$, then the second equation of (2.7) implies $\Pi_x(\tau_{\max}) = \Pi_x(0)$. Then, (2.7) simplifies to the well-known continuous-time regulator equations $A\Pi_x - \Pi_x S + P + BR = 0$, $C\Pi_x + Q = 0$

in the constant unknown (Π_x, R) (see [23]), and by taking $\Sigma = I_m$ and again focusing on τ-independent regulators, (2.8) reduces to $\Phi\Pi_\xi - \Pi_\xi S = 0$, $R = \Gamma\Pi_\xi$ in the constant unknown Π_ξ.

It is useful to introduce a hybrid steady-state generator system that is able to produce all of the ideal steady-state trajectories for achieving hybrid output regulation. The *Hybrid Regulation Equations* play a crucial role in defining these trajectories, and thus this system. In particular, with $(\Pi_x(\cdot), R(\cdot))$ a solution of (2.7), and initial conditions $(\tau_0, w_0) \in \mathcal{W}$, let $E \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ be the hybrid time domain associated with system (2.1)-(2.2). Pick $u = R(\tau)w$ and take the initial conditions as $\tau(0,0) = \tau_0$, $w(0,0) = w_0$ and $x(0,0) = \Pi_x(\tau(0,0))w(0,0)$. Then we can define $\tilde{x} = x - \Pi_x(\tau)w$ and write

$$
\begin{aligned}
\dot{\tilde{x}} &= A\tilde{x}, \quad \tau \in [0, \tau_{\max}], \\
\tilde{x}^+ &= M\tilde{x}, \quad \tau \in \{\tau_{\max}\}.
\end{aligned}
\tag{2.9}
$$

This system has an equilibrium point at $\tilde{x} = 0$, therefore, since we picked $x(0,0) = \Pi_x(\tau(0,0))w(0,0)$, $x(t,j) = \Pi_x(\tau(t,j))w(t,j)$ for all $(t,j) \in E$. Importantly, this further leads to the third equation of (2.7) resulting in $e(t,j) = 0$ for all $(t,j) \in E$. Thus, $x(t,j) = \Pi_x(\tau(t,j))w(t,j)$ and $u(t,j) = R(\tau(t,j))w(t,j)$, $(t,j) \in E$, represent ideal steady-state trajectories for the state, x, and input, u, in order to fulfill the regulation objective.

Since (τ_0, w_0) is arbitrary in $\mathcal{W}$, it is apparent that any regulator (2.4) solving the problem of hybrid output regulation must be able to generate all possible signals generated by the output y_w of the system

$$\begin{array}{rclrclr} \dot{\tau} & = & 1, & \dot{w} & = & Sw, & (\tau, w) \in \mathcal{W}, \\ \tau^+ & = & 0, & w^+ & = & Jw, & (\tau, w) \in \mathcal{W} \bigcap (\{\tau_{\max}\} \times \mathbb{R}^s), \\ y_w & = & R(\tau)w, & & & & \end{array} \tag{2.10}$$

when the input e of (2.4) is identically zero. This property is referred to as the hybrid internal model property, and is guaranteed by the existence of a solution $\Pi_\xi(\cdot)$ of (2.8).

In fact, using similar arguments to those above, let $E \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ be the hybrid time domain associated with (τ, w) subsystem (2.1)-(2.2), with the initial condition (τ_0, w_0) taken arbitrarily in $\mathcal{W}$, and to the regulator (2.4), with $e = 0$ and the initial condition $\xi(0,0) = \Pi_\xi(\tau(0,0))w(0,0)$. Furthermore, let $\Pi_x(\cdot)$ be a solution of (2.8). Then the regulator dynamics guarantee that $\xi(t,j) = \Pi_\xi(\tau(t,j))w(t,j)$ for all $(t,j) \in E$, while the third equation of (2.8) guarantees that $u(t,j) = R(\tau(t,j))w(t,j)$ for all $(t,j) \in E$, in other words, the regulator (2.4) has the hybrid internal model property.

2.3 Regulator Design

The goal here is to design the regulator parameters $(\Phi(\cdot), \Gamma(\cdot), \Sigma)$ such that the regulator has the internal model property (ie. (2.8) are satisfied), while also designing the regulator parameters $(\Phi(\cdot), \Lambda(\cdot), \Gamma(\cdot), K(\cdot), \Sigma, \Delta)$ such that (SR) and (NR) are fulfilled. In doing so, we split the regulator into an internal model and a stabilizer (ie. $\xi = \mathrm{col}(\xi_{\mathrm{im}}, \xi_{\mathrm{st}})$), as is done in the continuous-time domain. The regulator parameters $\Phi(\cdot)$, Σ and $\Gamma(\cdot)$ are partitioned according to

$$\Phi(\tau) = \begin{pmatrix} \Phi_{\mathrm{im}}(\tau) & \Phi_{\Delta}(\tau) \\ 0 & \Phi_{\mathrm{st}}(\tau) \end{pmatrix}, \Sigma = \begin{pmatrix} \Sigma_{\mathrm{im}} & \Sigma_{\Delta} \\ 0 & \Sigma_{\mathrm{st}} \end{pmatrix}, \Gamma(\tau) = \begin{pmatrix} \Gamma_{\mathrm{im}}(\tau) & \Gamma_{\mathrm{im}}(\tau) \end{pmatrix}.$$

Thus, we have an *internal model* unit of the form

$$\begin{aligned} \dot{\tau} &= 1, & \dot{\xi}_{\mathrm{im}} &= \Phi_{\mathrm{im}}(\tau)\xi_{\mathrm{im}} + \Phi_{\Delta}(\tau)\xi_{\mathrm{st}} + \Lambda_{\mathrm{im}}(\tau)e, \\ \tau^{+} &= 0, & \xi_{\mathrm{im}}^{+} &= \Sigma_{\mathrm{im}}\xi_{\mathrm{im}} + \Sigma_{\Delta}\xi_{\mathrm{st}} + \Delta_{\mathrm{im}}e, \end{aligned}$$

and a *stabilizer* unit of the form

$$\begin{aligned} \dot{\tau} &= 1, & \dot{\xi}_{\mathrm{st}} &= \Phi_{\mathrm{st}}(\tau)\xi_{\mathrm{st}} + \Lambda_{\mathrm{st}}(\tau)e, \\ \tau^{+} &= 0, & \xi_{\mathrm{st}}^{+} &= \Sigma \mathrm{st}\xi_{\mathrm{st}} + \Delta_{\mathrm{st}}e, \end{aligned}$$

where the flow and jump conditions are given by $(\tau, \xi, e) \in [0, , \tau_{\max}] \times \mathbb{R}^m \times \mathbb{R}$ and $(\tau, \xi, e) \in \{\tau_{\mathrm{im}}\} \times \mathbb{R}^m \times \mathbb{R}$, respectively, with the input

$$u = \Gamma_{\mathrm{im}}(\tau)\xi_{\mathrm{im}} + \Gamma_{\mathrm{st}}(\tau)\xi_{\mathrm{st}} + K(\tau)e.$$

In this framework, if the internal model can be designed such that the triplet $(\Phi_{\mathrm{im}}(\cdot), \Gamma_{\mathrm{im}}(\cdot), \Sigma_{\mathrm{im}})$ allows for the equations

$$\begin{aligned}
\frac{d\Pi'_\xi(\tau)}{d\tau} &= \Phi_{\mathrm{im}}(\tau)\Pi'_\xi(\tau) - \Pi'_\xi(\tau)S, \\
0 &= \Sigma_{\mathrm{im}}\Pi'_\xi(\tau_{\max}) - \Pi'_\xi(0)J, \\
R(\tau) &= \Gamma_{\mathrm{im}}(\tau)\Pi'_\xi(\tau),
\end{aligned} \tag{2.11}$$

to have a continuously differentiable solution $\Pi'_\xi(\cdot)$, then (2.8) are solved by a continuously differentiable solution $\Pi_\xi(\cdot)$, with $\Pi_\xi(\tau) = \mathrm{col}(\Pi'_\xi(\tau), 0)$ for any choice of the stabilizer parameters $(\Phi_{\mathrm{st}}, \Phi_\Delta, \Lambda, \Gamma_{\mathrm{st}}, \Sigma_{\mathrm{st}}, \Sigma_\Delta, K, \Delta)$. This allows the stabilizer parameters to be chosen such that (SR) and (NR) hold.

2.3.1 Sufficient Conditions for Internal Model Design

Here we present sufficient conditions for the design of the internal model unit. As shown in theorem 1 the internal model must be dependent on the clock, in general, but it is natural in comparing the internal model results for linear systems with periodic state jumps with results for continuous-time systems to wonder if there are scenarios for which the internal model can be designed independently of the clock variable, τ. In this section we first discuss the general τ-dependent model, then we present results regarding τ-independent models. In both subsections we will present sufficient conditions that allow for straightforward design of the internal model.

τ-Dependent Internal Model Design

Following the ideas of canonical internal model design for continuous-time systems (see [42]), we choose the pair $(\Phi_{\rm im}(\cdot), \Gamma_{\rm im}(\cdot))$ such that $\Phi_{\rm im}(\tau) = F_{\rm im} + G_{\rm im}\Gamma_{\rm im}(\tau)$ with $(F_{\rm im}, G_{\rm im}) \in \mathbb{R}^{\nu\times\nu} \times \mathbb{R}^{\nu\times 1}$, $\nu > 0$, a pair to be designed. The internal model property equations (2.11) thus become

$$\begin{aligned} \frac{d\Pi'_\xi(\tau)}{d\tau} &= F\Pi'_\xi(\tau) - \Pi'_\xi(\tau)S + G_{\rm im}R(\tau), \\ 0 &= \Sigma_{\rm im}\Pi'_\xi(\tau_{\max}) - \Pi'_\xi(0)J, \\ R(\tau) &= \Gamma_{\rm im}(\tau)\Pi'_\xi(\tau). \end{aligned} \tag{2.12}$$

The solution $\Pi'_\xi(\cdot)$ of the first two equations of (2.12) turns out to be

$$\Pi'_\xi(\tau) = (\exp(F_{\rm im}\tau)\Pi'_\xi(0) + L(\tau))\exp(-S\tau),$$

with the boundary constraint

$$0 = \Sigma_{\rm im}\exp(F_{\rm im}\tau_{\max})\Pi'_\xi(0) - \Pi'_\xi(0)J\exp(S\tau_{\max}) + \Sigma_{\rm im}L(\tau_{\max}),$$

where $L(\tau)$ is the solution to the linear matrix differential equation

$$L(0) = 0, \qquad \frac{dL(\tau)}{d\tau} = F_{\rm im}L(\tau) + G_{\rm im}R(\tau)\exp(S\tau).$$

See the proof of Lemma 1 in Appendix A of [39] for details. The boundary constraint for $\Pi'_\xi(\tau)$ is a Sylvester equation, which admits a unique solution $\Pi'_\xi(0)$ if and only if the eigenvalues of the matrices $\Sigma_{\rm im}\exp(F_{\rm im}\tau_{\max})$ and $J\exp(S\tau_{\max})$

are disjoint. This leaves us with the task of satisfying the third equation of (2.12) via appropriate design of a continuous $\Gamma_{\mathrm{im}} : [0, \tau_{\max}] \to \mathbb{R}^{1\times\nu}$. The following proposition is presented in [39] along with its proof.

Proposition 1 *Let F_{im} and Σ_{im} be chosen such that the eigenvalues of $\Sigma_{\mathrm{im}} \exp(F_{\mathrm{im}}\tau_{\max})$ and $J \exp(S\tau_{\max})$ are disjoint, so that the first two equations of (2.12) admit a continuously differentiable solution $\Pi'_\xi : [0, \tau_{\max}] \to \mathbb{R}^{\nu\times s}$. Let $\nu \geq s$ and the pair $(F_{\mathrm{im}}, G_{\mathrm{im}})$ be controllable. If there exists a positive $r \leq \nu$ such that the rank of $\Pi'_\xi(\tau) = r$ for all $\tau \in [0, \tau_{\max}]$, then there exists a continuous function $\Gamma_{\mathrm{im}} : [0, \tau_{\max}] \to \mathbb{R}^{1\times\nu}$ such that*

$$R(\tau) \;=\; \Gamma_{\mathrm{im}}(\tau)\Pi'_\xi(\tau)\forall\tau \in [0, \tau_{\max}].$$

In fact, it is possible to take $\Gamma_{\mathrm{im}}(\tau) = R(\tau)\Pi'_\xi(\tau)^\dagger$, where $\Pi'_\xi(\tau)^\dagger \in \mathbb{R}^{s\times\nu}$ represents the Moore-Penrose pseudo-inverse of $\Pi'_\xi(\tau)$.

τ-Independent Internal Model Design

We now look at conditions under which the internal model can be designed independently of the clock variable, τ. The conditions here are inspired by example 1 of [17] and appear in [39]. Here we cover the τ-independent case briefly, as it has little impact on the further work herein, but refer the interested reader to [39] for greater detail. First, it is clear that it must be the case that the hybrid regulator equations (2.7) must allow for a constant solution $R(\tau) = R$. With this in mind we can look for conditions under which there exists a constant triplet $(\Phi_{\mathrm{im}}, \Gamma_{\mathrm{im}}, \Sigma_{\mathrm{im}})$ such that the equations

$$0 = \Phi_{\mathrm{im}}\Pi'_\xi - \Pi'_\xi S, \quad 0 = \Sigma_{\mathrm{im}}\Pi'_\xi - \Pi'_\xi J, \quad R = \Gamma_{\mathrm{im}}\Pi'_\xi, \tag{2.13}$$

admit a constant solution Π'_ξ.

It turns out that the existence of solution to these equations is affected by observability properties of the hybrid steady-state generator system (2.10). The conditions will be presented in the following two propositions, but first it useful to define the following notation. We denote the *unobservability subspace* of the pair (S, R) by $\mathcal{N}(S, R) := \mathrm{Ker}\mathcal{O}$ where $\mathcal{O}$ is the observability matrix of the pair (S, R). Furthermore, the observability index of the pair (S, R) is denoted by ν, and T represents any $s \times s$ non-singular matrix whose last $s - \nu$ columns span $\mathrm{Ker}\mathcal{O}$. The following two propositions are taken from [39].

Proposition 2 *Let the hybrid regulation equations (2.7) be solvable with a constant non-zero $R(\tau) = R$. If $\mathcal{N}(S, R)$ is invariant for J, ie.,*

$$J\mathcal{N}(S, R) \subseteq \mathcal{N}(S, R)$$

then there exists a triplet $(\Phi_{\rm im}, \Gamma_{\rm im}, \Sigma_{\rm im}) \in \mathbb{R}^{\nu\times\nu} \times \mathbb{R}^{1\times\nu} \times \mathbb{R}^{\nu\times\nu}$ *with* $(\Phi_{\rm im}, \Gamma_{\rm im})$ *observable and a matrix* Π'_ξ *satisfying (2.13). In fact it is possible to take*

$$\begin{aligned}\Phi_{\rm im} &= \begin{pmatrix} 0 & I_{\nu-1\times\nu-1} \\ -p_0 & (-p_1 \cdots - p_{\nu-1}) \end{pmatrix}, \\ \Gamma_{\rm im} &= \begin{pmatrix} 1 & 0_{1\times\nu-1} \end{pmatrix}, \quad \Sigma_{\rm im} = \mathcal{O}_o J_o \mathcal{O}_o^{-1},\end{aligned} \tag{2.14}$$

with $\lambda^\nu + p_{\nu-1}\lambda^{\nu-1} + \cdots + p_1\lambda + p_0$ *the characteristic polynomial of* $[TST^{-1}]_{\nu\times\nu}$, $\mathcal{O}_o = [\mathcal{O}T^{-1}]_{\nu\times\nu}$ *and* $J_o = [TJT^{-1}]_{\nu\times\nu}$.

There is also dual result to this, which can be obtained by swapping the roles of S and J, and $\Phi_{\rm im}$ and $\Sigma_{\rm im}$.

Proposition 3 *Let the hybrid regulation equations (2.7) be solvable with a constant non-zero* $R(\tau) = R$ *and let* $\mathcal{N}(J, R)$ *be the unobservability subspace of the pair* (J, R). *If*

$$S\mathcal{N}(J, R) \subseteq \mathcal{N}(J, R),$$

then there exists a triplet $(\Phi_{\rm im}, \Gamma_{\rm im}, \Sigma_{\rm im}) \in \mathbb{R}^{\nu\times\nu} \times \mathbb{R}^{1\times\nu} \times \mathbb{R}^{\nu\times\nu}$ *with* $(\Sigma_{\rm im}, \Gamma_{\rm im})$ *observable and a matrix* Π'_ξ *satisfying (2.13). In fact, it is possible to take* $\Phi_{\rm im} = \mathcal{O}_o S_o \mathcal{O}_o^{-1}$, $\Gamma_{\rm im}$ *as in (2.14), and* $\Sigma_{\rm im}$ *as the* $\Phi_{\rm im}$ *in (2.14), where* $\lambda^\nu + p_{\nu-1}\lambda^{\nu-1} + \cdots + p_1\lambda + p_0$ *is the characteristic polynomial of* $[TJT^{-1}]_{\nu\times\nu}$, $\mathcal{O}_o = [\mathcal{O}T^{-1}]_{\nu\times\nu}$ *and* $S_o = [TST^{-1}]_{\nu\times\nu}$.

2.4 The Effect of Relative Degree of the Plant on the Hybrid Regulation Equations

It turns out that the relative degree of the plant plays an important role in the hybrid output regulation problem. The reason for this can be seen intuitively as a result of enforcing an identically zero regulation error at jump times via the control input u, which enters only during flows. The pertinent part of this statement is that this requires the ability to enforce discontinuities on the output $y = Cx$ of the system according to the jumps of the exogenous signal Qw. Because of this, we will take a moment here to apply a transform $x \to \xi := Tx$ that puts the triplet (A, B, C) of system (2.1)-(2.2) into Brunovsky's canonical form. Specifically, let $r \leq n$ be the relative degree of the triplet (A, B, C), ie. the lowest $r > 0$ such that $CA^{r-1}B \neq 0$, then with r in hand let $T := \begin{pmatrix} T_1^T & T_2^T \end{pmatrix}^T$ with $T_2 := \begin{pmatrix} C^T & A^TC^T & \cdots & (A^{r-1})^TC^T \end{pmatrix}^T \in \mathbb{R}^{r \times n}$ and $T_1 \in \mathbb{R}^{n-r \times n}$ chosen in such a way that T is non-singular. Then, we can rewrite and partition the transformed matrices A, P, M and N of (2.1)-(2.2) as follows.

$$
\begin{aligned}
TAT^{-1} &= \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad & TMT^{-1} &= \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}, \\
TP &= \begin{pmatrix} P_1 \\ P_2 \end{pmatrix}, & TN &= \begin{pmatrix} N_1 \\ N_2 \end{pmatrix}.
\end{aligned}
$$

Then by partitioning $\xi = (z, \mathbf{y})$, the system becomes,

$$\begin{aligned} \dot{z} &= A_{11}z + A_{12}\mathbf{y} + P_1 w, \\ \dot{\mathbf{y}} &= A_{21}z + A_{22}\mathbf{y} + B_2 u + P_2 w, \end{aligned} \tag{2.15}$$

during flows and

$$\begin{aligned} z^+ &= M_{11}z + M_{12}\mathbf{y} + N_1 w, \\ \mathbf{y}^+ &= M_{21}z + M_{22}\mathbf{y} + N_2 w, \end{aligned} \tag{2.16}$$

during jumps, where the matrix parameters A_{21}, A_{21}, B_2 are of the form

$$A_{22} = \begin{pmatrix} \mathbf{1}_{\mathrm{sd}} \\ \bar{A}_{22} \end{pmatrix}, \quad A_{21} = \begin{pmatrix} 0 \\ \bar{A}_{21} \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 \\ b \end{pmatrix},$$

where $\mathbf{1}_{\mathrm{sd}}$ is the matrix with all zero elements except along the super-diagonal, which is filled with 1's, $\bar{A}_{22}$ and $\bar{A}_{21}$ are row vectors and $b \in \mathbb{R}$ is non-zero.

Note that now the output of the original plant, $y = Cx$, is the first element of the vector $\mathbf{y}$, this is a direct result of the transform into Brunovsky's canonical form. Furthermore, with the goal of achieving zero error, recall (2.3), we can write down the dynamics of the error system via the transform $\mathbf{y} \to \mathbf{e} = \mathbf{y} + Vw$, where

$$V = \begin{pmatrix} V_1 \\ \vdots \\ V_r \end{pmatrix}, \text{ with the rows, } V_i, \text{ such that } V_i = QS^{i-1} + \sum_{j=2}^{i} P_{2(j-1)} S^{i-j}.$$

The error system dynamics are then

$$\begin{aligned} \dot{z} &= A_{11}z + A_{12}\mathbf{e} + \bar{P}_1 w, \\ \dot{\mathbf{e}} &= A_{21}z + A_{22}\mathbf{e} + B_2 u + \bar{P}_2 w, \end{aligned} \tag{2.17}$$

during flows and

$$\begin{aligned} z^+ &= M_{11}z + M_{12}\mathbf{e} + \bar{N}_1 w, \\ \mathbf{e}^+ &= M_{21}z + M_{22}\mathbf{e} + \bar{N}_2 w, \end{aligned} \tag{2.18}$$

during jumps. The matrix parameters $\bar{P}_1$, $\bar{P}_2$, $\bar{N}_1$ and $\bar{N}_2$ are as follows, where the form of $\bar{P}_2$ is of particular note,

$$\bar{P}_1 = P_1 - A_{12}V,\ \bar{P}_2 = \begin{pmatrix} 0_{r-1\times s} \\ \bar{P}_{2r} \end{pmatrix},\ \bar{N}_1 = N_1 - M_{12}V,\ \bar{N}_2 = N_2 + VJ - M_{22}V,$$

with $\bar{P}_{2r} = P_{2r} + V_r S - \bar{A}_{22}V$.

Using this transformed system, and thanks to the structure of $\bar{P}_2$, it should be clear that we can present sufficient conditions under which the hybrid regulation equations (2.7) admit a solution based on the zero dynamics of the plant, i.e. the sub-state z (see [39] and [18]). In fact, the following proposition regarding the data of the zero dynamics holds.

Proposition 4 *When $r < n$, the hybrid regulator equations (2.7) are solvable if*

$$\mathrm{eig}(M_{11}\exp(A_{11}\tau_{\max})) \cap \mathrm{eig}(J\exp(S\tau_{\max})) = \emptyset$$

and the unique solution Z of the Sylvester equation

$$M_{11}\exp(A_{11}\tau_{\max})Z - ZJ\exp(S\tau_{\max}) = H_1$$

is also a solution of

$$M_{21} \exp(A_{11}\tau_{\max})Z = H_2.$$

When $r = n$, the hybrid regulator equations (2.7) are solvable if

$$M_{22}V - VJ - N_2 = 0.$$

See [39] for the proof of the above proposition.

2.5 Stabilizing the Relative Degree One Case

Designing a stabilizer for the case where the plant is Brunovsky's canonical form and has relative degree one is fairly simple. The internal model triplet $(\Phi_{\rm im}(\cdot), \Sigma_{\rm im}, \Gamma_{\rm im})$ is chosen so that

$$\Phi_{\rm im}(\tau) = F_{\rm im} + G_{\rm im}\Gamma_{\rm im}(\tau),$$

with $(F_{\rm im}, G_{\rm im})$ an arbitrary controllable pair fulfilling the conditions of Proposition 1. The remaining design parameters for the regulator can be taken as

$$(\Phi, \Lambda, \Sigma, \Delta, \Gamma) = (\Phi_{\rm im}, \Lambda_{\rm im}, \Sigma_{\rm im}, \Delta_{\rm im}, \Gamma_{\rm im})$$

with the choices

$$\Lambda_{\rm im} = G_{\rm im}K, \quad \Delta_{\rm im} = 0, \quad K = -\kappa. \tag{2.19}$$

Then by choosing κ appropriately, with an additional constraint on the design of $(F_{\text{im}}, \Sigma_{\text{im}})$, the hybrid output regulation problem is solved according to the following proposition from [39].

Proposition 5 *Let* $(F_{\text{im}}, G_{\text{im}}, \Sigma_{\text{im}}, \Gamma_{\text{im}}(\cdot))$ *be fixed according to Proposition 1 with* $(F_{\text{im}}, \Sigma_{\text{im}})$ *chosen such that* $\text{eig}(\Sigma_{\text{im}} \exp(F_{\text{im}}\tau_{\max})) \in \mathcal{D}_1$*. Furthermore, assume that* $\text{eig}(M_{11} \exp(A_{11}\tau_{\max})) \in \mathcal{D}_1$*. Then there exists* $\kappa^* > 0$ *such that for all* $\kappa \geq \kappa^*$ *the closed-loop system resulting from the choices 2.19 fulfills the stability requirement,* $\text{eig}(\mathcal{J}_{\text{cl}}\phi_{\text{cl}}(\tau_{\max})) \in \mathcal{D}_1$*.*

This proposition clearly leaves a lot of room for improvement, as it is limited to the relative degree one case for the plant. Although, it turns out that using a high-gain stabilization method is still viable for higher-relative degree systems. This stabilizer design is covered in Chapter 3.

2.6 Robust Output Regulation

One downfall of the proposed internal model design method outlined here is a lack of robustness to parametric uncertainties in (2.1)-(2.2). If we we say that the matrices in (2.1)-(2.2) depend on a constant uncertain vector μ, then the solution $R(\cdot)$ of the regulation equations (2.7) will, in general, depend on the uncertainty μ. Since, $R(\tau)$ then enters into the internal model design through

$\Gamma(\tau)$ this causes a lack of robustness. This type of robustness is what Francis and Wonham call structural stability in the continuous-time case (see [23], [24]).

Recall that in the continuous-time case the internal model property equations reduce to $\Phi_{\text{im}}\Pi_\xi - \Pi_\xi S = 0$, $R = \Gamma_{\text{im}}\Pi_\xi$. Then, it turns out that by taking $(\Phi_{\text{im}}, \Gamma_{\text{im}})$ as in (2.14) the solution to the internal model property equations can be taken as $\Pi_\xi = \begin{pmatrix} R^T & S^T R^T & \dots & S^{\nu-1T} R^T \end{pmatrix}^T$, thus limiting the uncertainties to Π_ξ, thereby achieving the desired robustness.

Robust internal model design will be returned to in Chapter 5, since the general design methodology proves to be somewhat complex.

2.7 Conclusions

In this chapter we have laid the groundwork for talking about output regulation for hybrid systems with periodic jump times. The problem is defined explicitly in Section 2.1. Then the fundamental theorem relating the hybrid regulation equations and the internal model property to the problem of output regulation is presented. Sufficient conditions for the design of the internal model unit of the regulator are presented.

A few things are left open from this work, though. The first major missing piece is the design of the stabilizer unit of the regulator for systems of relative degree greater than one. A constructive method for designing the stabilizer unit based continuous-time high-gain feedback is presented in the next chapter. The major findings in that chapter were first presented in [17], and later in more detail in [18].

Furthermore, although sufficient conditions for the design of the internal model are covered, it turns out that a more guided approach to the internal model design can be very useful. There are certain exosystem-plant pairings that do not lend themselves to "easy" design, but can be dealt with if more guidance is given. In particular, the spline tracking problem presented in [13] is one such exosystem. A general and constructive internal model design approach is presented in [16], also see [15]. Lastly, the question of designing robust internal models, in the sense of structural stability in the parlance of Francis and Wonham (see [23] and [24]), is left open by [39] and [38]. As such, the design of internal models that are robust with respect to perturbations to plant data is also presented in [16], also see [15].

Chapter 3

Stabilizer Design

In designing the regulator to achieve hybrid output regulation we are able to separate the stabilizer design from the internal model design, see section 2.3 for details (also see [39], [38]). In this chapter we discuss the design of the stabilizer unit with regards to achieving output regulation. For the purposes of this chapter the plant is assumed to already be in Brunovsky's canonical form, the details of which are covered in Section 2.4. It is shown that a high gain observer can be used to estimate the appropriate sub-state of the plant, then high gain feedback of the estimate can be used to achieve the stability requirement for hybrid output regulation (also see [17] and [18]).

Section 3.1 contains the class of systems and the problem. Section 3.2 gives the regulator design. Following this, the main result of the chapter is presented in Section 3.3 followed by some examples. Finally, an additional result based on regulation through jumps in presented in Section 3.4 followed by an example.

3.1 The Class of Systems and The Problem

Assume we have a system in the Brunovsky canonical form

$$\left.\begin{aligned} \dot{\tau} &= 1, \quad \dot{w} = Sw, \\ \dot{z} &= A_{11}z + A_{12}y + P_1 w, \\ \dot{y} &= A_{21}z + A_{22}y + B_2 u + P_2 w, \end{aligned}\right\} (\tau, w, z, y) \in [0, \tau_{\max}] \times W \times \mathbb{R}^n \times \mathbb{R}^m, \tag{3.1}$$

$$\left.\begin{aligned} \tau^+ &= 0, \quad w^+ = Jw, \\ z^+ &= M_{11}z + M_{12}y + N_1 w, \\ y^+ &= M_{21}z + M_{22}y + N_2 w, \end{aligned}\right\} (\tau, w, z, y) \in \{\tau_{\max}\} \times W \times \mathbb{R}^n \times \mathbb{R}^m, \tag{3.2}$$

where $W \subset \mathbb{R}^s$ is compact and the set $[0, \tau_{\max}] \times W$ is (forward) invariant for (τ, w), and the matrix parameters A_{21}, A_{21}, B_2 are of the form $A_{22} = \begin{pmatrix} \mathbf{1}_{\mathrm{sd}} \\ \bar{A}_{22} \end{pmatrix}$, $A_{21} = \begin{pmatrix} 0 \\ \bar{A}_{21} \end{pmatrix}$, $B_2 = \begin{pmatrix} 0 \\ b \end{pmatrix}$, where $\mathbf{1}_{\mathrm{sd}}$ is the matrix with all zero elements except the super-diagonal that is filled with 1's, $\bar{A}_{22}$ and $\bar{A}_{21}$ are row vectors and $b \in \mathbb{R}$ is non-zero. Furthermore, we refer to the sub-state (z, x) as the plant and the sub-state w as the exosystem, both of which jump in sync with the clock variable, τ.

In this form, the goal of output regulation is to regulate the variable

$$e_1 = y_1 + Qw,\; e_1 \in \mathbb{R} \tag{3.3}$$

i.e., we want to find a regulator that processes only the error, e_1, and asymptotically steers it to zero.

The internal model design has already been discussed in Chapter 2, and will be further discussed in Chapter 5. So far, we have shown that the internal model unit can be designed separately from the stabilizer unit while guaranteeing that the overall regulator will have the internal model property (see Section 2.3). Our goal is now to design the stabilizer unit to satisfy the stability requirement for hybrid output regulation (see Chapter 2, Theorem 1).

3.2 The Regulator Design

In this chapter we assume that the internal model has been designed of the form

$$\begin{aligned}
&\left.\begin{aligned}\dot{\tau} &= 1\\ \dot{\eta} &= F\eta + Gu\end{aligned}\right\} (\tau,\eta) \in [0,\tau_{\max}] \times \mathbb{R}^{\nu},\\
&\left.\begin{aligned}\tau^{+} &= 0\\ \eta^{+} &= \Sigma_{\mathrm{im}}\eta\end{aligned}\right\} (\tau,\eta) \in \{\tau_{\max}\} \times \mathbb{R}^{\nu},\\
&u = \Gamma(\tau)\eta + v,
\end{aligned} \tag{3.4}$$

where $\nu \in \mathbb{N}$, F, G and Σ_{im} are matrices, and v is a residual control input. Furthermore, we assume that the internal model has been designed to have the internal model property (see Theorem 1).

It turns out that we are able to use a high-gain observer to observe the sub-state y via the measured variable $e_1 = y_1 + Qw$ (see [50] for the continuous-time high-gain observer). This observer can then be utilized in a high-gain feedback law to achieve the stability requirement of Theorem 1. Thus, the stabilizer unit is of the form

$$\left.\begin{array}{rcl} \dot{\tau} & = & 1, \\ \dot{\xi} & = & \Phi_{\mathrm{st}}\xi + \Lambda_{\mathrm{st}} e_1, \end{array}\right\} (\tau, \xi) \in [0, \tau_{\max}] \times \mathbb{R}^m,$$
$$\left.\begin{array}{rcl} \tau^+ & = & 0, \\ \xi^+ & = & \Sigma_{\mathrm{st}}\xi, \end{array}\right\} (\tau, \xi) \in \{\tau_{\max}\} \times \mathbb{R}^m, \tag{3.5}$$

with the residual control input, v, chosen as

$$v = K\xi. \tag{3.6}$$

The matrix parameter Φ_{st}, Λ_{st}, Σ_{st} and K are chosen as

$$\begin{array}{rclcrcl} K & = & -sgn(b)\kappa \begin{pmatrix} k_1 & \dots & k_{m-1} & 1 \end{pmatrix}, & & \Lambda_{st} & = & \begin{pmatrix} c_1 g \\ \vdots \\ c_m g^m \end{pmatrix}, \\ \Phi_{st} & = & \begin{pmatrix} -\Lambda_{st} & \begin{matrix} I_{m-1} \\ 0_{1\times(m-1)} \end{matrix} \end{pmatrix}, & & \Sigma_{\mathrm{st}} & = & M_{22}, \end{array} \tag{3.7}$$

where the coefficients c_i are designed such that $s^m + c_1 s^{m-1} + \ldots + c_{m-1}s + c_m$ is a Hurwitz polynomial, while the coefficients k_i must be chosen such that $\bar{M}\exp(A_C T) \in \mathcal{D}_1$, where

$$A_C = \begin{pmatrix} 0_{(m-2)\times 1} & & I_{m-2} & \\ & & & \\ -k_1 & -k_2 & \ldots & -k_{m-1} \end{pmatrix}, \; \bar{M} = \begin{pmatrix} I_{m-1} & 0 \end{pmatrix} M_{22} \begin{pmatrix} I_{m-1} \\ \\ 0 \end{pmatrix}. \quad (3.8)$$

In the case where $\bar{M} = I$, the coefficients k_i, must simply be chosen such that $s^{m-1} + k_1 s^{m-2} + \ldots + k_{m-2}s + k_{m-1}$ is a Hurwitz polynomial. This is notably the case when the plant is a classical continuous-time linear system.

3.3 High Gain Observers work for Continuous-Time Plants

The main result of this chapter is to show that the stabilizer unit (3.5) achieves the stability requirement for hybrid output regulation. In doing so we assume an internal model of the form (3.4) having the internal model property. Specifically, we assume the following.

In order for the hybrid regulation equations (2.7) to be solvable we should make the following assumptions, since the system is in Brunovsky's canonical form (see Section 2.4). The first assumption achieves the non-resonance requirement for hybrid output regulation.

Assumption 1 (Non-Resonance) *The following holds:*

$$\mathrm{eig}(M_{11}\mathrm{exp}(A_{11}\tau_{\max})) \cap \mathrm{eig}(J\mathrm{exp}(S\tau_{\max})) = \emptyset.$$

Following this assumption, we let $\Pi_z(\tau) : [0, \tau_{\max}] \to \mathbb{R}^{n\times s}$ be the continuously differentiable function that is the unique solution of

$$\begin{aligned} \frac{d\Pi_z(\tau)}{d\tau} &= A_{11}\Pi_z(\tau) - \Pi_z(\tau)S - A_{12}V + P_1, \\ 0 &= M_{11}\Pi_z(\tau_{\max}) - \Pi_z(0)J - M_{12}V + N_1\,, \end{aligned}$$

where $V = \begin{pmatrix} V_1^T & \dots & V_r^T \end{pmatrix}^T$ with the rows, V_i, such that $V_i = QS^{i-1} + \sum_{j=2}^{i} P_{2(j-1)}S^{i-j}$.

Then, the function $\Pi_z(\tau)$ must satisfy the following assumption, which is a direct result of attempting to achieve a tracking goal over jumps via a continuous-time control input. Specifically, it comes about due to possible jumps of Qw potentially forcing matching jumps of Cx. This is tied heavily to the relative degree of the system, again further details can be found in Section 2.4.

Assumption 2 *The matrix equation $M_{21}\Pi_z(\tau_{\max}) + N_2 + VJ - M_{22}V = 0$ is satisfied.*

Furthermore, in order to achieve the stability requirement via high-gain feedback we must make a minimum-phase assumption on the plant.

Assumption 3 (Minimum-Phase) *The eigenvalues of* $M_{11}\exp(A_{11}\tau_{\max})$ *are inside the unit disc.*

Finally, it will be assumed that the internal model has been designed according to the canonical form presented in Section 2.3. The details are specified in the following assumption.

Assumption 4 (Internal Model) *The internal model (3.4) has been designed such that the pair* (F, G) *is controllable, with* $\mathrm{eig}(\Sigma\exp(F\tau_{\max})) \in \mathcal{D}_1$ *and with* $\mathrm{eig}(\Sigma\exp(F\tau_{\max})) \cap \mathrm{eig}(J\exp(S\tau_{\max})) = \emptyset$*. Furthermore, let* $\Pi : [0, \tau_{\max}] \to \mathbb{R}^\nu \times \mathbb{R}^s$ *be the continuously differentiable function that is the unique solution of*

$$\begin{aligned} \frac{d\Pi(\tau)}{d\tau} &= F\Pi(\tau) - \Pi(\tau)S + GR(\tau), \\ 0 &= \Sigma\Pi(\tau_{\max}) - \Pi(0)J, \end{aligned}$$

where $\mathrm{rank}(\Pi(\tau)) = r < \nu$ *for all* $\tau \in [0, \tau_{\max}]$ *and* $R(\tau) = -B_2^{-1}(A_{21}\Pi_z(\tau) + \bar{P}_2)$*, with* $\bar{P}_2$ *as defined in Section 2.4. Finally, the parameter* Γ *has been chosen as* $\Gamma(\tau) = R(\tau)\Pi^\dagger(\tau)$*, where* $\Pi^\dagger(\tau)$ *is the Moore-Penrose pseudo-inverse of* $\Pi(\tau)$*.*

With these assumptions in hand we present the main result of the chapter regarding the closed loop stability of the regulator achieving hybrid output regulation. This is done by analyzing the closed loop system obtained via the error system resulting from the coordinate transformations

$$\begin{aligned} z \to \tilde{z} &:= z - \Pi_z(\tau)w, \\ y \to e &:= y + Vw. \end{aligned}$$

and the regulator consisting of the internal model unit and feedback (3.4) and stabilizer unit (3.5) with the residual control input (3.6). Specifically, the error system is

$$\left.\begin{aligned} \dot{\tau} &= 1, \quad \dot{w} = Sw, \\ \dot{\tilde{z}} &= A_{11}\tilde{z} + A_{12}e, \\ \dot{e} &= A_{21}\tilde{z} + A_{22}e + B_2(u - R(\tau)w), \end{aligned}\right\} (\tau, w, \tilde{z}, e) \in [0, \tau_{\max}] \times W \times \mathbb{R}^n \times \mathbb{R}^m,$$
$$\left.\begin{aligned} \tau^+ &= 0, \quad w^+ = Jw, \\ \tilde{z}^+ &= M_{11}\tilde{z} + M_{12}e, \\ e^+ &= M_{21}\tilde{z} + M_{22}e, \end{aligned}\right\} (\tau, w, \tilde{z}, e) \in \{\tau_{\max}\} \times W \times \mathbb{R}^n \times \mathbb{R}^m. \tag{3.9}$$

Theorem 2 *Let Assumptions 1-4 be satisfied. Then there exists $\kappa^* > 0$ and for each $\kappa \geq \kappa^*$ there exists $g^* > 0$, such that for each $g \geq g^*$, the error system and exosystem (3.9) with the regulator consisting of the internal model unit (3.4) and the stabilizer unit (3.5) with the state $(\tilde{z}, e, \tau, w, \eta, \xi)$ has the set $\{0\} \times \{0\} \times \Upsilon \times \{0\}$ globally exponentially stable, where $\Upsilon = \{(\tau, w, \eta) \in [0, T] \times \mathcal{W} \times \mathbb{R}^\nu : \eta = \Pi(\tau)w\}$.*

Proof: In proceeding with this proof we drop the exosystem dynamics from the analysis, since the exosystem is not affected by and does not affect the remainder of the system after an appropriate coordinate transformation.

Perform the following changes of coordinates. The first takes advantage of the high gain stabilization design, the second eliminates the steady-state effect of the exosystem on the observer from the analysis.

$$\begin{aligned} e_m \to \tilde{e}_m &:= e_m + k_1 e_1 + \ldots + k_{m-1} e_{m-1}, \\ \xi \to \chi &:= \xi - b^{-1} G \tilde{e}_m - \Pi(\tau) w. \end{aligned}$$

Note that $(\tilde{z}, [e_1 \ldots e_{m-1}], \tilde{e}_m, \chi) = 0$, $\tau \in [0, T]$, and $w \in \mathcal{W}$ if and only if $(\tilde{z}, e, \tau, w, \xi) \in \{0\} \times \{0\} \times \Upsilon$.

During flows this gives

$$\begin{aligned} \dot{\tilde{z}} &= A_{11}\tilde{z} + A_{12} e_1, \\ \dot{e}_1 &= e_2, \\ &\vdots \\ \dot{e}_{m-2} &= e_{m-1}, \\ \dot{e}_{m-1} &= \tilde{e}_m - k_1 e_1 - \ldots - k_{m-1} e_{m-1}, \\ \dot{\tilde{e}}_m &= (\Gamma(\tau) G + k_{m-1} + \alpha_m)\tilde{e}_m + [\ p \quad 0\ ] e + \bar{A}_{21}\tilde{z} + b\Gamma(\tau)\chi + bK\eta, \\ \dot{\chi} &= F\chi - b^{-1} G([\ p \quad 0\ ] e + \bar{A}_{21}\tilde{z}) + b^{-1}(FG - G(\alpha_m + k_{m-1}))\tilde{e}_m, \\ \dot{\eta} &= \Phi_{st}\eta + \Lambda_{st} e_1, \end{aligned}$$

where α_i are the elements of $\bar{A}_{22}$, and

$$\begin{aligned} p &= [\ \alpha_1 - k_1(\alpha_m + k_{m-1}) \quad \dots \quad \alpha_{m-1} - k_{m-1}(\alpha_m + k_{m-1})\ ] \cdots \\ &\quad +[\ 0 \quad k_1 \quad k_2 \quad \dots \quad k_{m-2}\ ]. \end{aligned}$$

From here onwards we use the notation $\mathbf{e} = [\ e_1 \quad \dots \quad e_{m-1}\ ]^T$, along with using the previously defined A_C. In this vein we also define $L = [\ 0_{1\times(m-2)} \quad 1\ ]^T$ and $B_C = [\ 0_{1\times(m-1)} \quad 1\ ]^T$.

The following coordinate transform puts the observer, η, into error coordinates, by following the original idea in the continuous-time literature [21], we change coordinates from $\eta \to \tilde{\eta}$, where $\tilde{\eta}$ is defined as

$$\tilde{\eta} := D_g(\eta - e),$$

with $D_g = \mathrm{diag}(g^{m-1}, \dots, g^0)$.

With K, Φ_{st} and Λ_{st} chosen as in (3.7) this change of variables results in the closed-loop system

$$
\begin{aligned}
\dot{\tilde{z}} &= A_{11}\tilde{z} + A_{12}e_1, \\
\dot{\mathbf{e}} &= A_C\mathbf{e} + L\tilde{e}_m \\
\dot{\tilde{e}}_m &= (\alpha_m + k_{m-1} + \Gamma(\tau)G - |b|\kappa)\tilde{e}_m \dots \\
&\quad +[\begin{array}{cc} p & 0 \end{array}]e + \bar{A}_{21}\tilde{z} + b\Gamma(\tau)\chi + bKD_g^{-1}\tilde{\eta}, \qquad (3.10) \\
\dot{\chi} &= F\chi - b^{-1}G([\begin{array}{cc} p & 0 \end{array}]e + \bar{A}_{21}\tilde{z}) + b^{-1}(FG - G(\alpha_m + k_{m-1}))\tilde{e}_m \\
\dot{\tilde{\eta}} &= gH\tilde{\eta} + B_C(q(e,\tilde{z},\chi,\tilde{e}_m) - bKD_g^{-1}\tilde{\eta}), \\
\dot{\tau} &= 1,
\end{aligned}
$$

during flows, i.e. for $(\tilde{z}, \mathbf{e}^T, \tilde{e}_m, \chi, \tilde{\eta}, \tau) \in \mathbb{R}^n \times \mathbb{R}^{m-1} \times \mathbb{R} \times \mathbb{R}^\nu \times \mathbb{R}^m \times [0, \tau_{\max}]$, where

$$
\begin{aligned}
q(e,\tilde{z},\chi,\tilde{e}_m) &= -\textstyle\sum_{i=1}^{m-1} \alpha_i e_i + \alpha_m(k_1e_1 + \dots + k_{m-1}e_{m-1}) \dots \\
&\quad -\bar{A}_{21}\tilde{z} - b\Gamma(\tau)\chi + (\kappa - \alpha_m - \Gamma(\tau)G)\tilde{e}_m,
\end{aligned}
$$

During jumps,

$$
\begin{aligned}
\tilde{z}^+ &= M_{11}\tilde{z} + M_{12}\begin{bmatrix} I_{m-1} \\ -k_1 \cdots - k_{m-1} \end{bmatrix}\mathbf{e} + M_{12}B_C\tilde{e}_m, \\
\mathbf{e}^+ &= \bar{M}\mathbf{e}, \\
\tilde{e}_m^+ &= \tilde{M}\tilde{e}_m + [k_1 \;\ldots\; k_{m-1}]\left(\bar{M} - I_{m-1}\right)\mathbf{e}, \\
\chi^+ &= \Sigma_{im}\chi \ldots \\
&\quad + b^{-1}(\Sigma_{im}G - G\tilde{M})\tilde{e}_m - b^{-1}G[k_1 \quad \ldots \quad k_{m-1}]\left(\bar{M} - I_{m-1}\right)\mathbf{e}, \\
\tilde{\eta}^+ &= M_{22}\tilde{\eta}, \\
\tau^+ &= 0,
\end{aligned}
\tag{3.11}
$$

for $(\tilde{z}, \mathbf{e}^T, \tilde{e}_m, \chi, \tilde{\eta}, \tau) \in \mathbb{R}^n \times \mathbb{R}^{m-1} \times \mathbb{R} \times \mathbb{R}^\nu \times \mathbb{R}^m \times \{\tau_{\max}\}$. The matrix parameter $\tilde{M}$ is defined as $\tilde{M} = B_C^T M_{22} B_C$, and H is the Hurwitz matrix defined as

$$
H = \begin{bmatrix} -c_1 & \\ & I_{m-1} \\ \vdots & \\ & 0_{1\times(m-1)} \\ -c_m & \end{bmatrix}.
$$

The closed-loop system described by (3.10)-(3.11) has a desirable structure that allows for the easy application of Proposition 10 from Appendix A.1. The Lyapunov analysis of the closed-loop system is performed in two steps. First, ignore the $\tilde{\eta}$ dynamics, and choose κ to be large enough to stabilize the $(\mathbf{e}, \tilde{z}, \chi, \tilde{e}_m, \tau)$ dynamics. Then, re-account for $\tilde{\eta}$ and choose g to be large enough such that the overall closed-loop system, (3.10)-(3.11), with the state

$(\tilde{z}, \mathbf{e}, \tilde{e}_m, \chi, \tilde{\eta}, \tau)$ has the set $\{0\} \times \{0\} \times \{0\} \times \{0\} \times \{0\} \times [0, \tau_{\max}]$ globally exponentially stable.

Pick $v_1 = (\mathbf{e}, \tilde{z}, \chi)$ and $v_2 = \tilde{e}_m$ and ignore $\tilde{\eta}$. The v_1 and v_2 dynamics can be written as

$$
\begin{aligned}
\dot{v}_1 &= \begin{bmatrix} A_C & 0 & 0 \\ A_{12} & A_{11} & 0 \\ -b^{-1}Gp & -b^{-1}G\bar{A}_{21} & F \end{bmatrix} v_1 + \begin{bmatrix} B_C \\ 0_{n\times 1} \\ b^{-1}(FG - G(\alpha_m + k_{m-1})) \end{bmatrix} v_2, \\
\dot{v}_2 &= (\alpha_m + k_{m-1} + \Gamma(\tau)G - |b|\kappa)v_2 + \begin{bmatrix} p & \bar{A}_{21} & b\Gamma(\tau) \end{bmatrix} v_1, \\
\dot{\tau} &= 1,
\end{aligned}
$$

during flows, when $(v_1, v_2, \tau) \in \mathbb{R}^{(m-1)+n+\nu} \times \mathbb{R} \times [0, \tau_{\max}]$, and

$$
\begin{aligned}
v_1^+ &= \begin{bmatrix} \bar{M} & 0 & 0 \\ M_{12} \begin{bmatrix} I_{m-1} \\ -k_1 \cdots - k_{m-1} \end{bmatrix} & M_{11} & 0 \\ -b^{-1}G \begin{bmatrix} k_1 & \dots & k_{m-1} \end{bmatrix} \left(\bar{M} - I_{m-1}\right) & 0 & \Sigma_{im} \end{bmatrix} v_1 \dots \\
&\quad + \begin{bmatrix} 0 \\ M_{12}B_C \\ \Sigma_{im} b^{-1}G - b^{-1}G\tilde{M} \end{bmatrix} v_2, \\
v_2^+ &= \tilde{M}v_2 + \begin{bmatrix} \begin{bmatrix} k_1 & \dots & k_{m-1} \end{bmatrix} \left(\bar{M} - I_{m-1}\right) & 0_{1\times n} & 0_{1\times\nu} \end{bmatrix} v_1, \\
\tau^+ &= 0,
\end{aligned}
$$

during jumps, when $(v_1, v_2, \tau) \in \mathbb{R}^{(m-1)+n+\nu} \times \mathbb{R} \times \{\tau_{\max}\}$.

Due to the minimum phase assumption on the plant and the design choices of K, Σ_{im} and F, this fits in the framework of Proposition 10. Therefore, κ can be chosen large enough so that the system with state (v_1, v_2) has the set $\{0\} \times \{0\}$ globally exponentially stable, and thus that $J_{cl} \exp(A_{cl}\tau_{\max}) \in \mathcal{D}_1$, where

$$A_{cl} = \begin{bmatrix} A_C & 0 & 0 & B_C \\ A_{12} & A_{11} & 0 & 0 \\ -\frac{1}{b}Gp & -\frac{1}{b}G\bar{A}_{21} & F & \frac{1}{b}(FG - G(\alpha_m + k_{m-1})) \\ p & 0 & 0 & \alpha_m + k_{m-1} + \Gamma(\tau)G - |b|\kappa \end{bmatrix}$$

and

$$J_{cl} = \begin{bmatrix} \bar{M} & 0 & 0 & 0 \\ M_{12}\begin{bmatrix} I_{m-1} \\ -k_1 \cdots - k_{m-1} \end{bmatrix} & M_{11} & 0 & M_{12}B_C \\ -b^{-1}G\begin{bmatrix} k_1 & \dots & k_{m-1} \end{bmatrix}\left(\bar{M} - I_{m-1}\right) & 0 & \Sigma_{im} & \Sigma_{im}b^{-1}G - b^{-1}G\tilde{M} \\ \begin{bmatrix} k_1 & \dots & k_{m-1} \end{bmatrix}\left(\bar{M} - I_{m-1}\right) & 0 & 0 & \tilde{M} \end{bmatrix}.$$

With this established there is one last step to show global exponential stability for the entire closed-loop system described by (3.10)-(3.11), namely the $\tilde{\eta}$ dynamics must be re-accounted for.

Take $v_1 = (\mathbf{e}, \tilde{z}, \chi, \tilde{e}_m)$ and $v_2 = \tilde{\eta}$. Then, during flows, i.e. $(v_1, v_2, \tau) \in \mathbb{R}^{m+n+\nu} \times \mathbb{R}^m \times [0, \tau_{\max}]$,

$$
\begin{aligned}
\dot{v}_1 &= A_{cl} v_1 + \begin{bmatrix} 0_{(m-1+n+s)\times 1} \\ bKD_g^{-1} \end{bmatrix} v_2 \\
\dot{v}_2 &= gHv_2 + \begin{bmatrix} 0_{m\times 1} \\ 1 \end{bmatrix} (q(e, \tilde{z}, \chi, \tilde{e}_m) - bKD_g^{-1} v_2) \\
\dot{\tau} &= 1
\end{aligned}
$$

and during jumps, i.e. $(v_1, v_2, \tau) \in \mathbb{R}^{m+n+\nu} \times \mathbb{R}^m \times \{\tau_{\max}\}$,

$$
\begin{aligned}
v_1^+ &= J_{cl} v_1 \\
v_2^+ &= M_{22} v_2 \\
\tau^+ &= 0
\end{aligned}
$$

Once again, this system fits into the framework of Proposition 10. Therefore, it can be concluded that the closed-loop system described by (3.10)-(3.11) with state $([e_1, \ldots, e_{m-1}], z, \chi, \tilde{e}_m, \tilde{\eta}, \tau)$ has the set $\{0\}\times\{0\}\times\{0\}\times\{0\}\times\{0\}\times[0, \tau_{\max}]$ globally exponentially stable. ■

3.3.1 Example: Stabilizing a Relative Degree 2 System and Achieving Hybrid Output Regulation

As an example of how to apply the regulator designed here, we provide the following. Consider a plant with the relative degree two transfer function:

$$\frac{Y(s)}{\widetilde{U}(s)} = \frac{a_2 s^2 + ds + k}{s^2(a_1 a_2 s^2 + k(a_1 + a_2))}.$$

With $a_2, d, k > 0$, the plant is minimum phase. For our simulations we take $a_1 = 10$, $a_2 = 1$, $k = 1$ and $d = 1$. Assume that there is a disturbance additive with the control signal, such that:

$$\tilde{u} = u - w,$$

where u is the control signal and w is a disturbance generated by the exosystem:

$$\left.\begin{aligned} \dot{w} &= 0 \\ \dot{\tau} &= 1 \end{aligned}\right\} \quad (w, \tau) \in \mathbb{R} \times [0, \tau_{\max}],$$

$$\left.\begin{aligned} w^+ &= -w \\ \tau^+ &= 0 \end{aligned}\right\} \quad (w, \tau) \in \mathbb{R} \times \{\tau_{\max}\},$$

where τ is a clock variable governing the exosystem's jumps.

Furthermore, we can write the state space realization of the transfer function in Brunovsky's canonical form as follows,

$$\begin{aligned}\dot{z} &= \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} z + \begin{pmatrix} 0 & 10 \\ 0 & 0 \end{pmatrix} y, \\ \dot{y} &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} y + \begin{pmatrix} 0 & 0 \\ 0.1 & -0.11 \end{pmatrix} z + \begin{pmatrix} 0 \\ 0.1 \end{pmatrix} (u - w).\end{aligned}$$

with $e_1 = y_1$.

Following the steps laid out in this dissertation, we can design a regulator to achieve global exponential stability of the origin of the plant in the presence of this disturbance.

Begin by choosing the pair (F, G) as:

$$F = \begin{pmatrix} -10 & -50 \\ 1 & 0 \end{pmatrix}, \quad G = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

The considered exosystem satisfies the assumptions of Proposition 2, namely the the pair (S, R) is observable, so we can take Γ to be τ-invariant, where:

$$\Pi_\eta = \begin{pmatrix} 0 \\ \frac{1}{50} \end{pmatrix}, \quad \Sigma_{\mathrm{im}} = -1, \quad \Gamma = \begin{pmatrix} 0 & 50 \end{pmatrix}.$$

Then, pick the Hurwitz polynomial coefficients $k_1 = 1$ and $(c_1, c_2) = (4, 4)$ in order to construct K, Λ_{st} and Φ_{st} as in (3.7). Finally, guided by Theorem 2, we pick κ sufficiently large and, subsequently, l sufficiently large. By simulation,

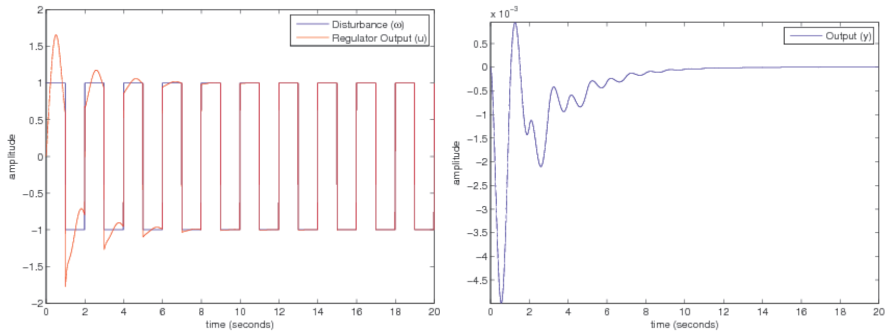

Figure 3.1: Disturbance and Regulator Output; Plant Output and Internal State

we find that $\kappa = 50$ and $l = 70$ is sufficient for stability. The remainder of the regulator is constructed based on these choices. The results are shown in Figure 3.1.

3.4 Regulator with stabilization through jumps

Recall that in Section 2.3.1 the idea of τ-independent internal models was presented. We can expand the stabilizer design to incorporate these ideas. For the case where the pair (S, R) is observable, the stabilizer can be designed exactly as in Section 3.2. But, if that observability assumption does not hold, and instead the pair (J, R) is observable, then it turns out that a slightly expanded stabilizer unit is needed for the regulator. The use of this observability property is why we call this method "stabilization through jumps."

"Stabilization through jumps" can be pursued if the following assumption holds.

Assumption 5 *The pair (J, R) is observable.*

In this context the regulator can be taken as

$$\left.\begin{aligned} \dot{\xi} &= S\xi \\ \dot{\eta} &= \Phi_{\mathrm{st}}\eta + \Lambda_{\mathrm{st}}e_1 \end{aligned}\right\} (\tau, \xi, \eta) \in [0, \tau_{\max}] \times \mathbb{R}^{\nu} \times \mathbb{R}^{m+1},$$
$$\left.\begin{aligned} \xi^+ &= J\xi + \Delta_{\eta}\eta \\ \eta^+ &= \Sigma_{\mathrm{st}}\eta \end{aligned}\right\} (\tau, \xi, \eta) \in \{\tau_{\max}\} \times \mathbb{R}^{\nu} \times \mathbb{R}^{m+1}, \tag{3.12}$$

with the feedback law

$$u = R\xi + K\eta,$$

where $\nu = s$,

$$\begin{aligned} \Phi_{\mathrm{st}} &= \begin{bmatrix} -\Lambda_{\mathrm{st}} & \begin{matrix} I_m \\ 0_{1\times m} \end{matrix} \end{bmatrix}, \; \Lambda_{\mathrm{st}} = \begin{bmatrix} c_1 g \\ \vdots \\ c_{m+1} g^{m+1} \end{bmatrix}, \\ K &= -sgn(b)\kappa \begin{bmatrix} k_1 & \ldots & k_{m-1} & 1 & 0 \end{bmatrix}, \end{aligned} \tag{3.13}$$

with the coefficients k_i chosen such that $(\bar{M} \exp(A_C \tau_{\max})) \in \mathcal{D}_1$, where $\bar{M}$ and A_C are as in (3.8), and

$$\Sigma_{\mathrm{st}} = \mathrm{blkdiag}(M_{22},\, 1)\,, \quad \Delta_{\eta} = \frac{G}{b}\begin{pmatrix} 0_{1\times m} & 1 \end{pmatrix} - GK, \tag{3.14}$$

where, with Assumption 5 in mind, G is chosen such that $((J+GR)\exp(S\tau_{\max})) \in \mathcal{D}_1$.

It is worth noting that, in this case, the dimension of the high-gain observer $\dot{\eta} = \Phi_{\rm st}\eta + \Lambda_{\rm st}e$ is $m+1$ instead of m, and that the value of η is also used in the computation of ξ^+. The additional state in the high-gain observer is meant to estimate $\dot{e}_m$. As will be clarified in the proof of the forthcoming Theorem 3, this "extra" estimation is crucial in order to stabilize the internal model through the jump channel by means of injection of the "output" term $R\xi$. With this regulator in mind we propose the following theorem.

Theorem 3 *Let Assumptions 1-5 be satisfied. Furthermore, let the regulator parameters be chosen as in (3.13)-(3.14). Then there exists $\kappa^* > 0$ and for each $\kappa \geq \kappa^*$ there exists $g^* > 0$ such that for each $g \geq g^*$ the closed-loop system with the error system and exosystem (3.9) and τ-independent regulator (3.12) with the state $(\tilde{z}, e, \tau, w, \xi, \eta)$ has the set $\{0\} \times \{0\} \times \Upsilon \times \{0\}$ globally exponentially stable, where $\Upsilon = \{(\tau, w, \xi) \in [0, \tau_{\max}] \times \mathcal{W} \times \mathbb{R}^{\nu} : \xi = w\}$.*

Proof: The following proof is fairly similar to that already presented for Theorem 2, with some minor differences in the changes of variables. The crux of the proof will once again be the use of Proposition 10, though.

We begin with the following coordinate changes

$$
\begin{aligned}
e \to \tilde{e}_m &= e_m + k_1 e_1 + \cdots + k_{m-1} e_{m-1}, \\
\xi \to \tilde{\xi} &= \xi - w, \\
\eta \to \tilde{\eta} &= D_g \left(\eta - \begin{pmatrix} e \\ e^{(m)} \end{pmatrix} \right).
\end{aligned}
$$

These changes result in the closed system with the flow dynamics

$$
\begin{aligned}
\dot{\tilde{z}} &= A_{11}\tilde{z} + A_{12} e_1, \\
\dot{\mathbf{e}} &= A_C \mathbf{e} + L\tilde{e}_m \\
\dot{\tilde{e}}_m &= (\alpha_m + k_{m-1} - |b|\kappa)\tilde{e}_m + \begin{bmatrix} p & 0 \end{bmatrix} e + \bar{A}_{21}\tilde{z} + bR\tilde{\xi} + bKD_g^{-1}\tilde{\eta}, \\
\dot{\tilde{\xi}} &= S\tilde{\xi}, \\
\dot{\tilde{\eta}} &= gH\tilde{\eta} - \begin{bmatrix} 0_{m\times 1} \\ \dot{e}^{(m)} \end{bmatrix},
\end{aligned}
\tag{3.15}
$$

and the jump dynamics

$$
\begin{aligned}
\tilde{z}^+ &= M_{11}\tilde{z} + M_{12} \begin{bmatrix} I_{m-1} \\ -k_1 \cdots - k_{m-1} \end{bmatrix} \mathbf{e} + M_{12} B_C \tilde{e}_m, \\
\mathbf{e}^+ &= \bar{M}\mathbf{e}, \\
\tilde{e}_m^+ &= \tilde{M}\tilde{e}_m + \begin{bmatrix} k_1 & \dots & k_{m-1} \end{bmatrix} \left(\bar{M} - I_{m-1}\right) \mathbf{e}, \\
\tilde{\xi}^+ &= (J + GR)\tilde{\xi} + \Xi_e \mathbf{e} + \Xi_{\tilde{e}_m}\tilde{e}_m + \tfrac{G}{b}\bar{A}_{21}\tilde{z} + \begin{bmatrix} 0_{1\times m} & 1 \end{bmatrix} \tilde{\eta}, \\
\tilde{\eta}^+ &= H_{\tilde{\eta}}\tilde{\eta} + H_e \mathbf{e} + H_{\tilde{e}_m}\tilde{e}_m + H_z z + H_{\tilde{\xi}}\tilde{\xi},
\end{aligned}
\tag{3.16}
$$

for $(\tilde{z}, \mathbf{e}^T, \tilde{e}_m, \tilde{\xi}, \tilde{\eta}) \in D$, where the relevant matrices and the equations for $\dot{e}^{(m)}$ and $\dot{e}$ used in the previous dynamics are defined in Appendix A.2 for those who are interested, but their exact definition does not matter for the remainder of the proof.

Once again we can fit this closed loop system into the framework of Proposition 10 to finish our proof. Start by picking $v_1 = (e_1, \ldots, e_{m-1}, \tilde{z}, \tilde{\xi})$, $v_2 = \tilde{e}_m$. This gives

$$\begin{aligned}
\dot{v_1} &= \begin{bmatrix} A_C & 0 & 0 \\ \begin{bmatrix} A_{12} & 0_{n\times(m-2)} \end{bmatrix} & A_{11} & 0 \\ 0 & 0 & S \end{bmatrix} v_1 + \begin{bmatrix} B_C \\ 0 \\ 0 \end{bmatrix} v_2, \\
\dot{v_2} &= (\alpha_m + k_{m-1} - |b|\kappa) v_2 + \begin{bmatrix} p & \bar{A}_{21} & bR(\tau) \end{bmatrix} v_1,
\end{aligned}$$

during flows, and

$$\begin{aligned}
v_1^+ &= \begin{bmatrix} \bar{M} & 0 & 0 \\ M_{12} \begin{bmatrix} I_{m-1} \\ -k_1 \quad \ldots \quad k_{m-1} \end{bmatrix} & M_{11} & 0 \\ \Xi_x & \frac{1}{b} G B x_{mz} & J + GR(T) \end{bmatrix} v_1 + \begin{bmatrix} 0 \\ M_{12} B_C \\ \Xi_{\tilde{e}_m} \end{bmatrix} v_2, \\
v_2^+ &= \tilde{M} v_2 + \begin{bmatrix} \begin{bmatrix} k_1 & \ldots & k_{m-1} \end{bmatrix} (\bar{M} - I_{m-1}) & 0 & 0 \end{bmatrix} v_1,
\end{aligned}$$

during jumps. Thanks to the minimum phase assumption, as well as the choices of the coefficients k_i and of the matrix G, this system fits into the structure of

Proposition 10, where κ is the tunable high-gain parameter. This shows that the eigenvalues of $\mathcal{J}_{cl}\exp(\mathcal{A}_{cl}T)$ are inside the unit disk, where $\mathcal{J}_{cl}$ and $\mathcal{A}_{cl}$ are

$$\mathcal{J}_{cl} = \begin{bmatrix} \begin{bmatrix} \bar{M} & 0 & 0 \\ M_{12}\begin{bmatrix} & I_{m-1} & \\ -k_1 & \dots & k_{m-1} \end{bmatrix} & M_{11} & 0 \\ \Xi_e & \frac{1}{b}G\bar{A}_{21} & J + GR(T) \end{bmatrix} & \begin{bmatrix} 0 \\ M_{12}B_C \\ \Xi_{\tilde{e}_m} \end{bmatrix} \\ \begin{bmatrix} \begin{bmatrix} k_1 & \dots & k_{m-1} \end{bmatrix}(\bar{M} - I_{m-1}) & 0 & 0 \end{bmatrix} & \tilde{M} \end{bmatrix},$$

and

$$\mathcal{A}_{cl} = \begin{bmatrix} \begin{bmatrix} A_C & 0 & 0 \\ \begin{bmatrix} A_{12} & 0_{n\times(m-2)} \end{bmatrix} & A_{11} & 0 \\ 0 & 0 & S \end{bmatrix} & \begin{bmatrix} B_C \\ 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} p & \bar{A}_{21} & bR(\tau) \end{bmatrix} & \alpha_m + k_{m-1} - |b|\kappa \end{bmatrix}.$$

Now, use the same proposition, but with κ fixed, to show that the overall closed loop system has the desired set globally exponentially stable. To this end, choose $v_1 = (\tilde{z}, \mathbf{e}^T, \tilde{\xi}, \tilde{e}_m)$ and $v_2 = (\tilde{\eta})$. This gives

$$
\begin{aligned}
\dot{v_1} &= \mathcal{A}_{cl} v_1 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ bKD_g^{-1} \end{bmatrix} v_2, \\
\dot{v_2} &= gHv_2 - \begin{bmatrix} 0_{m\times 1} \\ 1 \end{bmatrix} bKD_g^{-1} gHv_2 + f(v_1),
\end{aligned}
$$

during flows, where $f(v_1)$ is a linear function of v_1 that is bounded independently of g. And, during jumps

$$
\begin{aligned}
v_1^+ &= \mathcal{J}_{cl} v_1 + \begin{bmatrix} 0 \\ 0 \\ \begin{bmatrix} 0_{1\times m} & 1 \end{bmatrix} \\ 0 \end{bmatrix} v_2, \\
v_2^+ &= H_{\tilde{\eta}} v_2 + \begin{bmatrix} H_e & H_{\tilde{z}} & H_{\tilde{\xi}} & H_{\tilde{e}_m} \end{bmatrix} v_1.
\end{aligned}
$$

Once again this system fits into the framework of Proposition 10. This time, g acts as the tunable parameter, with the matrix H being Hurwitz. As such, the desired stability property has been proven. ■

3.4.1 Example: Stabilization Through Jumps

In this section we develop a numerical example in order to highlight the method for developing a high gain observer to achieve stabilization through jumps. We consider a continuous-time plant with relative degree three in the form (3.1)-(3.2) with $M_{11} = I_2$, $M_{12} = 0_{2\times3}$, $N_1 = 0$, $M_{21} = 0_{3\times2}$, $M_{22} = I_3$, $N_2 = 0$, $P_1 = 0$, $P_2 = \begin{pmatrix} -1 & 0 \end{pmatrix}$, $b = 1$, $\bar{A}_{21} = \begin{pmatrix} 1 & 2 \end{pmatrix}$, $\bar{A}_{22} = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$ and

$$A_{11} = \begin{pmatrix} -2 & -4 \\ 1 & 0 \end{pmatrix}, \qquad A_{12} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

The exosystem parameters are

$$S = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

with $\tau_{\max} = 1$.

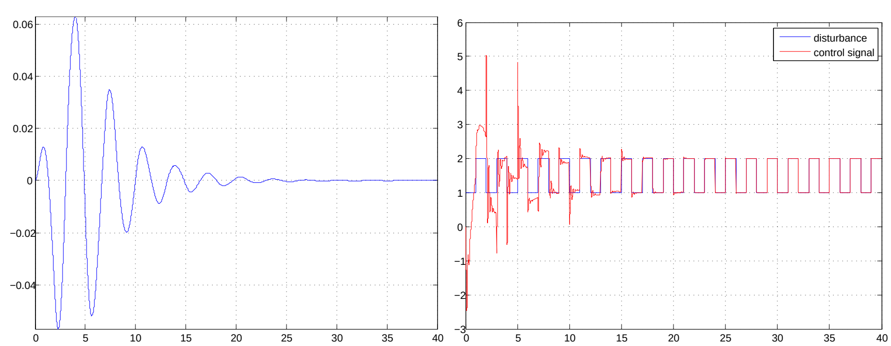

Figure 3.2: Jump regulator error; Jump regulator control and disturbance signals

Note that the pair (S, R) is not observable, but the pair (J, R) is. With the pair (J, R) being observable, the design methodology in Section 3.4 applies. We thus choose the regulator as in (3.12). First, pick the parameters k_i and c_i as coefficients of Hurwitz polynomials, as in (3.13). Here, they are chosen such that

$$K = -\kappa \begin{pmatrix} 2 & 1 & 1 & 0 \end{pmatrix}, \quad \Lambda_{\mathrm{st}} = \begin{pmatrix} 4g & 6g^2 & 4g^3 & 1g^4 \end{pmatrix}^T.$$

Furthermore, G is chosen as $G = \begin{pmatrix} -\frac{1}{2} & -\frac{3}{2} \end{pmatrix}^T$ to satisfy $((J+GR)\exp(S\tau_{\max})) \in \mathcal{D}_1$. Lastly, the tuning parameters are chosen as $\kappa = 15$ and $g = 40$ via simulation to achieve global exponential stability.

With (3.13)-(3.14) in mind, these choices completely define the regulator (3.12). Simulation produces the plots showing the error, e_1 and a comparison of the control signal, u, and disturbance signal, Rw, in Figure 3.2.

3.4.2 Conclusions

In this chapter we have presented a stabilizer unit for output regulation for SISO hybrid systems and exosystems under a periodic dwell time constraint. The high gain observer used here poses some additional constraints on the jump map of the plant. This same observer technique can also be used for a class of MIMO systems; this will be shown in Chapter 5.

We are still left with the task of presenting practical examples that can fit into this framework. To this end, it is shown that spline trajectories can be generated by hybrid exosystems.

Chapter 4

Applying Hybrid Output Regulation to the Problem of Spline Trajectory Tracking

In this chapter we take a slight detour from progressing the theory of hybrid output regulation to discuss a pertinent example. This example helps to motivate the general internal model design presented in Chapter 5. It turns out that spline trajectories can be generated by a hybrid exosystem with periodic jump times. The spline generating exosystem can then be used to create an internal model to achieve robust tracking of the spline trajectory. In this scenario, the robustness we refer to is analogous to the property that Francis and Wonham call structural stability ([24]), which deals with parametric uncertainties in the plant data.

Spline interpolation is widely adopted in the robotic literature ([44]) in order to generate reference signals that smoothly interpolate way-points by avoiding Runge's phenomenon, which usually appears while using polynomial interpolation. Splines have been shown to be effective in path generation of mobile robots [33], in the aerospace domain [47], and in many other applicative fields where efficient trajectory planning is a key requirement. Furthermore, they have been shown to be efficiently computable when dealing with actuation constraints in an optimal manner, see [20].

This chapter begins by showing how splines can be generated using a hybrid exosystem in Section 4.1. We then proceed to show that the curent methods of internal model design fail to cover the spline exosystem, thus motivating the developments of Chapter 5.

4.1 Splines as Hybrid Exosystems

We are interested in cyclic time signals $y^\star(t)$ obtained by periodically concatenating the basic signal, $\mathcal{B}(t)$, with a period given by T, where

$$y^\star(t) = \mathcal{B}(t - iT), \qquad i = \left\lfloor \frac{t}{T} \right\rfloor, \; t \geq 0\,, \; T > 0,$$

with $\mathcal{B} : [0, T] \to \mathbb{R}$ a sufficiently smooth function. The basic signal $\mathcal{B}(t)$ is thought of as being generated by using splines that pass through N points $\{p_1, p_2, \dots, p_N\}$ at time instances $\{t_1, t_2, \dots, t_N\}$, namely $\mathcal{B}(t_k) = p_k$, $k = 1, \dots, N$. It is assumed that $t_1 = 0$, $t_N = T - T/N$ and $t_{k+1} - t_k = T/N$ for all

$k = 1, \ldots, N-1$. We also assume that $\mathcal{B}(\cdot)$ is such that $\mathcal{B}(0) = \mathcal{B}(T) = p_1$ so that $y^\star : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is a continuous function (see Figure 4.1).

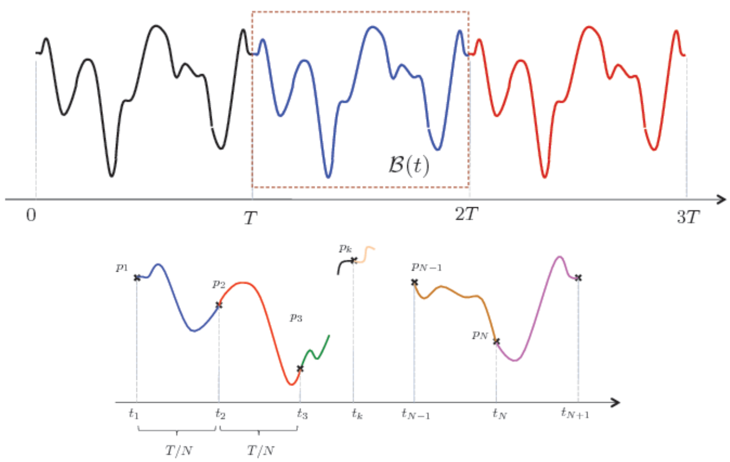

Figure 4.1: Periodically concatenated splines.

The idea behind spline generation (see [41]) is to interpolate N polynomials $\mathcal{P}_k(t) : [t_k, t_{k+1}] \to \mathbb{R}$ of suitable order to guarantee that $\mathcal{P}_k(t_k) = p_k$, $\mathcal{P}_k(t_{k+1}) = p_{k+1}$, $k = 1, \ldots, N$ (with $t_{N+1} = T$ and $p_{N+1} = p_1$) and to smooth the time derivatives of the signal $y^\star(t)$ at the times t_k, $k = 1, \ldots N+1$. For instance, by using polynomials of order three, simple algebraic arguments can be used to show that it is possible to design the four coefficients of the N polynomials in such a way that the first and second time derivative (velocities and accelerations) of $y^\star(t)$ are continuous at $t = t_k$, $k = 2, \ldots, N+1$. Smoother signals can be obtained by increasing the order of the polynomials $\mathcal{P}_k$. From now on, the polynomials $\mathcal{P}_k$ are assumed to be fixed in order to have continuity of $y^\star(t)$, $\dot{y}^\star(t)$ and $\ddot{y}^\star(t)$ for all $t \geq 0$.

Next, we are interested in computing $y^\star(t)$ as the output of a hybrid linear exosystem of the form

$$\begin{aligned} &\left.\begin{array}{rcl} \dot{\tau} &=& 1 \\ \dot{w} &=& Sw \end{array}\right\} (\tau, w) \in [0, \tau_{\max}] \times \mathbb{R}^s, \\ &\left.\begin{array}{rcl} \tau^+ &=& 0 \\ w^+ &=& Jw \end{array}\right\} (\tau, w) \in \{\tau_{\max}\} \times \mathbb{R}^s, \\ &y^\star(t) = Qw(t, j), \end{aligned} \tag{4.1}$$

where $Q = \begin{pmatrix} Q_1 & 0_{1\times N} \end{pmatrix}$, the matrices S and J are to be defined, and $\tau_{\max} := T/N$. The clock variable τ determines the polynomial switching times. The construction of the exosystem is aided by breaking it into two sub-states, $w_1 \in \mathbb{R}^4$, $w_2 \in \mathbb{R}^N$, with the dimension of the sub-state w_1 dependent on the order of the polynomials that are used as the basic spline functions.

Let $S_1 \in \mathbb{R}^4 \times \mathbb{R}^4$ be the matrix whose elements are all zero except along the super-diagonal, which is filled with ones, and let $Q_1 = \begin{pmatrix} 1 & 0_{1\times 3} \end{pmatrix}$. Furthermore, let

$$\mathcal{Q} := \begin{pmatrix} Q_1 \\ Q_1 S_1 \\ Q_1 e^{S_1 \tau_{\max}} \\ Q_1 S_1 e^{S_1 \tau_{\max}} \end{pmatrix}, \ \mathbf{v}_k := \begin{pmatrix} p_k \\ v_k \\ p_{k+1} \\ v_{k+1} \end{pmatrix}, \ k = 1, \dots, N,$$

with $v_k = \lim_{t \to t_k^+} \dot{\mathcal{P}}_k(t)$ and $v_{k+1} = \lim_{t \to t_{k+1}^-} \dot{\mathcal{P}}_k(t)$. The initial condition of the component w_1 of the exosystem is set to $w_1(t_1) = \mathcal{Q}^{-1}\mathbf{v}_1$, so that $y^\star(t) = Q_1 w_1(t)$ for $t \in [t_1, t_2]$.

Then we have to identify the switching rule of the state of the exosystem at times t_k, $k = 2, \dots, N+1$ in order to reproduce the reference at times $t > t_2$. To this purpose, we observe that the value $w_1^+(t_{k+1})$ needed to reproduce $\mathcal{P}_{k+1}(t)$ with $t \in [t_{k+1}, t_{k+2}]$, $k = 1, \dots N-1$, is

$$w_1^+(t_{k+1}) = \mathcal{Q}^{-1}\mathbf{v}_{k+1}. \tag{4.2}$$

We observe that p_{k+1} and v_{k+1} can be expressed as a function of $w_1(t_{k+1})$ since $p_{k+1} = Q_1 w_1(t_{k+1})$, and $v_{k+1} = Q_1 S_1 w_1(t_{k+1})$. It is possible to express v_{k+2} as function of p_k, p_{k+1}, p_{k+2}, v_k and v_{k+1}, by imposing continuity in the acceleration at time t_{k+1}. In fact, by imposing $Q_1 S_1^2 w_1(t_{k+1}) = Q_1 S_1^2 w_1(t_{k+1})^+$, and using

$$w_1(t_{k+1}) = e^{S_1 \tau_{\max}} w_1(t_k)^+ = e^{S_1 \tau_{\max}} \mathcal{Q}^{-1}\mathbf{v}_k \,,$$

one obtains

$$Q_1 S_1^2 e^{S_1 \tau_{\max}} \mathcal{Q}^{-1} \mathbf{v}_k = Q_1 S_1^2 \mathcal{Q}^{-1} \mathbf{v}_{k+1},$$

which, solved for v_{k+2}, yields

$$v_{k+2} = \Gamma \begin{pmatrix} p_k & p_{k+1} & p_{k+2} & v_k & v_{k+1} \end{pmatrix}^T,$$

with $\Gamma = \begin{pmatrix} -3/\tau_{\max} & 0 & 3/\tau_{\max} & -1 & -4 \end{pmatrix}$. By embedding the previous relation in (4.2) we obtain

$$w_1^+(t_{k+1}) = L \begin{pmatrix} p_k & p_{k+1} & p_{k+2} & v_k & v_{k+1} \end{pmatrix}^T,$$

where

$$L = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ \frac{6}{\tau_{\max}^2} & \frac{-6}{\tau_{\max}^2} & 0 & \frac{2}{\tau_{\max}} & \frac{4}{\tau_{\max}} \\ \frac{-18}{\tau_{\max}^3} & \frac{12}{\tau_{\max}^3} & \frac{6}{\tau_{\max}^3} & \frac{-6}{\tau_{\max}^2} & \frac{-18}{\tau_{\max}^2} \end{pmatrix}.$$

Now, we observe that $p_k = Q_1 e^{-S_1 \tau_{\max}} w_1(t_{k+1})$, $p_{k+1} = Q_1 w_1(t_{k+1})$, $v_k = Q_1 S_1 e^{-S_1 \tau_{\max}} w_1(t_{k+1})$, $v_{k+1} = Q_1 S_1 w_1(t_{k+1})$. To write a relation of the form $w^+(t_{k+1}) = J w(t_k)$ we are thus left to express p_{k+2} as a function of the state of the exosystem. By preserving the linearity of the exosystem, this can be done by "enriching" the exosystem with additional states $w_2 \in \mathbb{R}^N$ governed by the following dynamics (implementing a shift register)

$$\begin{aligned}\dot{w}_2 &= 0,\\ w_2^+ &= J_{22}w_2,\end{aligned}$$

where

$$J_{22} = \begin{pmatrix} 0_{1\times N-1} & 1 \\ I_{N-1} & 0_{N-1\times 1} \end{pmatrix}, \tag{4.3}$$

with the initial condition $w_2(t_1) = \begin{pmatrix} 1 & 0_{1\times N-1} \end{pmatrix}^T$. In this way $p_{k+2} = Pw_2(t_{k+1})$, with $P = \begin{pmatrix} p_3 & \dots & p_N & p_1 & p_2 \end{pmatrix}$, and

$$w_1^+(t_{k+1}) = L \begin{pmatrix} Q_1 e^{-S_1\tau_{\max}} & 0 \\ Q_1 & 0 \\ 0 & P \\ Q_1 S_1 e^{-S_1\tau_{\max}} & 0 \\ Q_1 S_1 & 0 \end{pmatrix} \begin{pmatrix} w_1(t_{k+1}) \\ w_2(t_{k+1}) \end{pmatrix}.$$

Overall, the exosystem, (4.1), takes the form

$$\left.\begin{array}{rcl}\dot{\tau} & = & 1 \\ \dot{w}_1 & = & S_1 w_1 \\ \dot{w}_2 & = & 0\end{array}\right\} (\tau, w) \in [0, \tau_{\max}] \times \mathbb{R}^s,$$

$$\left.\begin{array}{rcl}\tau^+ & = & 0 \\ w_1^+ & = & J_{11} w_1 + J_{12} w_2 \\ w_2^+ & = & J_{22} w_2\end{array}\right\} (\tau, w) \in \{\tau_{\max}\} \times \mathbb{R}^s,$$

with $y^\star = Q_1 w_1$, where

$$S_1 = \begin{pmatrix} 0_{3\times 1} \; I_3 \\ 0_{1\times 4} \end{pmatrix}, \; J_{11} = \begin{pmatrix} I_3 \; 0_{3\times 1} \\ L_1 \end{pmatrix}, \; J_{12} = \begin{pmatrix} 0_{3\times N} \\ L_2 \end{pmatrix}, \tag{4.4}$$

with L_1 and L_2 appropriately defined, and J_{22} defined as in (4.3). The matrices S and J are then implicitly defined.

4.2 Spline Tracking Using an Internal Model

Following the methods provided in Chapter 2 our goal is to design a regulator that will achieve hybrid output regulation for the following system continuous-time system in Brunovsky's canonical form.

$$\dot{y} \;=\; Ay + bu, \tag{4.5}$$

where $u \in \mathbb{R}$ and $y \in \mathbb{R}$, are the control input and the measured plant sub-state, respectively.

The regulated variable is the error, e, defined as

$$e = y - y^* = y - Qw,$$

where y^* is the spline based periodic signal introduced in Section 4.1. The goal is to find a hybrid regulator that processes only the error, e, and steers it asymptotically to zero.

Note that the system dealt with here is a very simplified version of that covered by the general framework. This is just to simplify the example by not having to deal with any extraneous information, including an observer in the stabilizer, while still showing the motivating properties of the spline generating exosystem.

The error dynamics of the system are

$$\dot{e} = Ae + b(u - R(\tau)w),$$

where $R(\tau) = \frac{-1}{b}(AQ - QS)$.

According to Section 2.3 we should be able to construct a regulator of the form

$$\left.\begin{aligned}\dot{\tau} &= 1\\ \dot{\eta} &= F_{\text{im}}\eta + G_{\text{im}}u\end{aligned}\right\} (\tau,\eta)\in[0,\tau_{\max}]\times\mathbb{R}^{\nu},$$
$$\left.\begin{aligned}\tau^{+} &= 0\\ \eta^{+} &= \Sigma_{\text{im}}\eta\end{aligned}\right\} (\tau,\eta)\in\{\tau_{\max}\}\times\mathbb{R}^{\nu}, \tag{4.6}$$
$$u = \Gamma_{\text{im}}(\tau)\eta + v,$$

where $\nu \in \mathbb{N}$, F_{im}, G_{im}, Σ_{im} are matrices, $\Gamma_{\text{im}} : [0,\tau_{\max}] \to \mathbb{R}^{1\times\nu}$ is a *continuously differentiable* function, and v is a residual control input, all to be designed.

Following Propositions 1 and 5, we can take $(F_{\text{im}}, G_{\text{im}})$ as a controllable pair, $(F_{\text{im}}, \Sigma_{\text{im}})$ such that $\operatorname{eig}(\Sigma_{\text{im}}\exp(F_{\text{im}}\tau_{\max})) \in \mathcal{D}_1$, and $v = -\kappa$ with κ large enough in order to achieve hybrid output regulation.

The problems with this design show up when Proposition 1 is looked at more closely. Specifically, Proposition 1 states that there exists a unique continuously differentiable function $\Pi_\eta : [0,\tau_{\max}] \to \mathbb{R}^{\nu\times(4+N)}$, which satisfies the internal model property equations (see Theorem 1)

$$\begin{aligned}\frac{d\Pi_\eta(\tau)}{d\tau} &= F_{\text{im}}\Pi_\eta(\tau) - \Pi_\eta(\tau)S + G_{\text{im}}R(\tau),\\ 0 &= \Sigma_{\text{im}}\Pi_\eta(\tau_{\max}) - \Pi_\eta(0)J.\end{aligned} \tag{4.7}$$

Furthermore, the function $\Gamma_{\text{im}}(\tau) = R(\tau)\Pi_\eta^{\dagger}(\tau)$, where $\Pi_\eta^{\dagger}(\tau)$ is the Moore-Penrose pseudo-inverse of $\Pi_\eta(\tau)$, is such that

$$\Gamma_{\text{im}}(\tau)\Pi_\eta(\tau) = R(\tau).$$

Recall that any regulator which has the hybrid internal model property must satisfy these equations, with $\Gamma_{\text{im}}(\tau)$ continuous.

Although the previous result shows that an error feedback regulator enforcing an asymptotically zero error can always be designed, it is not conclusive about the fact that the regulator is continuous during flows. As a matter of fact, there is no guarantee that $\Pi_\eta^\dagger(\tau)$, and thus $\Gamma_{\text{im}}(\tau)$, is continuous. By following [48], it turns out that the existence of an $r = \text{rank}(\Pi_\eta(\tau))$, such that $r \leq 4 + N$ for all $\tau \in [0, \tau_{\max}]$ is a sufficient condition under which the function $\Pi_\eta^\dagger(\tau)$, and thus $\Gamma_{\text{im}}(\tau)$, is continuous. The fulfillment of such a sufficient condition is, in general, affected by all the matrices entering in (4.7), among which the pair (S, J) define the hybrid exosystem.

4.3 A More Guided Design is Needed

Interestingly enough, simulation results show that in the case where the pair (S, J) has the specific form presented in Section 4.1 for spline generation, the function $\Pi_\eta^\dagger(\tau)$ is *not* continuous for generic choices of the matrix A of the plant, (4.5). Namely, the $\text{rank}(\Pi_\eta(\tau))$ changes in the interval $[0, \tau_{\max}]$. We can give evidence of this via simulation.

For these purposes consider the plant (4.5) and regulator (4.6) defined the by the following parameter choices, $A = 1$, $b = 1$ for the plant, which leads to $R(\tau) = R = \begin{pmatrix} -1 & 1 & 0 & \dots & 0 \end{pmatrix}$.

The regulator parameters are chosen according to $\Sigma_{\text{im}} = I$,

$$F = \begin{pmatrix} f_1 & \dots & f_8 \\ \\ I_7 & 0_{7\times 1} \end{pmatrix}, \quad G = \begin{pmatrix} 1 \\ \\ 0_{7\times 1} \end{pmatrix},$$

where $(f_1, \dots, f_8)$ are the coefficients such that the eigenvalues of F satisfy

$$\text{eig}(F) = -(1, 1, 2, 3, 5, 5, 6, 10).$$

Furthermore, the way-points of the exosystem are chosen as $(-1, 1, -2, 1)$; these fully define the exosystem parameters according to Section 4.1.

These choices lead to a solution $\Pi_\eta(\tau)$ to (4.7) that has a non-constant rank, as shown by the ratio of the maximum singular value of $\Pi_\eta(\tau)$ to the minimum singular value of $\Pi_\eta(\tau)$. This ratio is depicted in Figure 4.2. Figure 4.2 also shows the discontinuities in $R(\tau)\Pi_\eta(\tau)^\dagger$. Since $R(\tau)\Pi_\eta(\tau)^\dagger$ is not a scalar function we plot one element of it to show its discontinuity.

This result precludes the use of the τ-dependent regulator from Section 2.3. The design of an internal model that is continuous during flows for the spline-based exosystem is thus more elaborate and is covered by the general internal model design method presented in Chapter 5.

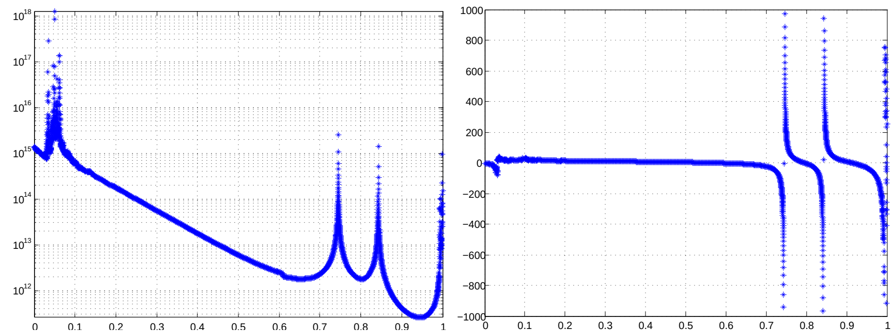

Figure 4.2: Ratio of largest to smallest singular value of $\Pi_\eta(\tau)$; $R(\tau)\Pi_\eta(\tau)^\dagger$

4.4 Conclusions

In this chapter we have motivated the need for a more guided approach to internal model design for hybrid exosystems in order to achieve an internal model design that is continuous during flows. The work in this chapter was originally published in [13], where a method for designing hybrid internal models that work for a more restrictive hybrid framework is also covered. Here we forgo discussing said method, and instead discuss the generalized internal model design that it inspired in Chapter 5.

Chapter 5

General Internal Model Design

In this chapter we consider a class of MIMO linear systems and exosystems that are subject to jumps according to a known clock that satisfies a dwell-time constraint. In addition to the expansion of including a class of MIMO systems, some other shortcomings of previous chapters are addressed here. Most notably, a general method for the design of internal models for these systems is addressed here. This is of particular interest because the design in Chapter 2 (see also [39]) relies on a technical assumption for its suggested method of internal model design, but, as was shown in Chapter 4, for the problem of tracking a spline trajectory (see [13]) this technical assumption can be problematic. The method presented here gets rid of this technical assumption and can in fact be applied to the problem of tracking spline trajectories.

The internal model developed here builds on a "visibility property" of the so-called hybrid steady-state generator system, namely the hybrid system that generates the ideal control input able to keep the regulation error identically zero. In this way we give a consistent design method for hybrid internal models applicable in general. The internal model designed is similar to a state observer, but with the alternate goal of reproducing the output of the hybrid steady state generator, as opposed to the entire state.

Furthermore, the internal models designed here address the issue of robustness normally associated with the idea of an internal model. The robustness property referred to here is what Francis and Wonham call structural stability in their seminal papers regarding output regulation and the internal model principle for continuous-time systems (see [23] and [24]). It turns out that in the hybrid setting, some additional attention is needed to achieve robustness. Here we present a method that is sufficient for dealing with linearly parametrized uncertainties that may affect the internal model.

The final goal of this chapter is to present a practical example where this framework and the ideas can be used. In doing so we address the issue of a quad-rotor tracking a periodic spline trajectory in a plane. The problem of tracking spline trajectories in this framework was first addressed in [13], which required some additional restrictive assumptions on the zero-dynamics of the plant, but those restrictions have also been addressed here. In fact the design proposed here is motivated by the methods used to track spline trajectories there, but works much more generally.

The chapter is laid out as follows. Section 5.1 lays out the framework and the hybrid output regulation problem, along with introducing the concept of a hybrid steady-state generator for the relevant exosystems and specifying the robustness goals. Section 5.2 details how the hybrid asymptotic internal model property is achieved by the proposed regulator, and presents the internal model design. Section 5.3 presents the practical example of robust tracking of a spline reference trajectory by a quad-rotor. Some relevant propositions and proofs are saved for the appendix.

5.1 Framework

5.1.1 Hybrid output regulation problem

In this chapter we deal with hybrid linear systems whose flow and jump relations are modeled by

$$\left.\begin{aligned} \dot{\tau} &= 1 \\ \dot{x} &= Ax + Bu + Pw \end{aligned}\right\} (\tau, x, u) \in [0, \tau_{\max}] \times \mathbb{R}^n \times \mathbb{R}^m$$
$$\left.\begin{aligned} \tau^+ &= 0 \\ x^+ &= Mx + Nw \end{aligned}\right\} (\tau, x, u) \in \{\tau_{\max}\} \times \mathbb{R}^n \times \mathbb{R}^m \qquad (5.1)$$

in which $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input, w is an exogenous variable, and τ plays the role of a clock variable that is reset every $\tau_{\max}$ instances of time, with $\tau_{\max}$ representing a dwell-time between two consecutive state jumps. Consistent with output regulation frameworks, the exogenous variable w is thought of as generated by an exosystem that, in the proposed hybrid setting, is modeled as

$$\left.\begin{array}{rcl} \dot{\tau} & = & 1 \\ \dot{w} & = & S(\tau)w \end{array}\right\} \quad (\tau, w) \in [0, \tau_{\max}] \times \mathbb{R}^s$$
$$\left.\begin{array}{rcl} \tau^+ & = & 0 \\ w^+ & = & Jw \end{array}\right\} \quad (\tau, w) \in \{\tau_{\max}\} \times W \tag{5.2}$$

in which $W \subset \mathbb{R}^s$ is a compact set. In the following we assume that the set $[0, \tau_{\max}] \times W$ is forward invariant for (5.2). Associated to systems (5.1) and (5.2), there is a regulation error $e \in \mathbb{R}^m$ modeled by

$$e = Cx + Qw\,. \tag{5.3}$$

In this context the problem we are interested in is to design an error feedback hybrid regulator of the form

$$\left.\begin{array}{rcl}\dot{\tau} & = & 1 \\ \dot{\eta} & = & F(\tau)\eta + G(\tau)e\end{array}\right\} \quad (\tau,\eta) \in [0,\tau_{\max}] \times \mathbb{R}^{\nu}$$

$$\left.\begin{array}{rcl}\tau^{+} & = & 0 \\ \eta^{+} & = & \Phi\eta + \Sigma e\end{array}\right\} \quad (\tau,\eta) \in \{\tau_{\max}\} \times \mathbb{R}^{\nu}$$

$$u = \Gamma\eta + Ke$$

so that the trajectories of the closed-loop system originating from initial conditions in $[0,\tau_{\max}] \times W \times \mathbb{R}^n \times \mathbb{R}^{\nu}$ are bounded and $\lim_{t+j\to\infty} e(t,j) = 0$. It is worth noting that the clock variable τ is assumed to be available for the design of the regulator, with the latter that is in general a time-varying system. Tracking and/or disturbance rejection of exosystem-generated reference/disturbance signals can be clearly cast in the previous framework.

In particular, note that the problem of tracking a trajectory generated by the hybrid exosystem (5.2) with the output of a continuous-time plant is covered by this framework. In this case, $M = I_n$, $N = 0$ in (5.1), and $-Qw$ is the reference signal generated by (5.2) to be tracked by the output Cx of the system. This is the case considered in the example presented in Section 5.3.

In Chapter 2 (see [39]), it has been shown that a necessary condition for the solution to the problem of hybrid linear regulation formulated above is that there exist continuous functions $\Pi : [0, \tau_{\max}] \to \mathbb{R}^n$ and $R : [0, \tau_{\max}] \to \mathbb{R}^m$ that are solutions of the *Hybrid Regulator Equations*

$$\begin{aligned} \frac{d\Pi(\tau)}{\tau} &= A\Pi(\tau) - \Pi(\tau)S(\tau) + P + BR(\tau) \\ 0 &= M\Pi(\tau_{\max}) - \Pi(0)J + N \\ 0 &= C\Pi(\tau) + Q \,. \end{aligned} \tag{5.4}$$

In the following we assume that the previous equations admit a solution $(\Pi(\cdot), R(\cdot))$. Specifically, the control design will build on the solution $R(\tau)$ as clarified later. The previous equations generalize the well-known regulator equations for continuous-time systems (see [23] and [24]). The latter, indeed, are obtained from (5.4) considering $M = I_n$, $J = I_s$, $N = 0$, a time-invariant exosystem (namely $S(\tau) \equiv S$), and constant unknowns $\Pi(\tau) \equiv \Pi$, $R(\tau) = R$. Under these conditions, in fact, the second equation of (5.4) is automatically fulfilled and the first and last equations reduce to $A\Pi - \Pi S + P + BR = 0$, $C\Pi + Q = 0$, that are the conventional regulator equations. Equations (5.4) also generalize regulator equations proposed in the context of output regulation for continuous-time linear systems in the presence of periodic exosystems (see [52]).

Conditions under which the hybrid regulator equations admit a solution involves a mix of "non-resonance conditions" and "compatibility conditions" between the relative degree of the regulated plant and the hybrid exosystem. More insight on the solution and on the underlying conditions can be given by expressing the triplet (A,B,C) in the so-called Brunowsky canonical form. In particular, by letting $(r_1,\ldots,r_m)$ the vector relative degree of the triplet (A,B,C), standard facts show that there exist a change of variable of the form $\mathrm{col}(z,\xi,y_r)=Tx$, with $y_r\in\mathbb{R}^m$, $\xi\in\mathbb{R}^{r-m}$, $r=r_1+\ldots+r_m$, $z\in\mathbb{R}^{n-r}$ such that the flow dynamics of regulated plant in (5.1) are similar to the system

$$
\begin{array}{rcl}
\dot z &=& A_{11}\,z+A_{12}\,L\xi+P_1\,w\\
\dot\xi &=& A_{22}\xi+A_{23}y_r+P_2\,w\\
\dot y_r &=& A_{31}\,z+A_{32}\xi+A_{33}y_r+\bar B\,u+P_3\,w\\
e &=& L\xi+Qw
\end{array}
\tag{5.5}
$$

where $\xi=\mathrm{col}\begin{pmatrix}\xi_1 & \ldots & \xi_m\end{pmatrix}$, $\xi_i=\mathrm{col}\begin{pmatrix}\xi_{i,1} & \ldots & \xi_{i,r_i-1}\end{pmatrix}$, $i=1,\ldots,m$,

$$A_{22}=\mathrm{blkdiag}(S_1,\,\ldots\,,S_m)\,,\quad A_{23}=\mathrm{blkdiag}(B_1,\,\ldots\,,B_m)\,,$$

$$L=\mathrm{blkdiag}(L_1,\,\ldots\,,L_m)$$

with $(S_i,B_i,L_i)\in\mathbb{R}^{(r_i-1)\times(r_i-1)}\times\mathbb{R}^{(r_i-1)\times 1}\times\mathbb{R}^{1\times(r_i-1)}$ a triplet in prime form, that is S_i is a shift matrix (all 1's on the upper diagonal and all 0's elsewhere), $B_i^T=(0\cdots 0\ \ 1)$, $L_i=(1\ 0\cdots 0\ )$, $i=1,\ldots,m$, $\bar B$ is the "high-frequency" matrix of the system, and where P_1, P_2 and P_3 are appropriately defined matrices. As for the jump dynamics, we cannot expect any special structure. The x^+ expression in (5.1) in the new coordinates thus takes the generic form

$$\begin{aligned} z^+ &= M_{11}\, z + M_{12}\, \xi + M_{13} y_r + N_1\, w \\ \xi^+ &= M_{21} z + M_{22} \xi + M_{23} y_r + N_2\, w \\ y_r^+ &= M_{31}\, z + M_{32} \xi + M_{33} y_r + N_3\, w \end{aligned}$$

where M_{ij} and N_j are appropriately defined matrices.

In these coordinates the solution of the regulator equations takes a particular form. In particular, partitioning $\Pi(\tau)$ as $\mathrm{col}(\Pi_z(\tau)\,,\ \Pi_\xi(\tau)\,,\ \Pi_{y_r}(\tau))$ coherently with the state partition in the new coordinates, it turns out that the first and the last equation of (5.4) impose that $\Pi_\xi(\tau)$ is necessarily given by

$$\Pi_\xi(\tau) = \begin{pmatrix} \Pi_{\xi_1}(\tau) \\ \vdots \\ \Pi_{\xi_m}(\tau) \end{pmatrix}, \quad \Pi_{\xi_i}(\tau) = \begin{pmatrix} \Pi_{\xi_{i,1}}(\tau) \\ \vdots \\ \Pi_{\xi_{i,r_i-1}}(\tau) \end{pmatrix} \tag{5.6}$$

where the $\Pi_{\xi_{i,j}}(\tau)$ are recursively defined as

$$\begin{aligned} \Pi_{\xi_{i,1}} &= -Q_i \\ \Pi_{\xi_{i,j}}(\tau) &= \frac{d\Pi_{\xi_{i,j-1}}(\tau)}{d\tau} + \Pi_{\xi_{i,j-1}}(\tau) S(\tau) - P_{2i,j-1} \end{aligned} \tag{5.7}$$

for $i = 1, \ldots, m$, $j = 2, \ldots, r_i - 1$, in which $P_{2i,j}$ represents the j-th rows of P_{2i} having partitioned P_2 as $\mathrm{col}(P_{21}\,, \ldots\,, P_{2m})$ coherently with the partition of ξ. Furthermore, by the first equation of (5.4), we have that $\Pi_{y_r}(\tau)$ is necessarily given by

$$\Pi_{y_r}(\tau) = \frac{d\Pi_{\xi_{i,r_i-1}}(\tau)}{d\tau} + \Pi_{\xi_{i,r_i-1}}(\tau)S(\tau) - P_{2i,r_i-1}\,. \tag{5.8}$$

With $\Pi_\xi(\tau)$ and $\Pi_{y_r}(\tau)$ fixed, the remaining unknowns $\Pi_z(\tau)$ and $R(\tau)$ then result from the first two equations in (5.4). In particular $\Pi_z(\tau)$ is constrained to be a solution of

$$\begin{aligned} \frac{d\Pi_z(\tau)}{d\tau} &= A_{11}\,\Pi_z(\tau) - A_{12}Q - \Pi_z(\tau)S + P_1 \\ 0 &= M_{11}\,\Pi_z(\tau_{\max}) - \Pi_z(0)\,J + M_{12}\,\Pi_\xi(\tau_{\max}) + M_{13}\,\Pi_{y_r}(\tau_{\max}) + N_1 \end{aligned} \tag{5.9}$$

with the additional constraint

$$\begin{aligned} &\begin{pmatrix} M_{21} \\ M_{31} \end{pmatrix} \Pi_z(\tau_{\max}) - \begin{pmatrix} \Pi_\xi(0) \\ \Pi_{y_r}(0) \end{pmatrix} J \ldots \\ &\quad + \begin{pmatrix} M_{22} \\ M_{32} \end{pmatrix} \Pi_\xi(\tau_{\max}) + \begin{pmatrix} M_{23} \\ M_{33} \end{pmatrix} \Pi_{y_r}(\tau_{\max}) + \begin{pmatrix} N_2 \\ N_3 \end{pmatrix} = 0\,. \end{aligned} \tag{5.10}$$

With $\Pi_z(\tau)$ also in hand, $R(\tau)$ is then determined by

$$\begin{aligned} R(\tau) &= \bar{B}^{-1}\big(\tfrac{d\Pi_{y_r}(\tau)}{d\tau} + \Pi_{y_r}(\tau)S(\tau) - A_{31}\Pi_z(\tau) - A_{32}\Pi_\xi(\tau)\ldots \\ &\qquad - A_{33}\Pi_{y_r}(\tau) - P_3(\tau)\big)\,. \end{aligned} \tag{5.11}$$

The solution of the hybrid regulation equations thus boils down to computing a $\Pi_z(\tau)$ fulfilling (5.9) and the additional constraint (5.10). In this respect the following result, proved in Lemma 1 of [39], details a "non resonance condition" under which a $\Pi_z(\tau)$ fulfilling (5.9) exits.

Lemma 1 *Let $\phi_S(\tau)$ be the state transition matrix of the time-varying system $\dot{w} = S(\tau)w$, namely the smooth matrix such that $d\phi_S(\tau)/d\tau = S(\tau)\phi_S(\tau)$ and $\phi_S(0) = I_s$. Assume that the* "non-resonance condition"

$$\mathrm{eig}(M_{11}\exp(A_{11}\tau_{\max})) \cap \mathrm{eig}(J\phi_S(\tau_{\max})) = \emptyset \tag{5.12}$$

holds true. Then for all $\Pi_\xi(\tau_{\max})$ and $\Pi_{y_r}(\tau_{\max})$ there exists a unique continuously differentiable solution $\Pi_z(\tau)$ to the equations (5.9).

As the eigenvalues of A_{11} coincide with the transmission zeros of the triplet (A, B, C), condition (5.12) can be interpreted as the generalization, in the considered hybrid setting, of the non-resonance condition between the zeros of the controlled plant and the modes of the exosystem well-known in the continuous-time domain.

With $\Pi_z(\cdot)$ also given, the hybrid regulation equations (5.4) are fulfilled if $(\Pi_z(\cdot)\,,\ \Pi_\xi(\cdot)\,,\ \Pi_{y_r}(\cdot))$ is such that the constraints (5.10) are satisfied. Those constraints can be regarded as "compatibility constraints" between the relative degree of the regulated plant and the hybrid exosystem. It essentially fixes a requirement on the continuity of the "reference signal" $Qw(t)$ during jumps in relation to the relative degree of the system. For example, in the case where the plant is a continuous-time system with unitary relative degree (namely $r_i = 1$, $i = 1, \ldots, m$), it is immediately seen that (5.10) are automatically fulfilled if

$QJ = Q$, namely if $(Qw)^+ = Qw$, which is a requirement on the continuity of the output reference signal during jumps. Henceforth we consider such "compatibility constraints" fulfilled, which implies that the hybrid regulation equations have a solution. The latter is given by $\Pi_\xi(\cdot)$ and $\Pi_{y_r}(\cdot)$ in (5.6), (5.7), (5.8), by the $\Pi_z(\cdot)$ that is a solution of (5.9) (according to Lemma 1), and by $R(\cdot)$ in (5.11).

We complete the section by presenting two assumptions under which the problem of hybrid output regulation will be solved. The first asks that the hybrid system is minimum-phase as detailed in Assumption 1 below. This assumption is not necessary for the internal model design, but is formulated in order to present a systematic robust design procedure for the regulator based on high-gain arguments. The second assumption simply asks that the square MIMO system (A, B, C) is invertible. In the Brunowsky canonical coordinates this assumption simply asks that the high frequency matrix $\bar{B}$ is invertible.

Assumption 6 (Minimum-Phase) *The matrices A_{11}, M_{11} are such that*

$$\mathrm{eig}(M_{11}\exp(A_{11}\tau_{\max})) \in \mathcal{D}_1.$$

Assumption 7 (Invertibility) *The matrix $\bar{B} \in \mathbb{R}^m \times \mathbb{R}^m$ is invertible.*

The following assumption must also be made in order to achieve the stabilization goal using a continuous-time high gain observer. Note that this assumption is not necessary for achieving the internal model property for the internal model design, so it is conceivable that a different stabilizer could be used with this internal model in order to achieve hybrid output regulation for hybrid plants.

Assumption 8 (Continuous-Time Plant) *The matrix parameters M_{ij} are such that $M_{ij} = I$ if $i = j$, but $M_{ij} = 0$ otherwise.*

5.1.2 The Hybrid Steady State Generator System and Robust Regulation

As shown in Chapter 2 (see [39]), the existence of a solution of the hybrid regulation equations is equivalent to the existence of steady state trajectories for the state and the input of (5.1) characterized by a regulation error (5.3) that is identically zero. In particular, with $\tau(t, j)$ and $w(t, j)$ the trajectories of (5.2) originating from the actual initial condition of the hybrid exosystem, it turns out that $x(t, j) = \Pi(\tau(t, j))w(t, j)$ and $u(t, j) = R(\tau(t, j))w(t, j)$ represent the ideal steady state trajectory and the ideal control input toward which the state of the plant and the output of the regulator must converge to in order to solve the problem at hand.

With regards to these trajectories, it is apparent that a crucial role in the design of internal model-based regulators is played by the "hybrid steady state generator system" (see Chapter 2) defined as the following hybrid system

$$
\begin{aligned}
&\left.\begin{aligned} \dot{\tau} &= 1 \\ \dot{w} &= S(\tau)w \end{aligned}\right\} \quad (\tau, w) \in [0, \tau_{\max}] \times \mathcal{W} \\
&\left.\begin{aligned} \tau^+ &= 0 \\ w^+ &= Jw \end{aligned}\right\} \quad (\tau, w) \in \{\tau_{\max}\} \times \mathcal{W} \\
&y_w = R(\tau)w
\end{aligned}
\tag{5.13}
$$

with output $y_w \in \mathbb{R}^m$. (Note that in Chapter 2, the set $\mathcal{W}$ is time-varying, but here it is not. Although, this notation might be confusing, it is merely a consequence of the notation that $(\tau, w) \in \mathcal{W}$ in Chapter 2, whereas here we say that $(\tau, w) \in [0, \tau_{\max}] \times \mathcal{W}$.) This system generates all the ideal steady state control inputs required of the regulator in order to keep the regulation error, e, identically zero. Due to the fact that the initial condition, $(\tau(0), w(0))$, of the exosystem is arbitrary on $[0, \tau_{\max}] \times \mathcal{W}$, it is in particular apparent that the "visible" dynamics of system (5.13) must be embedded into any regulator that solves the problem of output regulation.[1] This observation is at the foundation of the celebrated internal model principle (see [24] for continuous-time linear systems and [39] for hybrid linear systems).

[1] The concept of visibility here is used loosely and will be better specified later by following [19]. Intuitively, visible dynamics are state trajectories of (5.13) that show up on the output y_w and, as such, must be necessarily reproduced by the regulator.

An obstacle in embedding a copy of (5.13) into the regulator comes from the fact that the function $R(\cdot)$, as solution of the hybrid regulator equations, is in general affected by possible parametric uncertainties in the regulated plant. This fact raises an issue of *robust* design of the regulator. In order to deal with this issue it is worth introducing the class of systems that are "state-output" equivalent to (5.13) as formally defined in Definition 1 below. In the definition we refer to an "equivalent" system defined by

$$\begin{aligned} &\left.\begin{aligned} \dot{\tau} &= 1 \\ \dot{\mathbf{w}} &= \mathbf{S}(\tau)\mathbf{w} \end{aligned}\right\} \quad (\tau, \mathbf{w}) \in [0, \tau_{\max}] \times \mathbf{W} \\ &\left.\begin{aligned} \tau^+ &= 0 \\ \mathbf{w}^+ &= \mathbf{J}\mathbf{w} \end{aligned}\right\} \quad (\tau, \mathbf{w}) \in \{\tau_{\max}\} \times \mathbf{W} \\ &\mathbf{y_w} = \mathbf{R}(\tau)\mathbf{w} \end{aligned} \tag{5.14}$$

where $\mathbf{w} \in \mathbb{R}^{\mathbf{s}}$, $\mathbf{s} \in \mathbb{N}$, and $\mathbf{W}$ is a compact subset of $\mathbb{R}^{\mathbf{s}}$ with $[0, \tau_{\max}] \times \mathbf{W}$ invariant for (5.14). We note that (5.13) and (5.14) have the same hybrid time domain (see [28]) dependent on the initial condition $\tau(0)$. The crucial distinction between this "equivalent" system and the previous hybrid steady-state generator is that this system is allowed to have a different dimension. This is particularly relevant because it turns out that we can use duplication of the previous hybrid steady-state generator to achieve robustness, as will be shown soon.

Definition 1 *System (5.13) is* state-output equivalent *to system (5.14) if for any $\tau(0) \in [0, \tau_{\max}]$ and $w(0) \in \mathcal{W}$ there exists a $\mathbf{w}(0) \in \mathbf{W}$ such that, having denoted by $E \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ the corresponding hybrid time domain,*

$$y_w(t, j) = \mathbf{y_w}(t, j) \qquad \forall\, (t, j) \in E\,.$$

The presence of possible uncertainties in $R(\tau)$ can be overcome by defining an equivalent system in an appropriate way. This is certainly the case if $R(\tau)$ is linearly parametrized in the uncertainties as shown below. In particular, by partitioning $R(\cdot)$ input-wise as $R(\cdot) = \mathrm{col}(R_1(\cdot), \ldots, R_m(\cdot))$ with $R_i : [0, \tau_{\max}] \to \mathbb{R}^{1\times s}$, assume that there exist $p_i \in \mathbb{N}$ and *known* continuously differentiable functions $R_{ij} : [0, \tau_{\max}] \to \mathbb{R}^{1\times s}$, $i = 1, \ldots, m$, $j = 1, \ldots, p_i$, such that

$$R_i(\tau) = \sum_{j=1}^{p_i} R_{ij}(\tau)\mu_{ij} \tag{5.15}$$

where $\mu_{ij} \in \mathbb{R}$ are uncertain parameters ranging in a *known* compact set $[\underline{\mu}_{ij},\, \bar{\mu}_{ij}]$, $i = 1, \ldots, m$, $j = 1, \ldots, p_i$. Under this condition a hybrid steady state generator equivalent to (5.13) and not affected by uncertainties can be constructed as detailed in the next proposition.

Proposition 6 *With $p = \sum_{i=1}^{m} p_i$, $\mathbf{R}_i = (R_{i1}(\tau),\ \ldots,\ R_{i,p_i}(\tau))$, $i = 1, \ldots, m$, and $W_{ij} = \{\bar{w} \in \mathbb{R}^s \ :\ \bar{w} = \mu w,\, w \in \mathcal{W}\,,\ \mu \in [\underline{\mu}_{ij},\, \bar{\mu}_{ij}]\}$, $i = 1, \ldots, m$, $j = 1, \ldots, p_i$, let*

$$\mathbf{S}(\tau) = I_p \otimes S(\tau)\,, \quad \mathbf{J} = I_p \otimes J\,, \quad \mathbf{R}(\tau) = \mathrm{blockdiag}\left(\mathbf{R}_1(\tau),\ \ldots,\ \mathbf{R}_m(\tau)\right)\,,$$

$$\mathbf{W} = W_{11} \times \ldots \times W_{1p_1} \times \ldots \times W_{m1} \times \ldots \times W_{mp_m} .$$

Then system (5.14) is state-output equivalent to (5.13).

5.1.3 Main result about regulator design

In this section we present a general result underlining the design of the internal model based regulator. The result builds on the Hybrid Internal Model Property that is precisely defined in the following. This definition is given in different terms from the definition in Chapter 2 (and [39]), but in fact serves the same purpose.

Definition 2 *(***Hybrid Internal Model Property***) We say that a quadruplet* $(F_{\rm im}(\cdot), G_{\rm im}(\cdot), \Gamma_{\rm im}(\cdot), \Sigma_{\rm im})$*, where* $F_{\rm im} : [0, \tau_{\max}] \to \mathbb{R}^{\nu\times\nu}$*,* $G_{\rm im} : [0, \tau_{\max}] \to \mathbb{R}^{\nu\times m}$ *and* $\Gamma_{\rm im} : [0, \tau_{\max}] \to \mathbb{R}^{1\times\nu}$*,* $\nu \in \mathbb{N}$ *, are* continuously differentiable *functions and* $\Sigma_{\rm im}$ *is a matrix, has the hybrid internal model property relative to (5.14) if for some continuously differentiable function* $\Pi_\eta : [0, \tau_{\max}] \to \mathbb{R}^{\nu\times\mathbf{s}}$ *the set*

$$\mathcal{S} = \{(\tau, \mathbf{w}, \eta) \in [0, \tau_{\max}] \times \mathbf{W} \times \mathbb{R}^\nu : \eta = \Pi_\eta(\tau)\mathbf{w}\} \tag{5.16}$$

is globally exponentially stable for the hybrid system

$$\left.\begin{array}{rcl}\dot{\tau} & = & 1 \\ \dot{\mathbf{w}} & = & \mathbf{S}(\tau)\mathbf{w} \\ \dot{\eta} & = & F_{\mathrm{im}}(\tau)\eta + G_{\mathrm{im}}(\tau)\mathbf{R}(\tau)\mathbf{w}\end{array}\right\} (\tau, \mathbf{w}, \eta) \in [0, \tau_{\max}] \times \mathbf{W} \times \mathbb{R}^{\nu}$$
$$\left.\begin{array}{rcl}\tau^{+} & = & 0 \\ \mathbf{w}^{+} & = & \mathbf{J}\mathbf{w} \\ \eta^{+} & = & \Sigma_{\mathrm{im}}\eta\end{array}\right\} (\tau, \mathbf{w}, \eta) \in \{\tau_{\max}\} \times \mathbf{W} \times \mathbb{R}^{\nu} \tag{5.17}$$

and

$$\Gamma_{\mathrm{im}}(\tau)\Pi_{\eta}(\tau) = \mathbf{R}(\tau) \qquad \forall\, \tau \in [0, \tau_{\max}]. \tag{5.18}$$

Later in this section, and more extensively in Section 5.2, a systematic procedure for designing quadruplets with the hybrid internal model property will be presented. For the time being we show that, if such a quadruplet exists, then a regulator solving the problem at hand can be systematically designed. Toward this end we consider a hybrid regulator consisting of an internal model (η) and a stabilizer (χ) of the form

$$\left.\begin{aligned} \dot{\tau} &= 1 \\ \dot{\eta} &= F_{\mathrm{im}}(\tau)\eta + G_{\mathrm{im}}(\tau)u \\ \dot{\chi} &= \Phi_{\mathrm{st}}\chi + \Lambda_{\mathrm{st}}(L\xi + Qw) \end{aligned}\right\} (\tau,\eta,\chi) \in [0,\tau_{\max}] \times \mathbb{R}^{\nu} \times \mathbb{R}^{r},$$
$$\left.\begin{aligned} \tau^{+} &= 0 \\ \eta^{+} &= \Sigma_{\mathrm{im}}\eta \\ \chi^{+} &= \chi \end{aligned}\right\} (\tau,\eta,\chi) \in \{\tau_{\max}\} \times \mathbb{R}^{\nu} \times \mathbb{R}^{r}, \tag{5.19}$$
$$u = \Gamma_{\mathrm{im}}(\tau)\eta - \kappa \bar{B}^{-1} H\chi,$$

where $H = \mathrm{blkdiag}(H_1, \ldots, H_m)$, with $H_i = (k_{i1} \ldots k_{ir_i-1})$ such that the k_{ij} are coefficients of a Hurwitz polynomials $s^{r_i-1} + k_{i1}s^{r_i-2} + \cdots + k_{ir_i-1}$. The matrix parameters Φ_{st} and Λ_{st} are chosen as block diagonal matrices, where the blocks along the diagonal are

$$\Lambda_{\mathrm{st}_i} = \begin{pmatrix} c_{i1}g \\ \vdots \\ c_{ir_i}g^{r_i} \end{pmatrix}, \quad \Phi_{\mathrm{st}_i} = \begin{pmatrix} -\Lambda_{\mathrm{st}_i} & \begin{matrix} I_{r_i-1} \\ 0 \end{matrix} \end{pmatrix}.$$

The following result, whose proof is presented in Appendix A.3, provides the main guidelines for the design of (5.19).

Proposition 7 *Let Assumptions 6, 7 be fulfilled. Let (5.14) be a system that is state-output equivalent to the hybrid steady-state generator system (5.13) and let* *the quadruplet* $(F_{\text{im}}(\cdot), G_{\text{im}}(\cdot), \Gamma_{\text{im}}(\cdot), \Sigma_{\text{im}})$ *have the hybrid asymptotic internal model property relative to (5.14). Furthermore, let Assumption 8 hold. Then there exists a* $\kappa^* > 0$ *and for each* $\kappa \geq \kappa^*$ *there exists a* $g^* > 0$ *such that for each* $g \geq g^*$ *the regulator (5.19) solves the problem of hybrid output regulation.*

It is worth noting that in the case of a relative degree one system the stabilizer need only be a static system, since there is no need for an observer to track the unmeasured states. In this case the feedback law reduces to $u = \Gamma_{\text{im}}(\tau)\eta + \kappa\bar{B}^{-1}He$ with $H \in \mathbb{R}^{m\times m}$ a Hurwitz matrix and κ sufficiently large.

In the next section we discuss the design of the internal model unit.

5.2 Achieving the Hybrid Asymptotic Internal Model Property

In this section we develop the main result of this chapter, which is the problem of systematically designing a quadruplet $(F_{\text{im}}(\tau), G_{\text{im}}(\tau), \Gamma_{\text{im}}(\tau), \Sigma_{\text{im}})$ that has the hybrid internal model property according to Definition 2.

We approach the problem of designing a quadruplet $(F_{\text{im}}(\tau), G_{\text{im}}(\tau), \Gamma_{\text{im}}(\tau), \Sigma_{\text{im}})$ that fulfills the properties of Definition 2 by designing an observer for the dynamics of (5.14) that are "visible" on the output $\mathbf{y_w}$. Towards this end, in the next subsection we present a decomposition of system (5.14) that isolates visible and invisible dynamics. Our goal is to identify a hybrid system that is state-output equivalent to (5.14) and for which an asymptotic observer can be designed. The design of the hybrid asymptotic observer is dealt with in Section 5.2.2. This, in turn, will lead to immediately obtaining an "output reproducer" of system (5.14). For notational convenience, in the following part we drop the bold notation for system (5.14), by using $S(\tau)$, J, $R(\tau)$, W and y_w instead of $\mathbf{S}(\tau)$, $\mathbf{J}$, $\mathbf{R}(\tau)$, $\mathbf{W}$ and $\mathbf{y_w}$.

5.2.1 Isolating invisible dynamics

Towards the final goal of isolating visible and invisible dynamics of the *hybrid* system (5.14), we start by focusing on the flow dynamics by identifying dynamics that do not affect the output during flow. Consider the continuous-time time-varying linear system of the form

$$\begin{aligned} \dot{w} &= S(\tau)w, \qquad w \in \mathbb{R}^s \\ y_w &= R(\tau)w, \end{aligned} \tag{5.20}$$

defined on the interval $\tau \in [0, \tau_{\max}]$ and let $\phi(\tau, \tau_0)$ be the state transition matrix associated with $\dot{w} = S(\tau)w$. As the system is time-varying, a Kalman-like decomposition related to observability can be rigorously obtained by the arguments of [19]. The definition of an *invisible state* is crucial to that paper and is recalled with a slight adaptation for our purposes.

Definition 3 (Invisbile) *We say that a state $w \in \mathbb{R}^s$ is* invisible *in the interval $[0, \tau_{\max}]$ at time $\tau \in [0, \tau_{max}]$ if it is unobservable and unreconstructable at time τ in the specified time interval, namely if*

$$R(t)\phi(t,\tau)w(\tau) = 0 \qquad \textit{for all } t \in [0, \tau_{\max}] \,.$$

Note that the previous relation holds for all t in the interval, namely for $t \geq \tau$ (unobservability of the state w) and for $t \leq \tau$ (unreconstructability of the state w). In other words, all the output behavior, $y_w(t)$, originating from an invisible state at time τ is identically zero (in the interval) both forward (i.e. for $t \geq \tau$) and backward ($t \leq \tau$) in time. Furthermore, we define the invisible space as in the following.

Definition 4 *We let $\mathcal{L}(\tau)$ be the space of states that are invisible at time τ in the interval $[0, \tau_{\max}]$.*

Let Q be the Gramian associated to the system in the interval defined as

$$Q(\tau) = \int_0^{\tau_{\max}} \phi(t,\tau)^T \, R^T(t) R(t) \phi(t,\tau) dt \,.$$

This Gramian results from the similar Gramian in [19], but where the interval is fixed to $[0, \tau_{\max}]$ for our purposes.

The following result plays a crucial role in finding the change of variables that isolates the visible and invisible dynamics of (5.20).

Theorem 4 *The following holds:*

- $\mathcal{L}(\tau) = \mathrm{Ker}(Q(\tau))$;
- $\mathrm{dim}\mathcal{L}(\tau_1) = \mathrm{dim}\mathcal{L}(\tau_2) := s_{no}$ *for all* $\tau_1, \tau_2 \in [0, \tau_{\max}]$.

Proof: The proof follows the arguments in [19] specialized to the present context. First observe that for any $\tau_1, \tau_2 \in [0, \tau_{\max}]$ the following holds

$$Q(\tau_1) = \phi(\tau_2, \tau_1)^T \, Q(\tau_2) \, \phi(\tau_2, \tau_1) \, . \tag{5.21}$$

This can be seen directly as follows. Start by noting that

$$\phi(\tau_2, \tau_1)^T \, Q(\tau_2) \, \phi(\tau_2, \tau_1) = \phi(\tau_2, \tau_1)^T \int_0^{\tau_{\max}} \phi(t, \tau_2)^T \, R^T(t) R(t) \phi(t, \tau_2) dt \, \phi(\tau_2, \tau_1),$$

Then by the properties of the state transition matrix, the right side of the above equation can be rewritten as

$$\int_0^{\tau_{\max}} \phi(t, \tau_1)^T \, R^T(t) R(t) \phi(t, \tau_1) dt \, ,$$

which is $Q(\tau_1)$.

Regarding the first point, by definition of invisibility, $w \in \mathcal{L}(\tau)$ at time τ $\Rightarrow R(t)\phi(t, \tau)w(\tau) = 0$ for all $t \in [0, \tau_{\max}] \Rightarrow Q(\tau)w(\tau) = 0$, namely $w(\tau) \in \mathrm{Ker}Q(\tau)$. Now, suppose that $w(\tau) \in \mathrm{Ker}\; Q(\tau)$ at time τ, namely $Q(\tau)w(\tau) = 0$ and, in turn, $w(\tau)^T Q(\tau) w(\tau) = 0$. By definition of $Q(\cdot)$, this implies that

$$\begin{aligned} w(\tau)^T \int_0^{\tau_{\max}} \phi(t, \tau)^T \, R^T(t) R(t) \phi(t, \tau) dt \, w(\tau) &= 0, \\ \int_0^{\tau_{\max}} y_w^T(t) y_w(t) dt &= 0 \, , \end{aligned}$$

by the properties of the state-transition matrix. This then implies that $y_w(t) = 0$ for all $t \in [0, \tau_{\max}]$, namely $w(\tau)$ is invisible.

Regarding the second bullet, note that, by virtue of (5.21), $\text{rank}Q(\tau_1) = \text{rank}Q(\tau_2)$ since $\phi(\tau_2, \tau_1)$ is non-singular, then the result follows with $s_{no} = s - \text{rank}(Q(\tau))$. ■

We can now find a τ-dependent smooth change of variables $w = T(\tau)x$ such that, in the new coordinates, system (5.20) reads as

$$\begin{aligned} \dot{x}_{\text{no}} &= 0 \\ \dot{x}_{\text{o}} &= 0 \\ y_w &= R_{\text{o}}(\tau)x_{\text{o}}, \end{aligned} \tag{5.22}$$

with $x_{no} \in \mathbb{R}^{s_{no}}$, $x_o = \mathbb{R}^{s_o}$, where $s_o := s - s_{no}$, for some appropriately defined continuously differentiable function $R_{\text{o}}(\cdot) : [0, \tau_{\max}] \to \mathbb{R}^{m \times s_o}$ such that the gramian

$$Q_o(\tau) = \int_0^{\tau_{\max}} R_o^T(t)R_o(t)dt \tag{5.23}$$

associated with the x_o subsystem is non-singular for all $\tau \in [0, \tau_{\max}]$.

Pick any $\tau_0 \in [0, \tau_{\max}]$ and let $\{\text{v}_i\}_{i=1}^{s_{no}}$ be a basis of $\mathcal{L}(\tau_0)$ (namely a basis of $\text{Ker}(Q(\tau_0))$). Furthermore, let $\mathcal{C}(\tau_0)$ be the complement of $\mathcal{L}(\tau_0)$ relative to $\mathbb{R}^s$, namely the space such that $\mathcal{C}(\tau_0) \oplus \mathcal{L}(\tau_0) = \mathbb{R}^s$, and let $\{\text{v}_i\}_{i=s_o+1}^{s}$ be a basis of $\mathcal{C}(\tau_0)$.

In the following we derive a basis for $\mathcal{L}(\tau)$ and $\mathcal{C}(\tau)$, with $\mathcal{C}(\tau)$ such that $\mathcal{L}(\tau) \oplus \mathcal{C}(\tau) = \mathbb{R}^s$ for all $\tau \in [0, \tau_{\max}]$. Those bases are obtained by flowing forward and backward in time the bases of $\mathcal{L}(\tau_0)$ and $\mathcal{C}(\tau_0)$. To this end, it turns out that (see [19])

$$\mathcal{L}(\tau) = \text{span}\left\{ \phi(\tau, \tau_0) \begin{bmatrix} \mathrm{v}_1 & \dots & \mathrm{v}_{s_{no}} \end{bmatrix} \right\} \forall \tau \in [0, \tau_{\max}]. \tag{5.24}$$

To prove this, note that $\phi(\tau, \tau_0)\mathrm{v}_i \in \mathcal{L}(\tau)$ for all $i = 1, \dots, s_{no}$. As a matter of fact

$$\begin{aligned} Q(\tau)\phi(\tau, \tau_0)\mathrm{v}_i &= \phi(\tau, \tau_0)^{-T}\phi(\tau, \tau_0)^T Q(\tau)\phi(\tau, \tau_0)\mathrm{v}_i \\ &= \phi(\tau, \tau_0)^{-T} Q(\tau_0)\mathrm{v}_i = 0\,. \end{aligned}$$

Furthermore,

$$\text{rank}\ \phi(\tau, \tau_0) \begin{bmatrix} \mathrm{v}_1 & \dots & \mathrm{v}_{s_{no}} \end{bmatrix} = s_{no}\,.$$

Since $\dim\mathcal{L}(\tau) = s_{no}$, the previous facts prove (5.24). Similarly,

$$\mathcal{C}(\tau) = \text{span}\left\{ \phi(\tau, \tau_0) \begin{bmatrix} \mathrm{v}_{s_{no}+1} & \dots & \mathrm{v}_s \end{bmatrix} \right\} \tag{5.25}$$

for all $\tau \in [0, \tau_{\max}]$. As a matter of fact,

$$\text{rank}\ \phi(\tau, \tau_0) \begin{bmatrix} \mathrm{v}_{s_{no}+1} & \dots & \mathrm{v}_s \end{bmatrix} = s_o$$

and

$$\text{rank}\ \phi(\tau, \tau_0) \begin{bmatrix} \mathrm{v}_1 & \dots & \mathrm{v}_{s_{no}} & \mathrm{v}_{s_{no}+1} & \dots & \mathrm{v}_s \end{bmatrix} = s$$

by which, using (5.24), (5.25) follows.

By using the previous results we thus consider the (smooth) change of variable $w = T(\tau)x$ with

$$T(\tau) = \phi(\tau, \tau_0)V, \qquad V := \begin{bmatrix} \mathrm{v}_1 & \cdots & \mathrm{v}_{s_{no}} & \mathrm{v}_{s_{no}+1} & \cdots & \mathrm{v}_s \end{bmatrix}. \tag{5.26}$$

By construction it turns out that

$$S(\tau)w = \dot{w} = \dot{T}(\tau)x + T(\tau)\dot{x},$$

from which

$$\dot{x} = T(\tau)^{-1}(S(\tau)T(\tau) - \dot{T}(\tau))x\,.$$

Using

$$\dot{T}(\tau) = \dot{\phi}(\tau, \tau_0)V = S(\tau)\phi(\tau, \tau_0)V = S(\tau)T(\tau),$$

the previous relations yield

$$\dot{x} = 0\,.$$

Furthermore, by construction and by the definition of an invisible state space,

$$R(\tau)\phi(\tau, \tau_0)V = \begin{bmatrix} 0 & R_o(\tau) \end{bmatrix},$$

where

$$R_o(\tau) = R(\tau)\phi(\tau, \tau_0)\begin{bmatrix} \mathrm{v}_{s_{no}+1} & \cdots & \mathrm{v}_s \end{bmatrix}.$$

Simple arguments can be finally used to show $Q_o(\tau)$ is not singular for all τ in the interval. Rewrite (5.23) as

$$\mathcal{Q}_o(\tau_{\max}) = [\ \mathrm{v}_{s_{no}+1}\ \cdots\ \mathrm{v}_s\ ]^T \int_0^{\tau_{\max}} \phi(t,\tau)^T R^T(t) R(t)\phi(t,\tau)dt[\ \mathrm{v}_{s_{no}+1}\ \cdots\ \mathrm{v}_s\ ],$$

$$\mathcal{Q}_o(\tau_{\max}) = [\ \mathrm{v}_{s_{no}+1}\ \cdots\ \mathrm{v}_s\ ]^T \mathcal{Q}(\tau_{\max})[\ \mathrm{v}_{s_{no}+1}\ \cdots\ \mathrm{v}_s\ ].$$

Then by construction of $\mathcal{C}(\tau_0)$, $\mathcal{Q}_o(\tau_{\max})$ is non-singular.

We note that the subspace $\{x\ :\ x_o = 0\}$ is invariant and composed of invisible states. On the other hand the subsystem

$$\begin{aligned} \dot{x}_o &= 0 \\ y_w &= R_o(\tau)x_o \end{aligned}$$

is "visible" in the interval, namely the subspace of invisible states $\mathcal{L}_o(\tau)$ associated to the previous system is such that $\mathcal{L}_o(\tau) \equiv \{0\}$.

It is worth noting that the visibility property of the pair $(0, R_o(\cdot))$ does not imply, in general, that the pair is uniformly observable, namely that the observability matrix $\mathcal{O}_o(\tau)$ associated to the pair $(0, R_o(\tau))$ is non-singular for all τ in the interval.

Now consider the *hybrid* system (5.14), with the goal of identifying visible and invisible dynamics for this system. By applying the transformation matrix $T(\tau)$ discussed above, the jump relation of system (5.14) transforms as

$$x^+ = [T(\tau_{\max})^{-1}]^+ w^+ = T(0)^{-1} J w = T(0)^{-1} J T(\tau_{\max}) x,$$

where[2] $x = x(\tau_{\max})$ and $w = w(\tau_{\max})$. By partitioning $T(0)^{-1}JT(\tau_{\max})$ consistently with x we have

$$\begin{aligned} x_o^+ &= J_o\, x_o(\tau_{\max}) + J_{ono}\, x_{no}(\tau_{\max}), \\ x_{no}^+ &= J_{no}\, x_{no}(\tau_{\max}) + J_{noo}\, x_o(\tau_{\max}), \end{aligned} \tag{5.27}$$

where the matrices J_o, J_{ono}, J_{no} and J_{noo} do not have any special properties.

We note that, by construction, the hybrid system flowing according to (5.20) and jumping according to (5.27) is state-output equivalent to (5.14). Furthermore, we note that the x_{no} state component, which is invisible for the continuous-time system (5.20) during flows, might become visible for the hybrid system (5.14). As a matter of fact, the x_{no} component might show up during jumps by affecting x_0 through the jump relation $x_o^+ = J_o x_o + J_{ono} x_{no}$, thus affecting the output $y_w(t)$ in the "subsequent" flow interval. This means that, in the attempt to identify a system that is state-output equivalent to (5.14) and for which an asymptotic observer can be designed, it cannot be ignored.

This observation motivates the forthcoming developments in which the goal is to compute a system that is state-output equivalent to the hybrid system flowing according to (5.22) and jumping according to (5.27), by isolating the component of x_{no} that is also invisible during jumps. To this purpose, let $\Upsilon \in \mathbb{R}^{s_{no}} \times \mathbb{R}^{s_{no}}$ be the change of variables that puts the pair (J_{no}, J_{ono}) in observable canonical

[2] Here and in the following we compactly denote by $\xi(\tau_{\max})$ and $\xi(0)$ the value of a state variable ξ at the end and at the beginning of a generic time interval.

form. Namely,

$$\Upsilon J_{no}\Upsilon^{-1} = \begin{pmatrix} J'_{no} & 0 \\ \\ \star & \star \end{pmatrix}, \qquad J_{ono}\Upsilon^{-1} = \begin{pmatrix} J'_{ono} & 0 \end{pmatrix},$$

where

$$(J'_{no}, J'_{ono}) \in \mathbb{R}^{s'_{no}\times s'_{no}} \times \mathbb{R}^{(s-s_{no})\times s'_{no}}, \quad s'_{no} \geq 0\,,$$

is an observable pair, with $\star$ denoting generic blocks of no interest in the subsequent developments. By changing coordinates as $x_{no} \mapsto x'_{no} = \Upsilon x_{no}$ and by partitioning $x'_{no} = \mathrm{col}(x'_{noo}, x'_{nono})$ with $x'_{noo} \in \mathbb{R}^{s'_{no}}$, $x'_{nono} \in \mathbb{R}^{s_{no}-s'_{no}}$, it turns out that the dynamics of x_o and x'_{no} are described by the flow dynamics

$$\begin{aligned} \dot{x}_{\mathrm{o}} &= 0 \\ \dot{x}'_{\mathrm{no}} &= 0\,, \end{aligned}$$

and by the jump relation

$$\begin{aligned} x_o^+ &= J_o x_o + J'_{ono}x'_{no} \\ x'^+_{noo} &= J'_{no}x'_{noo} + J'_{noo}x_o \\ x'^+_{nono} &= \star \end{aligned}$$

where $J'_{noo} \in \mathbb{R}^{s'_{no}\times s_o}$ is the matrix obtained by extracting the first s'_{no} rows from the matrix ΥJ_{noo}, and where $\star$ denotes a linear combination of $x'_{noo}(0)$, $x'_{nono}(0)$, and $x_o(\tau_{\max})$ of no interest in the following. By keeping in mind that the output y_w is affected only by the x_{o} component, it is immediately seen that x'_{nono} has no effect on the output, neither during flows nor during jumps. Hence, we conclude that system (5.20), (5.27) is state-output equivalent to the hybrid system

$$\begin{aligned}
\dot{z}_o &= 0 \\
\dot{z}_{no} &= 0 \\
z_o^+ &= N_o z_o + N_{ono} z_{no} \\
z_{no}^+ &= N_{no} z_{no} + N_{noo} z_o \\
y_z &= R_o(\tau) z_o
\end{aligned} \tag{5.28}$$

where $z_o \in \mathbb{R}^{s_o}$, $z_{no} \in \mathbb{R}^{s'_{no}}$, $N_o = J_o$, $N_{ono} = J'_{ono}$, $N_{no} = J'_{no}$, $N_{noo} = J'_{noo}$.

All the state components of the previous system are visible. In the next section an asymptotic hybrid observer for this system is presented.

5.2.2 Design of the internal model

The goal of this subsection is to present a methodology for the design of the internal model having the output reproducer capabilities required in Proposition 1. The idea that is followed in the design is to construct a hybrid asymptotic observer for the dynamics of (5.28).

The design of the observer for the z_{no} part, which is invisible during flows but which show up during jumps, follows the intuition that a discrete-time observer could be designed using the "measurement" $z_o^+ - N_o z_o$ to construct an innovation term. As z_o^+ is not measurable we "inject" it in the z_{no} jump dynamics through the change of variables

$$z_{no} \mapsto \xi_{no} = z_{no} + K_2 z_o$$

with K_2 to be fixed. By also letting $\xi_o = z_o$, in the new coordinates system (5.28) reads as

$$\begin{array}{rcl} \dot{\xi}_o & = & 0 \\ \dot{\xi}_{no} & = & 0 \\ \xi_o^+ & = & \bar{N}_o \xi_o + \bar{N}_{ono} \xi_{no} \\ \xi_{no}^+ & = & \bar{N}_{noo} \xi_o + (N_{no} + K_2 N_{ono}) \xi_{no} \\ y_\xi & = & R_o(\tau) \xi_o \end{array} \tag{5.29}$$

where $\bar{N}_{ono} := N_{ono}$ and

$$\begin{array}{rcl} \bar{N}_o & := & (N_o - N_{ono} K_2) \\ \bar{N}_{noo} & := & N_{noo} - N_{no} K_2 + K_2 N_o - K_2 N_{ono} K_2 \,. \end{array}$$

Using the fact that the pair (N_{no}, N_{ono}) is observable we now choose K_2 such that

$$\mathrm{eig}(N_{no} + K_2 N_{ono}) \in \mathcal{D}_1 \,.$$

We now develop two different observers for (5.29) according to the properties fulfilled by the pair $(0, R_o(\tau))$.

The pair $(0, R_o(\tau))$ is not uniformly observable.

As remarked in Section 5.2.1, there is no guarantee that the "visible" pair $(0, R_o(\tau))$ is uniformly observable. The only guarantee is that the Gramian (5.23) is non-singular. In such a case the asymptotic properties of the observer are mainly obtained through jumps, by integrating the output y_w over the interval $[0, \tau_{\max}]$ to compute a "finite-time" estimate of ξ_o via inversion of the Gramian. Specifically, the following proposition holds.

Proposition 8 *Let* $\nu = 2s_o + s'_{no}$, $F_{\rm im}(\tau) \equiv 0_{\nu\times\nu}$, $\Gamma_{\rm im}(\tau) = \begin{pmatrix} 0_{m\times s_o} & R_o(\tau) & 0_{m\times s'_{no}} \end{pmatrix}$,

$$G_{\rm im}(\tau) = \begin{pmatrix} R_o^T(\tau) \\ \\ 0_{\nu - s_o \times m} \end{pmatrix}, \quad \Sigma_{\rm im} = \begin{pmatrix} 0_{s_o\times s_o} & 0_{s_o\times s_o} & 0_{s_o\times s'_{no}} \\ \bar{N}_o\, Q_o(0)^{-1} & 0_{s_o\times s_o} & \bar{N}_{ono} \\ \bar{N}_{noo}\, Q_o(0)^{-1} & 0_{s'_{no}\times s_o} & N_{no} + K_2 N_{ono} \end{pmatrix}.$$

Then the quadruplet $(F_{\rm im}, G_{\rm im}(\cdot), \Gamma_{\rm im}(\cdot), \Sigma_{\rm im})$ *has the hybrid internal model property.*

The proof of this proposition is presented in Appendix A.4. In the structure of the observer it is possible to isolate a state variable η_i, with the flow dynamics given by $\dot{\eta}_i = R_o^T(\tau) y_\xi$ and the jump map given by $\dot{\eta}_i^+ = 0$, whose goal is to estimate the state component ξ_o of (5.29) at the beginning ($\xi_o(0)$) and at the end ($\xi_o(\tau_{\max})$) of the time interval through the relation (A.12) and to use those estimates at the jump time to enforce a cascade structure of the error system (see the proof of the proposition). It is worth noting that the flow dynamics of the η_i variable are those of a pure integrator. This could open the door to some criticism about the sensitivity of the proposed observed to noise superimposed on the input u. In this respect a variant of the internal model structure presented in the previous proposition is to consider flow η_i dynamics of the form

$$\dot{\eta}_i = H\eta_i + R_o^T(\tau)u,$$

where H is a Hurwitz matrix that is introduced to filter possible noise superimposed on u. The previous construction continues to hold so long as we take care to substitute $Q_o(\cdot)$ in the expressions of Σ_{im} with the "filtered" Gramian

$$Q_o^f(\tau_{\max}) = \int_0^{\tau_{\max}} \exp(H(\tau_{\max} - t))\, R_o^T(t) R_o(t) dt\,.$$

The pair $(0, R_o(\tau))$ is uniformly observable.

An alternative observer design can be proposed if the visible pair $(0, R_o(\tau))$ is also uniformly observable in the interval. In the following the uniform observability condition is considered "output-wise" as formalized in the next assumption.

Assumption 9 *Let* $R_o(\tau) = \mathrm{col}(R_{o1}(\tau), \ldots R_{om}(\tau))$*. For all* $i = 1, \ldots, m$ *the observability matrices*

$$\mathcal{O}_i(\tau) = \begin{pmatrix} R_{oi}(\tau) \\ \dot{R}_{oi}(\tau) \\ \vdots \\ R_{oi}^{(s_o-1)}(\tau) \end{pmatrix}$$

are non-singular for all $\tau \in [0, \tau_{\max}]$.

Under this assumption, the observer for system (5.29) can be constructed by using high-gain tools to estimate, during flows, the observable component of the system. Instrumental to the design of the observer is the transformation of system (5.29) into a canonical observability form following the arguments in [1] (also see [46] and [45]).

Let $P_i(\tau) : [0, \tau_{\max}] \to \mathbb{R}^{s_o \times s_o}$, $i = 1, \ldots, m$, be defined as (see [1])

$$P_i(\tau)^{-1} = \begin{bmatrix} q_i(\tau) & \tilde{\mathcal{L}} q_i(\tau) & \ldots & \tilde{\mathcal{L}}^{s_o-1} q_i(\tau) \end{bmatrix}$$

where $q_i(\tau)$ is the last column of $\mathcal{O}_i(\tau)^{-1}$ and $\tilde{\mathcal{L}}(\cdot)$ is the differential operator

$$\tilde{\mathcal{L}} q_i(\tau) := -\dot{q}_i(\tau) \, .$$

By defining $\chi_o \in \mathbb{R}^{m s_o}$ and χ_{no} as

$$\chi_o = \mathbf{P}(\tau) \, \xi_o \, , \qquad \chi_{no} = \xi_{no}$$

with $\mathbf{P}(\tau) = \mathrm{col}\left(P_1(\tau), \cdots, P_m(\tau)\right)$, the arguments in [1] show that the flow dynamics of χ_o and χ_{no} are described by

$$\begin{aligned} \dot{\chi}_o &= (\mathbf{A} + \mathbf{r}(\tau)\mathbf{C})\,\chi_o \\ \dot{\chi}_{no} &= 0 \end{aligned} \tag{5.30}$$

where $\mathbf{A} = \mathrm{blkdiag}(A, \ldots, A)$, $\mathbf{C} = \mathrm{blkdiag}(C, \ldots, C)$,

$\mathbf{r}(\tau) = \mathrm{blkdiag}(r_1(\tau), \ldots, r_m(\tau))$, with

$$A = \begin{pmatrix} 0_{1\times s_o-1} & 0 \\ I_{s_o-1} & 0_{s_o-1\times 1} \end{pmatrix}, \quad C = \begin{pmatrix} 0_{1\times s_o-1} & 1 \end{pmatrix},$$

and with $r_i : [0, \tau_{\max}] \to \mathbb{R}^{s_o\times 1}$, $i = 1, \ldots, m$, appropriately defined smooth functions. Furthermore, $y_\xi = \mathbf{C}\chi_o$. Finally, the jump relations of χ_o and χ_{no} can be easily computed as

$$\begin{aligned} \chi_o^+ &= \mathbf{M}_o\,\chi_o + \mathbf{M}_{ono}\,\chi_{no} \\ \chi_{no}^+ &= (N_{no} + K_2 N_{ono})\,\chi_{no} + \mathbf{M}_{noo}\,\chi_o \end{aligned} \tag{5.31}$$

where $\mathbf{M}_o = \mathrm{blkdiag}(M_{o1}, \ldots, M_{om})$, $\mathbf{M}_{ono} = \mathrm{col}(M_{ono\,1}, \ldots, M_{ono\,m})$, $\mathbf{M}_{noo} = (M_{noo\,1}, \ldots, M_{noo\,m})$ with

$$M_{o\,i} = P_i(0)\,\bar{N}_o\,P_i(\tau_{\max})^{-1}, \quad M_{ono\,i} = P_i(0)\,\bar{N}_{ono}, \quad M_{noo\,i} = \bar{N}_{noo}\,P_i(\tau_{\max})^{-1},$$

$i = 1, \ldots, m$. The following proposition, proved in Appendix A.5, shows that an asymptotic observer for the χ_o and χ_{no} dynamics can always be designed.

It should also be noted that the change of variables is a Lyapunov Transformation (stability preserving map) if the following assumption holds (see [2] for LTV version or see [35] for the time-invariant version). This guarantees that the transformed system will have the same stability properties as the original system and is thus essential for this design method.

Assumption 10 *The matrices $P(\tau)$ and $P^{-1}(\tau)$ are bounded for all $\tau \in [0, \tau_{\max}]$.*

Proposition 9 *Let Assumptions 9 and 10 be satisfied. Furthermore, let $\mathbf{K}_1 = \mathrm{blkdiag}(K_1, \dots K_m)$ with*

$$K_1 = \begin{pmatrix} c_{s_o}\ell^{s_o} & c_{s_o-1}\ell^{s_o-1} & \dots & c_1\ell \end{pmatrix}^T$$

where the c_i's are coefficients of an Hurwitz polynomial and ℓ is a design parameter. Let $\nu = ms_o + s'_{no}$,

$$F_{\mathrm{im}} = \begin{pmatrix} \mathbf{A} + \mathbf{K}_1\mathbf{C} & 0_{ms_o \times s'_{no}} \\ 0_{s'_{no} \times ms_0} & 0_{s'_{no} \times s'_{no}} \end{pmatrix}, \quad G_{\mathrm{im}}(\tau) = \begin{pmatrix} \mathbf{r}(\tau) - \mathbf{K} \\ 0_{s'_{no} \times m} \end{pmatrix},$$

$$\Sigma_{\mathrm{im}} = \begin{pmatrix} \mathbf{M}_o & \mathbf{M}_{ono} \\ \mathbf{M}_{noo} & N_{no} + K_2 N_{ono} \end{pmatrix},$$

and $\Gamma_{\mathrm{im}} = (\mathbf{C}\,,\ 0_{m \times s'_{no}})$. Then there exists an $\ell^\star \geq 1$ such that for all $\ell \geq \ell^\star$ the quadruplet $(F_{\mathrm{im}}, G_{\mathrm{im}}(\tau), \Sigma_{\mathrm{im}}, \Gamma_{\mathrm{im}})$ has the hybrid internal model property.

5.3 Example: High-Precision Robust Tracking of a Spline Reference Signal by a UAV

We consider the problem of high-precision tracking of spline-generated trajectories in the lateral and longitudinal position of a rotary wing UAV such as a quad-rotor. The forthcoming developments can be easily generalized to other kinds of under-actuated vehicles, such as ducted-fans, helicopters, coaxials, multi-rotors. Denoting by (x, y, z) the lateral, longitudinal and vertical position of the UAV expressed with respect to an inertial reference frame, and by (θ, ϕ, ψ) the roll, pitch and yaw angles, it turns out that

$$M\begin{pmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{pmatrix} = \begin{pmatrix} C_\psi C_\theta & -S_\psi C_\phi + C_\psi S_\theta S_\phi & S_\phi S_\psi + C_\phi S_\theta C_\psi \\ S_\psi C_\theta & C_\phi C_\psi + S_\phi S_\theta S_\psi & -C_\psi S_\phi + S_\psi S_\theta C_\phi \\ -S_\theta & C_\theta S_\phi & C_\theta C_\phi \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ -K_T \end{pmatrix}\omega_e^2 + \begin{pmatrix} 0 \\ 0 \\ M\mathrm{g} \end{pmatrix}.$$

where M is the mass of the vehicle, g is acceleration due to gravity, K_T is the thrust coefficient (assumed equal for all the rotors), and ω_e^2 is the sum of the squares of the rotational speeds of the four propellers. In the previous parameters K_T is regarded as uncertain parameter. For the sake of simplicity we do not take into account all of the kinematics and dynamics of the vehicle by implicitly considering the roll and the pitch angles of the vehicle as virtual inputs for the lateral and longitudinal dynamics. [3]

We start by feedback linearising the vertical dynamics by choosing a proportional integral controller of the propeller speed of the form

$$\omega_e^2 = \frac{1}{K_T^o C_\theta C_\phi}(v_1 + M\mathrm{g} + \xi)\,, \qquad \dot{\xi} = v_2$$

in which K_T^o is a nominal value of K_T and v_1 and v_2 are two residual inputs to be chosen in order to stabilize the system $M\ddot{z} = -\mu(v_1 + M\mathrm{g} + \xi) + M\mathrm{g}$, $\dot{\xi} = v_2$ with $\mu = K_T/K_T^o$ an uncertain parameter. A robust dynamic controller processing the available measures (z, ξ) can be used for this purpose. Standard linear arguments then show that the resulting closed-loop vertical system is such that $z \to 0$ no matter what the actual value of μ is. As a consequence of this choice, the lateral and longitudinal dynamics read as

$$M \begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} = \mu\, R(\psi)\, V(\theta, \phi)\, (v_1 + M\mathrm{g} + \xi)$$

where $R(\psi)$ is the 2×2 elementary rotation matrix whose first and second row are respectively $(C_\psi \;\; -S_\psi)$, $(S_\psi\; C_\psi)$, and $V(\theta, \phi) = \mathrm{col}\left(\tan\theta\,,\; -\dfrac{\tan\phi}{C_\theta}\right)$. Note that $V(\theta, \phi)$ is invertible for all $\theta, \phi \in (-\pi/2, \pi/2)$. We thus design the two

[3] By following standard nomenclature in the literature of control of under-actuated vehicles, we just consider the design of the "outer loop". Standard "backstepping" tools can then be adopted to obtain the true torque input starting from a virtual control law for the roll and pitch angles.

(virtual) inputs (θ, ψ) such that

$$V(\theta, \phi) = \frac{1}{v_1 + M\mathrm{g} + \xi} R(\psi)^T u$$

where $u \in \mathbb{R}^2$ is a residual input, so as to obtain $M\mathrm{col}(\ddot{x}\,,\ \ddot{y}) = \mu u$. This system is clearly in the Brunowsky form (5.5) with $\bar{B} = \frac{\mu}{M} I_2$ and with all the other matrix parameters equal to zero. In particular the system has vector relative degree $(2, 2)$.

The goal for this system is to robustly track a spline trajectory which passes through the points $\{(1, 1), (1, 2), (1, 3), (2, 3), (2, 2), (2, 1)\}$ in a cycle (see Figure 5.1). The spline in the plane can be obtained by generating spline trajectories of the form indicated in Chapter 4 for each of the x and y coordinates. According to the theory in Section 4.1, each spline can be thought of as generated by an exosystem of dimension 10 (as $N = 6$). In particular, the reference signals for the x and y coordinates can be thought of as generated by $\dot{w}_x = S_x w_x$, $y_x^\star = Q_x w_x$ and $\dot{w}_y = S_y w_y$, $y_y^\star = Q_y w_y$ where $Q_x \in \mathbb{R}^{1\times 10}$, $Q_y \in \mathbb{R}^{1\times 10}$, $S_x \in \mathbb{R}^{10\times 10}$ and $S_y \in \mathbb{R}^{10\times 10}$ are constructed as in Section 4.1. The overall exosystem thus has dimension $s = 20$ with $S = \mathrm{blkdiag}(S_x,\, S_y)$ and the Q in (5.5) having the form $Q = \mathrm{blkdiag}(Q_x,\, Q_y)$. In this case the hybrid regulation equations can be easily solved, yielding an expression for $R(\tau)$ of the form

$$R(\tau) = \bar{B}^{-1}\, Q\, S^2\, .$$

Note that $R(\cdot)$ is affected by an uncertain parameter μ entering in $\bar{B}$. By bearing in mind Proposition 6, the fact that the uncertain parameter only affects $\bar{B}$, and thus $R(\cdot)$, in a scalar multiplicative way, implies that no duplication of $R(\tau)$ is needed for robust design, namely $\mathbf{R}(\tau) = R(\tau)$ and $\mathbf{s} = s = 20$.

Recall that in Chapter 4 we showed that it is not reliable to use the methods of internal model design from Chapter 2. The reason for this is because the solution to the hybrid internal model property equations (see Theorem 1) does not have constant rank, therefore the internal model obtained is not continuous during flows (see Proposition 1 and the pertinent Figure 4.2).

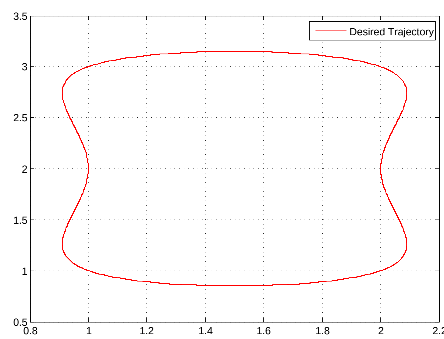

Figure 5.1: 2-D Quad-rotor Trajectory

Thus, we implement the design procedure presented in Section 5.2. Specifically, we follow the design procedure presented in Proposition 8. We define the errors $e_x = x - Q_x w_x$ and $e_y = y - Q_y w_y$ as the only available measurements, and thus design the high gain observer presented in Proposition 7 (see also [18]) to reconstruct velocities.

For this particular exosystem it turns out that the size of the sub-state that is visible through jumps, s_o, is four. This can be readily seen from the structure of (S, R). In this case in fact, the visible sub-state is the same as the observable sub-state. Recall (from Section 4.1) that (S_i, R_i) has the following form

$$S_i = \begin{pmatrix} S_1 & 0 \\ & \\ 0 & 0 \end{pmatrix}, \quad R_i = \begin{pmatrix} 0 & 0 & 1 & 0 & \dots & 0 \end{pmatrix},$$

where S_i is the 4×4 only containing ones on its super-diagonal. Then knowing that $S = \text{blkdiag}(S_x, S_y)$, $R = \text{blkdiag}(R_x, R_y)$, the visibility properties become obvious. This means that the sub-state η_o of the internal model is a vector of length four.

Furthermore, we can see that some additional states of the steady-state generator will be "observable" by the s_o sub-state via jumps. This sub-state is what is called s_{noo}. Since the matrices S and J undergo the transformation defined in (5.26) the structure becomes a little messy, but we can infer from our knowledge of the original jump map J, recall (4.4), that from L_2 and J_{22} some of this "observability" will exist. In fact for this example, s_{noo} turns out be a vector of length nine. This means that the sub-state η_{no} is a vector of dimension nine and we are able to "throw out" seven states, i.e. the exosystem is length twenty, but our internal model (η_o, η_{no}) need only be dimension eleven. Recall, also, that we need the additional sub-state η_i for the finite observer, so the full internal model $(\eta_i, \eta_o, \eta_{no})$ ends up being dimension fifteen. We should also comment on the intuition behind (L_2, J_{22}) here. Recall that J_{22} is simply the shift matrix, but since L_{22} depends on the points that the spline trajectory should pass through, the observability properties of the pair end up being dependent on the desired trajectory. This is interesting because it then means that there may be some trajectories that need smaller internal models despite passing through the same number of points, simply dependent on what the points actually are.

Proceeding with the internal model design, for this example we choose K_2 via solving an LQR problem with N_{no} and N_{ono} and using identity matrices for the weighting parameters. For the stabilizer feedback we choose $H = \text{blkdiag}(H_x, H_y)$ with both H_x and H_y taken as $\begin{pmatrix} 1 & 1 \end{pmatrix}$ and with the high gain parameter $\kappa = 10^2$.

The observer for the plant state is designed with the choices
$\Lambda_{\text{st}_i} = \text{col}\begin{pmatrix} 2g, & g^2 \end{pmatrix}$, with the high gain parameter $g = 10^2$, from which Λ_{st} and Φ_{st} are defined via Proposition 7. Recall from the discussion of the regulator design just above Proposition 7 that Λ_{st} and Φ_{st} are block diagonal matrices with Λ_{st_i} and Φ_{st_i} being the block elements. In this case $m = 2$ so there are two block elements.

The results from these regulator choices can be seen in Figures 5.2. In order to test the robustness of the regulator we simulate a large change to the parameter μ at time $t = 18$sec, with μ instantaneously switching from $\mu = 0.8$ to $\mu = 0.3$. From a physical view point, the decrease in the thrust coefficient can be motivated by battery discharge. Due to the high-gain feedback inherent in the stabilizer unit used for the regulator, there are some large spikes in the input from peaking. In fact we implement a saturation for the input to avoid some of the larger peaking effects.

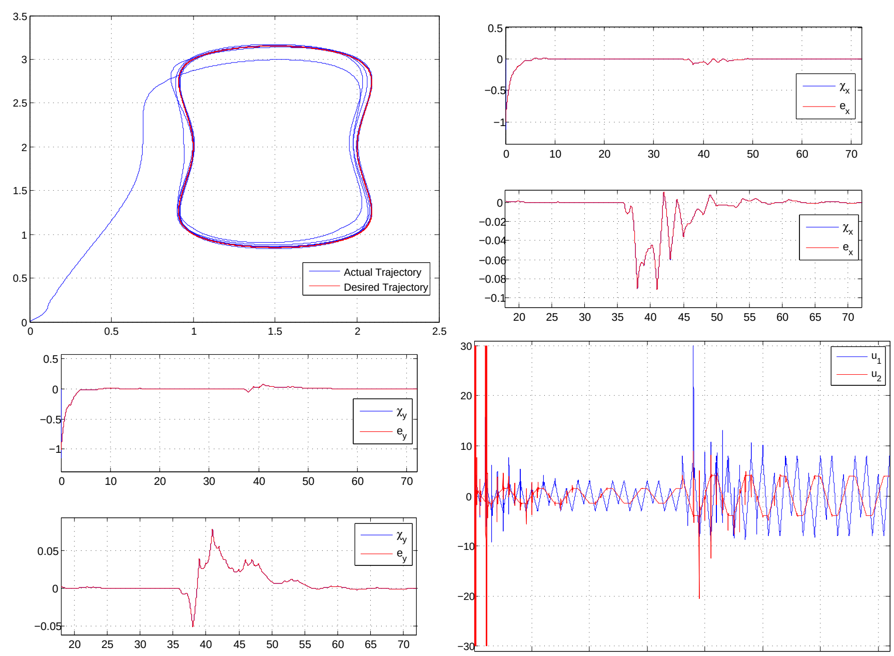

Figure 5.2: Internal model-based design. Actual Quad-Rotor Trajectory; Error in the x-coordinate; Error in the y-coordinate; Input $u(\tau)$

A further numerical analysis has been made by implementing a controller without an internal model, namely a controller of the form

$$u = R(\tau)w - \kappa\bar{B}^{-1}H\chi,$$

which uses a nominal feedforward term (the $R(\tau)$ in the regulator has been taken with $\mu = 1$) to nominally track the desired trajectory, along with the stabilization feedback, as above the variable χ is the sub-state of the regulator corresponding to the stabilizer unit. The results are shown in Figure 5.3 in which the large tracking error can be observed.

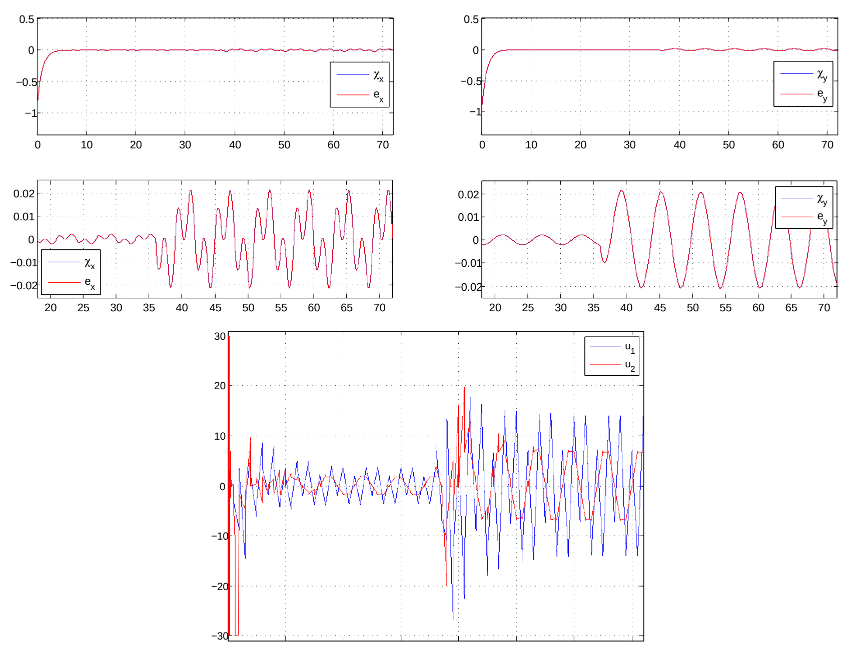

Figure 5.3: Regulator without internal model. Error in the x-coordinate; Error in the y-coordinate; input $u(\tau)$

5.4 Conclusion

We have successfully designed an internal model unit for achieving robust output regulation for a class of linear hybrid systems and exosystems with periodic jump times. This achieves the goal of generalizing work done in previous chapters. The quad-rotor example shows that there are possible practical applications for this work, especially in the area of attempting to achieve robust tracking goals, where plant parameters can be uncertain.

Chapter 6

Conclusions

We have developed a method for designing regulators that achieve robust output regulation for linear hybrid systems with periodic jump times. In doing so, a general method for developing internal models has been developed, along with guidelines for achieving robustness with respect to plant parameters. By this we mean that the regulator will achieve output regulation even in the presence of uncertain plant parameters. This is the important aspect of the internal model principle as originally developed by Francis and Wonham.

Future directions for work in hybrid output regulation could go in a few directions. Generalizing results from [12] to allow for unknown jump clocks for a larger set of systems could be one direction. Specifically the results of [12] limit the framework to relative degree one systems. It turns out that proving stability for higher relative degree systems for this case proves difficult as some of the cascade structure used in the original proof breaks down. Further work could go into developing output regulation for nonlinear hybrid systems. Preliminary steps have been taken in this direction in [8] and [14] (also see [49]). Additionally, in sticking with the theme of output regulation for periodically jumping systems, the framework could be extended to systems with an average dwell-time, where again [49] (see also [32]) could be useful.

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Appendix A

Appendices

A.1 Hybrid High-Gain Stabilization

Proposition 10 *Consider the system*

$$\left.\begin{array}{rcl}\dot{\tau} &=& 1,\\ \dot{v}_1 &=& A_1(\tau)v_1 + B_1(\tau)v_2,\\ \dot{v}_2 &=& A_2(\tau)v_2 + B_2(\tau)v_1 + \kappa H v_2,\end{array}\right\} \quad (\tau, v_1, v_2) \in [0,T] \times \mathbb{R}^\rho \times \mathbb{R}^\sigma,$$
$$\left.\begin{array}{rcl}\tau^+ &=& 0,\\ v_1^+ &=& J_1 v_1 + L_1 v_2,\\ v_2^+ &=& J_2 v_2 + L_2 v_1,\end{array}\right\} \quad (\tau, v_1, v_2) \in \{T\} \times \mathbb{R}^\rho \times \mathbb{R}^\sigma, \tag{A.1}$$

where $T > 0$, H is Hurwitz, $B_i(\tau)$ and $A_2(\tau)$ are bounded on $\tau \in [0,T]$ and $A_1(\tau)$ is continuous on $\tau \in [0,T]$. Furthermore there is a continuously differentiable, symmetric, matrix $P_1(\tau)$, such that

$$0 < p_{1a} I \leq P_1(\tau) \leq p_{1b} I, \forall \tau \in [0,T], \tag{A.2}$$

which satisfies the matrix differentiable equation

$$\begin{aligned} -\dot{P}_1(\tau) &= P_1(\tau)A_1(\tau) + A_1^T(\tau)P_1(\tau) + Q_1(\tau), \\ 0 &> J_1^T P_1(0) J_1 - P_1(T), \end{aligned} \tag{A.3}$$

where $Q_1(\tau)$ *is continuous, symmetric and positive definite, ie.*

$$Q_1(\tau) \geq q_1 I > 0, \forall \tau \in [0, T].$$

There exists $\kappa^* > 0$ *such that, for each* $\kappa \geq \kappa^*$*, the set* $[0, T] \times \{0\} \subset \mathbb{R}^{\rho+\sigma+1}$ *is globally exponentially stable.*

Proof: This proof breaks the system (A.1) into two subsystems. Simply take the v_1 dynamics as the first system and treat v_2 as its input, then do the reverse for the second system. Finally, the two subsystems are interconnected and it is shown that the interconnection is GES.

For a function β depending on a state ϕ that satisfies $\phi^+ = g(\phi)$, we define the shorthand notation $\beta^+ := \beta(g(\phi))$. Similarly, if β is continuously differentiable and ϕ satisfies $\dot{\phi} = f(\phi)$, we define the shorthand notation $\dot{\beta} :=< \nabla\beta(\phi), f(\phi) >$.

To show that the interconnection is GES, we find a Lyapunov function, Ψ, and positive constants, $\alpha_1, \ldots, \alpha_4$, such that:

$$\alpha_1 ||v||_2^2 \leq \Psi(v, \tau) \leq \alpha_2 ||v||_2^2, \tag{A.4}$$

$$\Psi^+ - \Psi \leq -\alpha_3 ||v||_2^2, \tag{A.5}$$

$$\dot{\Psi} \leq -\alpha_4 ||v||_2^2, \tag{A.6}$$

where $||\cdot||_2$ denotes the Euclidean norm and $v = (v_1, v_2)$. This implies that the system with state v has the origin GES, since this means that:

$$\begin{aligned} \Psi^+ &\leq \exp(-\lambda)\Psi, \\ \dot{\Psi} &\leq -\lambda\Psi, \end{aligned}$$

for some $\lambda > 0$. It then follows that

$$\Psi(v(t,j), \tau(t,j)) \leq \exp(-\lambda(t+j))\Psi(v(0,0), \tau(0,0)).$$

In turn, $||v(t,j)||_2$ can be bounded using the inequality (A.4).

A.1.1 System A

Consider the system

$$\left.\begin{aligned} \dot{\tau} &= 1, \\ \dot{v}_1 &= A_1(\tau)v_1 + B_1(\tau)v_2, \end{aligned}\right\} \quad (\tau, v_1) \in [0, T] \times \mathbb{R}^\rho,$$

$$\left.\begin{aligned} \tau^+ &= 0, \\ v_1^+ &= J_1 v_1 + L_1 v_2, \end{aligned}\right\} \quad (\tau, v_1) \in \{T\} \times \mathbb{R}^\rho.$$

The Lyapunov function candidate

$$W = v_1^T P_1(\tau) v_1,$$

where $P_1(\tau)$ satisfies the inequality (A.2) and the matrix differential equation (A.3), with $Q_1(\tau) \geq q_1 I > 0, \forall \tau \in [0,T]$, satisfies the following relations.

$$\begin{aligned}
\dot{W} &= \left(A_1(\tau)v_1 + B_1(\tau)v_2\right)^T P_1(\tau)v_1 + v_1^T P_1(\tau)\left(A_1(\tau)v_1 + B_1(\tau)v_2\right) + v_1^T \dot{P}_1(\tau)v_1, \\
&= v_1^T\Big(A_1(\tau)P_1(\tau) + P_1(\tau)A_1(\tau)\dot{P}_1(\tau)\Big)v_1 + \tfrac{1}{\gamma_1}v_1^T P_1^2(\tau)v_1 + \gamma_1 v_2^T B_1^T(\tau)B_1(\tau)v_2, \\
\dot{W} &= -v_1^T Q_1(\tau)v_1 + \tfrac{1}{\gamma_1}v_1^T P_1^2(\tau)v_1 + \gamma_1 v_2^T B_1^T(\tau)B_1(\tau)v_2.
\end{aligned}$$

For $\gamma_1 > 0$ fixed large enough, the following inequality holds,

$$\dot{W} \quad \leq \quad -c_1||v_1||_2^2 + c_2||v_2||_2^2. \tag{A.7}$$

Also,

$$\begin{aligned}
W^+ - W &= \left(J_1 v_1 + L_1 v_2\right)^T P_1(0)\left(J_1 v_1 + L_1 v_2\right) - v_1^T P_1(T)v_1, \\
&= v_1^T\left(J_1^T P_1(0)J_1 - P_1(T)\right)v_1 + \tfrac{1}{\gamma_2}v_1^T\left(J_1^T P_1^2(0)J_1\right)v_1 + \gamma_2 v_2^T L_1^T L_1 v_2.
\end{aligned}$$

For $\gamma_2 > 0$ fixed large enough, the following inequality holds,

$$W^+ - W \quad \leq \quad -d_1||v_1||_2^2 + d_2||v_2||_2^2. \tag{A.8}$$

A.1.2 System B

Consider the system

$$\left.\begin{aligned}
\dot{\tau} &= 1, \\
\dot{v}_2 &= A_2(\tau)v_2 + B_2(\tau)v_1 + \kappa H v_2,
\end{aligned}\right\} \quad (\tau, v_2) \in [0,T] \times \mathbb{R}^\sigma,$$

$$\left.\begin{aligned}
\tau^+ &= 0, \\
v_2^+ &= J_2 v_2 + L_2 v_1,
\end{aligned}\right\} \quad (\tau, v_2) \in \{T\} \times \mathbb{R}^\sigma.$$

The Lyapunov function candidate

$$V = \exp(\lambda\tau)v_2^T P_2 v_2,$$

where P_2 satisfies

$$-Q_2 = P_2 H + H^T P_2,$$

with $Q_2 > I > 0$, satisfies the following relations.

$$\begin{aligned}
\dot{V} &= \lambda\exp(\lambda\tau)v_2^T P_2 v_2 + \exp(\lambda\tau)\left(A_2(\tau)v_2 + B_2(\tau)v_1 + \kappa H v_2\right)^T P_2 v_2 \ldots \\
&\quad + \exp(\lambda\tau)v_2^T P_2\left(A_2(\tau)v_2 + B_2(\tau)v_1 + \kappa H v_2\right), \\
\dot{V} &= \exp(\lambda\tau)v_2^T\left(\lambda P_2 + A_2^T P_2 + P_2 A_2(\tau) + \kappa\left(H^T P_2 + P_2 H\right) + \gamma_3 I\right)v_2 \ldots \\
&\quad + \tfrac{1}{\gamma_3}\exp(\lambda\tau)v_1^T B_2^T(\tau)P_2^2 B_2(\tau)v_1.
\end{aligned}$$

Choose $\gamma_3 = \kappa$, then

$$\dot{V} \leq \exp(\lambda\tau)\left((\lambda c_3 + c_4 - \kappa c_5)\,||v_2||_2^2 + \tfrac{1}{\kappa}c_6||v_1||_2^2\right). \tag{A.9}$$

Furthermore,

$$\begin{aligned}
V^+ - V &= \left(J_2 v_2 + L_2 v_1\right)^T P_2\left(J_2 v_2 + L_2 v_1\right) - \exp(\lambda T)v_2^T P_2 v_2, \\
&= v_2^T\left(2J_2^T P_2 J_2 - \exp(\lambda T)P_2\right)v_2 + v_1^T L_2^T P_2 L_2 v_1.
\end{aligned}$$

Thus,

$$V^+ - V \leq (d_3 - \exp(\lambda T)d_4)\,||v_2||_2^2 + d_5||v_1||_2^2. \tag{A.10}$$

A.1.3 Interconnection

Now, we can combine the two Lyapunov function candidates as

$$\Psi = W + \ell V, \tag{A.11}$$

to create a Lyapunov function satisfying (A.4) for the system (A.1). Combining (A.8) with (A.10) gives

$$\begin{aligned}
\Psi^+ - \Psi &\leq -d_1||v_1||_2^2 + d_2||v_2||_2^2 + \ell\left(\left(d_3 - \exp(\lambda T)d_4\right)||v_2||_2^2 + d_5||v_1||_2^2\right), \\
\Psi^+ - \Psi &\leq \left(\ell d_5 - d_1\right)||v_1||_2^2 + \left(\ell\left(d_3 - \exp(\lambda T)d_4\right) + d_2\right)||v_2||_2^2.
\end{aligned}$$

Choose $\ell > 0$ small enough such that $\ell d_5 - d_1 < 0$, with ℓ fixed, choose $\lambda > 0$ large enough such that $\ell\left(d_3 - \exp(\lambda T)d_4\right) + d_2 < 0$. Then, the inequality (A.5) is fulfilled.

Furthermore, combining (A.7) with (A.9) gives

$$\begin{aligned}
\dot{\Psi} &\leq -c_1||v_1||_2^2 + c_2||v_2||_2^2 + \exp(\lambda\tau)\left(\left(\lambda c_3 + c_4 - \kappa c_5\right)||v_2||_2^2 + \tfrac{1}{\kappa}c_6||v_1||_2^2\right), \\
\dot{\Psi} &\leq \left(\exp(\lambda\tau)\tfrac{c_6}{\kappa} - c_1\right)||v_1||_2^2 + \left(\exp(\lambda\tau)\left(\lambda c_3 + c_4 - \kappa c_5\right) + c_2\right)||v_2||_2^2.
\end{aligned}$$

With $\lambda > 0$ previously fixed and $\tau \in [0, T]$, we are free to pick $\kappa > 0$ large enough such that

$$\max\left(\exp(\lambda\tau)\frac{c_6}{\kappa} - c_1, \exp(\lambda\tau)\left(\lambda c_3 + c_4 - \kappa c_5\right) + c_2\right) < 0.$$

Then, the inequality (A.6) is satisfied. ∎

A.2 Various Parameter Definitions from Chapter 3

In the proof of Theorem 3 we use the following parameters,

$$\dot{e}^{(m)} = \left(\begin{bmatrix} \alpha_1 & \dots & \alpha_m & 0 \end{bmatrix} + K\right)\begin{pmatrix} \dot{e} \\ 0 \end{pmatrix} + \\ +\bar{A}_{21}A_{11}\tilde{z} + \bar{A}_{21}Be_1 + b\left(R(\tau)S + \frac{dR(\tau)}{d\tau}\right)\tilde{\xi} + bKD_g^{-1}gH\tilde{\eta},$$

$$\dot{e} = \begin{bmatrix} \begin{matrix} 0_{(m-2)\times 1} & I_{m-2} \\ -k_1 \ \dots & -k_{m-1} \end{matrix} & 0_{(m-1)\times 1} \\ \begin{bmatrix} 0 & -k_1 & \dots & -k_{m-2} & 0 \end{bmatrix} + k_{m-1}\begin{bmatrix} k_1 & \dots & k_{m-1} & 0 \end{bmatrix} \end{bmatrix} e + \\ + \begin{bmatrix} 0_{(m-1)\times 1} \\ \dot{\tilde{e}}_m - k_{m-1}\tilde{e}_m \end{bmatrix},$$

$$\Xi_e = \tfrac{1}{b}G\begin{bmatrix} \alpha_1 & \dots & \alpha_m \end{bmatrix}\begin{bmatrix} I_{m-1} \\ -k_1 \cdots - k_{m-1} \end{bmatrix},$$

$$\Xi_{\tilde{e}_m} = \tfrac{1}{b}G\begin{bmatrix} \alpha_1 & \dots & \alpha_m \end{bmatrix}\begin{bmatrix} 0_{(m-1)\times 1} \\ 1 \end{bmatrix},$$

$$H_{\tilde{\eta}} = \begin{bmatrix} M_{22} & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0_{m\times 1} \\ 1 \end{bmatrix}\left(bKD_g^{-1}\left(I - \begin{bmatrix} M_{22} & 0 \\ 0 & I \end{bmatrix}\right) - R(0)G\begin{bmatrix} 0_{1\times m} & 1 \end{bmatrix}\right),$$

$$H_e = \begin{bmatrix} 0_{m\times m} \\ \begin{bmatrix} \alpha_1 \dots \alpha_m \end{bmatrix} (I - M_{22}) - R(0)G \begin{bmatrix} \alpha_1 \dots \alpha_m \end{bmatrix} - \bar{A}_{21} M_{12} \end{bmatrix} \cdot$$
$$\cdot \begin{bmatrix} I_{m-1} & \\ & 0_{m\times 1} \\ -k_1 \dots -k_{m-1} & \end{bmatrix},$$

$$H_{\tilde{e}_m} = \begin{bmatrix} 0_{m\times m} \\ \begin{bmatrix} \alpha_1 & \dots & \alpha_m \end{bmatrix} (I - M_{22}) - R(0)G \begin{bmatrix} \alpha_1 & \dots & \alpha_m \end{bmatrix} - \bar{A}_{21} M_{12} \end{bmatrix} \cdot$$
$$\cdot \begin{bmatrix} 0_{(m-1)\times 1} \\ 1 \end{bmatrix},$$

$$H_{\tilde{z}} = \begin{bmatrix} 0_{m\times 1} \\ 1 \end{bmatrix} \left(A_{12}\left(I - M_{11}\right) - R(0)GA_{12}\right),$$

and

$$H_{\tilde{\xi}} = \begin{bmatrix} 0_{m\times 1} \\ 1 \end{bmatrix} b\left(R(T) - R(0)(J + GR(T))\right).$$

A.3 Proof of Proposition 7

The closed loop system analyzed here is the combination of the plant in Brunovsky's canonical form (5.5) and the exosystem (5.2), along with the regulator consisting of an internal model unit and a stabilization unit (IM). From proposition 4 it is known that the internal model unit achieves the hybrid internal model property, therefore it is now fair to assume that the following equations are satisfied:

$$\begin{aligned}
\frac{d\Pi_z(\tau)}{d\tau} &= A_{11}\,\Pi_z(\tau) - A_{12}Q - \Pi_z(\tau)S + P_1 \\
0 &= M_{11}\,\Pi_z(\tau_{\max}) - \Pi_z(0)\,J + M_{12}\,\Pi_\xi(\tau_{\max}) + M_{13}\,\Pi_{y_r}(\tau_{\max}) + N_1
\end{aligned}$$

along with the constraint

$$0=\begin{pmatrix} M_{21} \\ M_{31} \end{pmatrix}\Pi_z(\tau_{\max})-\begin{pmatrix} \Pi_\xi(0) \\ \Pi_{y_r}(0) \end{pmatrix}J+\begin{pmatrix} M_{22} \\ M_{32} \end{pmatrix}\Pi_\xi(\tau_{\max})+\begin{pmatrix} M_{23} \\ M_{33} \end{pmatrix}\Pi_{y_r}(\tau_{\max})+\begin{pmatrix} N_2 \\ N_3 \end{pmatrix}$$

and the internal model property

$$\begin{aligned}
\frac{d\Pi_\eta(\tau)}{d\tau} &= F_{\mathrm{im}}\Pi_\eta(\tau) - \Pi_\eta(\tau)S(\tau) + G_{\mathrm{im}}R(\tau), \\
0 &= \Sigma_{\mathrm{im}}\Pi_\eta(\tau_{\max}) - \Pi_\eta(0)J, \\
R(\tau) &= \Gamma_{\mathrm{im}}(\tau)\Pi_{eta}(\tau).
\end{aligned}$$

With these equations in mind, we perform the following changes of variables in order to analyze the behavior of the the system in coordinates that are unaffected by the steady-state behavior of the exosystem:

$$\begin{aligned}
z \to \tilde{z} &= z - \Pi_z(\tau)w, \\
\xi \to \tilde{\xi} &= \xi - \Pi_\xi w, \\
y_r \to \tilde{y}_r &= y_r - \Pi_{y_r}(\tau)w.
\end{aligned}$$

This puts the system into the following form.

$$
\begin{aligned}
\dot{\tilde{z}} &= A_{11}\tilde{z} + A_{12}L\tilde{\xi}, \\
\dot{\tilde{\xi}} &= A_{22}\tilde{\xi} + A_{23}\tilde{y}_r, \\
\dot{\tilde{y}}_r &= A_{31}\tilde{z} + A_{32}\tilde{\xi} + A_{33}\tilde{y}_r + \bar{B}(u - R(\tau)w), \\
\dot{\eta} &= F_{\mathrm{im}}(\tau)\eta + G_{\mathrm{im}}(\tau)u, \\
\dot{\chi} &= \Phi\chi + \Lambda L\tilde{\xi},
\end{aligned}
$$

during flows, and

$$
\begin{aligned}
\tilde{z}^{+} &= \tilde{z}, \\
\tilde{\xi}^{+} &= \tilde{\xi}, \\
\tilde{y}_r &= \tilde{y}_r, \\
\eta^{+} &= \Sigma_{\mathrm{im}}\eta, \\
\chi^{+} &= \chi,
\end{aligned}
$$

during jumps, with the feedback $u = \Gamma_{\mathrm{im}}(\tau)\eta - \kappa\bar{B}^{-1}H\chi$, and with

$$
R(\tau) = \bar{B}^{-1}(\dot{\Pi}_{y_r}(\tau) + \Pi_{y_r}(\tau)S(\tau) - A_{31}\Pi_z(\tau) - A_{32}\Pi_\xi(\tau) - A_{3}3\Pi_{y_r}(\tau) - P_3).
$$

Note that all of the sub-systems are continuous-time, with the possible exception of the internal model unit.

The following transformation puts the plant into a form that is easily shown to have the desired asymptotically stable behavior by utilizing the choice of Hurwitz coefficients for the k_{ij}.

$$
\tilde{y}_r \to \tilde{e}_r = \tilde{y}_r + \bar{K}\tilde{\xi},
$$

where the matrix parameter $\bar{K}$ is the block diagonal matrix where the blocks, $\bar{K}_i$ on the diagonal are chosen as

$$\bar{K}_i = \begin{pmatrix} k_{i1} & \dots & k_{ir_i-1} \end{pmatrix}.$$

Additionally, the following transform is applied to put the stabilizer unit into error coordinates in order to show that it properly tracks the plant:

$$\chi \to \tilde{\chi} = \chi - G_{\mathrm{im}}(\tau)\bar{B}^{-1}\tilde{e}_r - \Pi_\eta(\tau)w.$$

After these coordinate transformations we are left with the following system, which has a nice cascade structure for a applying a Lyapunov analysis.

$$
\begin{aligned}
\dot{\tilde{z}} &= A_{11}\tilde{z} + A_{12}L\tilde{\xi}, \\
\dot{\tilde{\xi}}_i &= S_i\tilde{\xi}_i + B_i(\tilde{e}_r - (\begin{array}{ccc} k_{i1} & \dots & k_{ir_i-1} \end{array})\tilde{\xi}_i), \\
\dot{\tilde{e}}_r &= A_{31}\tilde{z} + (A_{33} - A_{33}\bar{K} + \bar{K}(A_{22} - A_{23}\bar{K}))\tilde{\xi} \dots \\
&\quad +(A_{33} + \bar{B}\Gamma_{\mathrm{im}}(\tau)G_{\mathrm{im}}(\tau)\bar{B}^{-1} + \bar{K}A_{23} - \kappa I_m)\tilde{e}_r, \\
\dot{\tilde{\eta}} &= F_{\mathrm{im}}(\tau)\tilde{\eta} + (F_{\mathrm{im}}(\tau)G_{\mathrm{im}}(\tau)\bar{B}^{-1} + G_{\mathrm{im}}(\tau)\Gamma_{\mathrm{im}}(\tau)G_{\mathrm{im}}(\tau)\bar{B}^{-1} + \tfrac{dG_{\mathrm{im}}(\tau)}{d\tau}\bar{B}^{-1} \dots \\
&\quad -G_{\mathrm{im}}(\tau)\bar{B}^{-1}(A_{33} + \bar{B}\Gamma_{\mathrm{im}}(\tau)G_{\mathrm{im}}(\tau)\bar{B}^{-1}))\tilde{e}_r \dots \\
&\quad -G_{\mathrm{im}}(\tau)\bar{B}^{-1}(A_{31}\tilde{z} + (A_{32} - A_{33}\bar{K})\tilde{\xi}) - \dots \\
&\quad G_{\mathrm{im}}(\tau)\bar{B}^{-1}\bar{K}((A_{22} - A_{23}\bar{K})\tilde{\xi} + A_{23}\tilde{e}_r), \\
\dot{\tilde{\chi}}_i &= g\tilde{H}_i\tilde{\chi}_i - \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} (A_{31_i}\tilde{z} + A_{32_i}\tilde{\xi} + A_{33_i}(\tilde{e}_{r_i} - (k_{i1} \dots k_{ir_i-1})\tilde{\xi}_i) \dots \\
&\quad +\bar{B}_i(\Gamma_{\mathrm{im}}(\tau)(\tilde{\eta} + G_{\mathrm{im}}(\tau)\bar{B}^{-1}\tilde{e}_r) - \kappa H D_g^{-1}\tilde{\chi} - \kappa I_m\tilde{e}_r)),
\end{aligned}
$$

during flows, with $\tilde{H}$ a Hurwitz matrix, and

$$\begin{aligned}
\tilde{z}^+ &= \tilde{z}, \\
\tilde{\xi}^+ &= \tilde{\xi}, \\
\tilde{e}_r^+ &= -\bar{K}\xi + \tilde{e}_r, \\
\tilde{\eta}^+ &= \Sigma_{\text{im}}\tilde{\eta} + \Sigma_{\text{im}}G_{\text{im}}(\tau_{\max})\bar{B}^{-1}\tilde{e}_r - G_{\text{im}}(0)\bar{B}^{-1}(-\bar{K}\tilde{\xi} + \tilde{e}_r), \\
\tilde{\chi}^+ &= \chi,
\end{aligned}$$

during jumps.

At this point we can take advantage of the small gain theorem presented in Proposition 10 by picking $v_1 = (\tilde{\xi}, \tilde{z}, \tilde{\eta})$ and $v_2 = (\tilde{e}_r)$, then tuning the feedback gain κ. Finally, apply Proposition 10 again with $v_1 = (\tilde{\xi}, \tilde{z}, \tilde{\eta}, \tilde{e}_r)$ and $v_2 = \tilde{\chi}$ to tune the observer gain parameter, g.

A.4 Proof of Proposition 8

By bearing in mind (5.17), the fact that system (5.14) is state-output equivalent to system (5.29), and the choice of $\Gamma_{\text{im}}(\tau)$, we prove the result by showing that the state η_o, and η_{no} of the system

$$\begin{aligned}
\dot{\eta}_i &= R_o^T(\tau)\, y_\xi\,, \quad \dot{\eta}_o = 0\,, \quad \dot{\eta}_{no} = 0 \\
\eta_i^+ &= 0 \\
\eta_o^+ &= \bar{N}_o\, Q_o(\tau_{\max})^{-1}\, \eta_i + \bar{N}_{ono}\, \eta_{no} \\
\eta_{no}^+ &= \bar{N}_{noo}\, Q_o(\tau_{\max})^{-1}\, \eta_i + (N_{no} + K_2 N_{ono})\, \eta_{no}
\end{aligned}$$

asymptotically estimate the variables ξ_o and ξ_{no} of (5.29). We first consider the η_i variable whose hybrid dynamics immediately yields

$$\eta_i(\tau_{\max}) = \int_0^{\tau_{\max}} R_o^T(\tau) R_o(\tau) d\tau\ \xi_o(0) = Q_o(\tau_{\max})\, \xi_o(0) = Q_o(\tau_{\max})\, \xi_o(\tau_{\max})\,. \tag{A.12}$$

By defining the error coordinates as $\tilde{\eta}_i = \eta_i - Q_o(\tau_{\max})\xi_o$, $\tilde{\eta}_o = \eta_o - \xi_o$, $\tilde{\eta}_{no} = \eta_{no} - \xi_{no}$, using the fact that $\xi_o(\tau_{\max}) = Q_o(\tau_{\max})^{-1}\eta_i(\tau_{\max})$, the error system is represented by

$$\begin{aligned}
\dot{\tilde{\eta}}_i &= R_o^T(\tau) R_o(\tau)\xi_o\,, \quad \dot{\tilde{\eta}}_o = 0\,, \quad \dot{\tilde{\eta}}_{no} = 0 \\
\tilde{\eta}_i^+ &= 0 \\
\tilde{\eta}_o^+ &= \bar{N}_{ono}\, \tilde{\eta}_{no} \\
\tilde{\eta}_{no}^+ &= (N_{no} + K_2 N_{ono})\, \tilde{\eta}_{no}.
\end{aligned}$$

Using the fact that $\mathrm{eig}(N_{no}+K_2N_{ono}) \in \mathcal{D}_1$ we immediately conclude that $\tilde{\eta}_o(t,j)$ and $\tilde{\eta}_{no}(t,j)$ converge to zero. In particular, partitioning the change of variables $T(\tau)^{-1}$ and Υ introduced in Section 5.2.1 as $T(\tau)^{-1} = \mathrm{col}(T_o(\tau)^{-1}, T_{no}(\tau)^{-1})$

and $\Upsilon = \mathrm{col}(\Upsilon_o, \Upsilon_{no})$ consistently with the partition $x = \mathrm{col}(x_o,\, x_{no})$ and $x'_{no} = \mathrm{col}(x'_{noo}, x'_{nono})$, it turns out that the hybrid internal model property of Definition 2 is fulfilled with a $\Pi_\eta(\tau)$ of the form

$$\Pi_\eta(\tau) = \begin{pmatrix} Q_o(\tau)T_o(\tau)^{-1} \\ T_o(\tau)^{-1} \\ \Upsilon_o T_{no}(\tau)^{-1} + K_2 T_o(\tau)^{-1} \end{pmatrix}.$$

A.5 Proof of Proposition 9

By bearing in mind (5.17), the fact that system (5.14) is state-output equivalent to system (5.29), the choice of $\Gamma_{\mathrm{im}}(\tau)$, and the fact that $y_w = \mathbf{C}\chi_o$, we prove the result by showing that the system

$$\begin{aligned} \dot\eta &= F_{\mathrm{im}}\eta + G_{\mathrm{im}}(\tau)y_\xi \\ \eta^+ &= \Sigma_{\mathrm{im}}\eta \end{aligned}$$

with $\eta = \mathrm{col}(\eta_o, \eta_{no})$, $\eta_o \in \mathbb{R}^{ms_o}$, $\eta_{no} \in \mathbb{R}^{s'_{no}}$, is an asymptotic observer of the (χ_o, χ_{no}) system (5.30)-(5.31). Letting $\tilde\eta_o = \eta_o - \chi_o$ and $\tilde\eta_{no} = \eta_{no} - \chi_{no}$, the error system reads as

$$\begin{aligned} \dot{\tilde\eta}_o &= (\mathbf{A} + \mathbf{K}_1\mathbf{C})\,\tilde\eta_o \\ \dot{\tilde\eta}_{no} &= 0 \\ \tilde\eta_o^+ &= \mathbf{M}_o\,\tilde\eta_o + \mathbf{M}_{ono}\,\tilde\eta_{no} \\ \tilde\eta_{no}^+ &= (N_{no} + K_2 N_{ono})\,\tilde\eta_{no} + \mathbf{M}_{noo}\,\tilde\eta_o \end{aligned} \tag{A.13}$$

By rescaling the $\tilde{\eta}_o$ variable as $\tilde{\eta}'_o = \mathbf{D}(\ell)\eta_o$, with

$$\mathbf{D}(\ell) = \mathrm{blkdiag}(D(\ell), \ldots, D(\ell)) \qquad D(\ell) = \mathrm{diag}\begin{pmatrix} 1 & \ell & \ldots & \ell^{s_o-1} \end{pmatrix}$$

system (A.13) transforms as

$$\begin{array}{rcl} \dot{\tilde{\eta}}'_o &=& \ell H\, \tilde{\eta}'_o \\ \dot{\tilde{\eta}}_{no} &=& 0 \\ \tilde{\eta}'^{+}_o &=& \mathbf{D}(\ell)\, \mathbf{M}_o\, \mathbf{D}^{-1}(\ell)\, \tilde{\eta}'_o + \mathbf{D}(\ell)\, \mathbf{M}_{ono}\, \tilde{\eta}_{no} \\ \tilde{\eta}^{+}_{no} &=& (N_{no} + K_2 N_{ono})\, \tilde{\eta}_{ono} + \mathbf{N}_{noo}\, \mathbf{D}^{-1}(\ell)\, \tilde{\eta}'_o \end{array} \tag{A.14}$$

in which H is Hurwitz. From this, using the fact that $\mathrm{eig}(N_{no} + K_2 N_{ono}) \in \mathcal{D}_1$, the result immediately follows by using the next Proposition 11 whose proof follows similar arguments to Proposition 10. In particular, partitioning the change of variables $T(\tau)^{-1}$ and Υ introduced in Section 5.2.1 as $T(\tau)^{-1} = \mathrm{col}(T_o(\tau)^{-1}, T_{no}(\tau)^{-1})$ and $\Upsilon = \mathrm{col}(\Upsilon_o, \Upsilon_{no})$ consistently with the partition $x = \mathrm{col}(x_o, x_{no})$ and $x'_{no} = \mathrm{col}(x'_{noo}, x'_{nono})$, it turns out that the hybrid internal model property of Definition 2 is fulfilled with a $\Pi_\eta(\tau)$ of the form

$$\Pi_\eta(\tau) = \begin{pmatrix} \mathbf{P}(\tau) T_o(\tau)^{-1} \\ \Upsilon_o T_{no}(\tau)^{-1} + K_2 T_o(\tau)^{-1} \end{pmatrix}.$$

A.6 Polynomial Growth vs. Exponential Decay

Proposition 11 *Consider the system*

$$\left.\begin{array}{rcl}\dot{\tau} & = & 1, \\ \dot{v}_1 & = & A_1 v_1, \\ \dot{v}_2 & = & \ell A_2 v_2,\end{array}\right\} \qquad (\tau, v_1, v_2) \in [0,T] \times \mathbb{R}^\rho \times \mathbb{R}^\sigma$$

$$\left.\begin{array}{rcl}\tau^+ & = & 0 \\ v_1^+ & = & J_1 v_1 + L_1(\ell) v_2 \\ v_2^+ & = & J_2(\ell) v_2 + L_2(\ell) v_1\end{array}\right\} \quad (\tau, v_1, v_2) \in \{T\} \times \mathbb{R}^\rho \times \mathbb{R}^\sigma$$

where, $T > 0$, the matrices $J_2(\ell)$, $L_1(\ell)$ and $L_2(\ell)$ are polynomially dependent on ℓ. If A_2 is Hurwitz and $J_1 \exp(A_1 T) \in \mathcal{D}_1$, then there exists $\ell^\star > 0$ such that, for each $\ell \geq \ell^\star$, the set $[0,T] \times \{0\} \subset \mathbb{R}^{\rho+\sigma+1}$ is globally exponentially stable.

Similar techniques are used for proving Proposition 10, so only a sketch of the proof for Proposition 11 is provided.

Proof: First, choose the Lyapunov function

$$W = \exp(-\epsilon\tau) v_1^T \exp(A_1(T-\tau))^T X \exp(A_1(T-\tau)) v_1,$$

where $\epsilon > 0$ is to be chosen and $X = X^T > 0$ is the solution to the relevant discrete Lyapunov equation. Then W can be bound along flows and jumps as

$$\begin{array}{rcl}\dot{W} & \leq & -c_2 ||v_1||_2^2, \\ W^+ - W & \leq & -d_3 ||v_1||_2^2 + d_4 p_2(\ell) ||v_2||_2^2,\end{array}$$

where, $c_i, d_i > 0$ are scalar constants, $||\cdot||_2^2$ denotes the squared Euclidean norm, and $p_i(\ell)$ denotes a scalar polynomial function of ℓ. Now, choose the Lyapunov function

$$V = \exp(\ell\lambda\tau)v_2^T P v_2,$$

where $\lambda > 0$ is to be chosen and $P = P^T > 0$ is the solution to the relevant Lyapunov equation. Then, we can bound it along flows and jumps as

$$\begin{aligned} \dot{V} &\leq \ell(\lambda c_1 - c_4)||v_2||_2^2, \\ V^+ - V &\leq (d_1 p_1(\ell) - \exp(\ell\lambda T)d_2)\,||v_2||_2^2 + d_5 p_{L_2}(\ell)||v_1||_2^2. \end{aligned}$$

Combining these Lyapunov functions as $\Psi = rW + V$, where $r > 0$ is to be chosen, gives the Lyapunov function for the full system. This leads to

$$\begin{aligned} \dot{\Psi} &\leq -rc_2||v_1||_2^2 + \ell(\lambda c_1 - c_4)||v_2||_2^2, \\ \Psi^+ - \Psi &\leq (-rd_3 + d_5 p_{L_2}(\ell))||v_1||_2^2 + \\ &\quad + (rd_4 p_2(\ell) + d_1 p_1(\ell) - \exp(\ell\lambda T)d_2)\,||v_2||_2^2. \end{aligned}$$

Therefore, we can pick $\lambda > 0$ such that $\lambda c_1 - c_4 < 0$, then pick $r = d_3^{-1}(d_5 p_{L_2}(\ell) + \ell)$. Finally, with r and λ fixed, we can choose $\ell > 0$ large enough such that $rd_4 p_2(\ell) + d_1 p_1(\ell) - \exp(\ell\lambda T)d_2 < 0$. ■